TANNAKIAN RECONSTRUCTION OF REDUCTIVE GROUP SCHEMES

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ABSTRACT. We give sharp criteria for when a reductive group scheme satisfies Tannakian reconstruction. When the base scheme is Noetherian, we explicitly identify its Tannaka group scheme.

1. INTRODUCTION

This note contains some observations on the category of finite-rank representations of a reductive group scheme.

To be precise, let S be an affine scheme and $G \to S$ be a flat affine group scheme. Let $\mathrm{Vect}(S)^G$ denote the category of G-equivariant vector bundles on S, i.e. finite projective $\mathcal{O}_S$-modules equipped with an $\mathcal{O}_G$-comodule structure. It embeds in the category $\mathrm{QCoh}(S)^G$ of G-equivariant quasi-coherent sheaves on S.

Write $\omega : \mathrm{Vect}(S)^G \to \mathrm{Vect}(S)$ for the forgetful functor. The presheaf $\underline{\mathrm{Aut}}^\otimes(\omega)$ of symmetric monoidal automorphisms of ω receives a natural map from G:

$$G \to \underline{\mathrm{Aut}}^\otimes(\omega). \tag{1.1}$$

It is known that (1.1) is an isomorphism when S is a Dedekind domain, by classical Tannakian reconstruction of Saavedra, Deligne, and Milne [SR72], [DMOS82], [Del90].

For a general affine scheme S, the morphism (1.1) may fail to be an isomorphism. The purpose of this note is to understand the source of this failure in the case of a reductive group scheme.

1.1. Summary of results

1.1.1. For any affine scheme S and reductive group scheme $G \to S$, our Theorem 2.0.1 asserts that the following conditions are equivalent:

(1) G satisfies the *strong resolution property*, i.e. every object of $\mathrm{QCoh}(S)^G$ is a G-equivariant quotient of a direct sum of objects in $\mathrm{Vect}(S)^G$;
(2) G satisifes *Tannakian reconstruction*, i.e. (1.1) is an isomorphism;
(3) G is *linear*, i.e. it is a closed subgroup scheme of $\mathrm{GL}_{n,S} \to S$ for some $n \geq 0$;
(4) The radical torus $\mathrm{Rad}(G)$ is *isotrivial*, i.e. it splits over a finite étale cover of S.

1.1.2. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are established in much greater generality by Schäppi [Sch13, Corollary 7.5.2], although we supply a direct proof in the case of flat affine group schemes. The implication (3) $\Rightarrow$ (1) is due to Thomason [Tho87, Theorem 2.18] when S is Noetherian and we explain the redundancy of this hypothesis. The equivalence (3) $\Leftrightarrow$ (4) is due to Gille [Gil21], which we do not reproduce.

In [Sch13, §8.2], Schäppi poses the following question: does there exist a flat affine group scheme such that $\mathrm{Vect}(S)^G$ does not generate $\mathrm{QCoh}(S)^G$ as an abelian category? This

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property is formally equivalent to the strong resolution property, so Theorem 2.0.1 answers Schäppi's question in the affirmative and produces explicit examples.

1.1.3. When S is furthermore connected and Noetherian, we determine the Tannaka group scheme $\underline{\mathrm{Aut}}^{\otimes}(\omega)$ of G.

To state the answer, we observe that the torus $\mathrm{Rad}(G)$ has a maximal isotrivial quotient $\mathrm{Rad}(G) \twoheadrightarrow \mathrm{Rad}(G)^f$. Let G^f be the push-out of G along this map. Then G^f is representable by a reductive group scheme. Our Theorem 3.2.3 constructs a canonical isomorphism:

$$G^f \cong \underline{\mathrm{Aut}}^{\otimes}(\omega) \tag{1.2}$$

of affine group schemes under G.

This result can be seen as a refinement of the equivalence between the isotriviality of $\mathrm{Rad}(G)$ and the Tannakian reconstruction of G. To my knowledge, it is the first instance where it is possible to explicitly identify a Tannaka group scheme which possibly differs from the original group scheme.

1.1.4. This paper is organized as follows. Section 2 proves the equivalence among criteria for Tannakian reconstruction of a reductive group scheme (Theorem 2.0.1). Section 3 identifies the Tannaka group $\underline{\mathrm{Aut}}^{\otimes}(\omega)$ in the Noetherian setting (Theorem 3.2.3).

1.2. Acknowledgements

2. Criteria for reconstruction

Let $S = \mathrm{Spec}(R)$ be an affine scheme and $G \to S$ be a flat affine group scheme. Hom-sets in the category $\mathrm{QCoh}(S)^G$ are denoted by $\mathrm{Hom}_G(-, -)$. We view $\mathcal{O}_G$ as an object of $\mathrm{QCoh}(S)^G$ via the group operation.

The goal of this section is to prove the following statement.

Theorem 2.0.1. *If* $G \to S$ *is reductive, the following are equivalent:*
  (1) G *satisfies the strong resolution property;*
  (2) G *satisfies Tannakian reconstruction;*
  (3) G *is linear;*
  (4) $\mathrm{Rad}(G)$ *is isotrivial.*

The implications $(1) \Rightarrow (2) \Rightarrow (3)$ are established in §2.1. The implication $(3) \Rightarrow (1)$ is the subject of §2.2. We quote [Gil21] for the equivalence $(3) \Leftrightarrow (4)$. Finally, we point out in Corollary 2.3.2 that these conditions are met when S is a normal domain (not assumed Noetherian).

2.1. $(1) \Rightarrow (2) \Rightarrow (3)$

2.1.1. For any $\mathcal{F} \in \mathrm{QCoh}(S)^G$, consider the comma category $\mathrm{Vect}(S)^G_{/\mathcal{F}}$ of pairs $(\mathcal{V}, f)$ where $\mathcal{V} \in \mathrm{Vect}(S)^G$ and $f : \mathcal{V} \to \mathcal{F}$ is a morphism in $\mathrm{QCoh}(S)^G$. There is a canonical morphism:

$$L_{\mathcal{F}} : \operatorname*{colim}_{(\mathcal{V}, f) \in \mathrm{Vect}(S)^G_{/\mathcal{F}}} \mathcal{V} \to \mathcal{F}. \tag{2.1}$$

The implications $(1) \Rightarrow (2) \Rightarrow (3)$ in Theorem 2.0.1 follow from the assertions below, which clarify the relationship among these conditions.

Proposition 2.1.2. *Let* $G \to S$ *be a flat affine group scheme. Then:*

(a) G *satisfies the strong resolution property if and only if* $L_{\mathcal{F}}$ *is bijective for all* $\mathcal{F} \in \mathrm{QCoh}(S)^G$;

(b) G *satisfies Tannakian reconstruction if and only if* $L_{\mathcal{O}_G}$ *is bijective;*

(c) *when* $G \to S$ *is of finite type,* G *is linear if and only if* $L_{\mathcal{O}_G}$ *is surjective.*

Proof of Proposition 2.1.2(a). Since every colimit in $\mathrm{QCoh}(S)^G$ is a quotient of a direct sum, bijectivity of $L_{\mathcal{F}}$ for all $\mathcal{F} \in \mathrm{QCoh}(S)^G$ implies the strong resolution property.

To prove the converse, we first observe that $L_{\mathcal{F}}$ is surjective under the hypothesis. It remains to prove that it is injective. Since the index category $\mathrm{Vect}(S)^G_{/\mathcal{F}}$ contains finite direct sums, it suffices to show that for an individual object $(\mathcal{V}, f) \in \mathrm{Vect}(S)^G_{/\mathcal{F}}$, an element $v \in \mathcal{V}$ with $f(v) = 0$ vanishes in the colimit.

Since $G \to S$ is flat, the R-submodule $\mathrm{Ker}(f) \subset \mathcal{V}$ inherits a G-module structure. The strong resolution property gives some $\mathcal{V}_1 \in \mathrm{Vect}(S)^G$ with a morphism $\mathcal{V}_1 \to \mathrm{Ker}(f)$ whose image contains v. The composition $\mathcal{V}_1 \to \mathcal{V} \to \mathcal{F}$ vanishes, showing that the map:

$$\mathcal{V}_1 \to \operatorname*{colim}_{(\mathcal{V},f) \in \mathrm{Vect}(S)^G_{/\mathcal{F}}} (\mathcal{V})$$

is zero, so in particular, v vanishes in the colimit. $\qquad\square$

2.1.3. Before proving assertion (b), we record an observation: for each $\mathcal{V} \in \mathrm{Vect}(S)^G$, there is a canonical isomorphism between the R-module of G-equivariant maps $\mathcal{V} \to \mathcal{O}_G$ and the R-linear dual of $\mathcal{V}$:

$$\mathrm{Hom}_G(\mathcal{V}, \mathcal{O}_G) \cong \mathcal{V}^\vee. \qquad (2.2)$$

Indeed, this map is defined by composing $f : \mathcal{V} \to \mathcal{O}_G$ with the counit $\epsilon : \mathcal{O}_G \to R$. Its inverse is given by composing the coaction map $\mathcal{V} \to \mathcal{V} \otimes \mathcal{O}_G$ with a given $\varphi \in \mathcal{V}^\vee$.

2.1.4. Let $\omega : \mathrm{Vect}(S)^G \to \mathrm{Vect}(S)$ denote the forgetful functor. For any affine S-scheme S', write $\omega_{S'}$ for the composition of ω with the natural functor $\mathrm{Vect}(S) \to \mathrm{Vect}(S')$.

The presheaf $\underline{\mathrm{Aut}}^\otimes(\omega)$ sending an affine S-scheme $S' = \mathrm{Spec}(R')$ to the group of automorphisms of $\omega_{S'}$ as a symmetric monoidal functor is representable by an affine group scheme (see [Del90, §4] or [Wed04, §2]):

$$\underline{\mathrm{Aut}}^\otimes(\omega) \cong \mathrm{Spec}(\mathrm{coend}(\omega^\vee \otimes_R \omega)).$$

Here, $\omega^\vee \otimes_R \omega$ denotes the functor:

$$(\mathrm{Vect}(S)^G)^{\mathrm{op}} \times \mathrm{Vect}(S)^G \to \mathrm{QCoh}(S), \quad \mathcal{V}_1, \mathcal{V}_2 \mapsto (\mathcal{V}_1)^\vee \otimes_R \mathcal{V}_2,$$

and $\mathrm{coend}(\omega^\vee \otimes_R \omega)$ is equipped with a natural Hopf algebra structure in $\mathrm{QCoh}(S)$.

There is a canonical map:

$$G \to \underline{\mathrm{Aut}}^\otimes(\omega), \qquad (2.3)$$

sending an S'-point of G to its action on $\mathcal{V} \otimes_R R'$ for all $\mathcal{V} \in \mathrm{Vect}(S)^G$. The condition that G satisfies Tannakian reconstruction translates to the assertion that (2.3) is an isomorphism of affine group schemes over S.

Proof of Proposition 2.1.2(b). To each object $(\mathcal{V}, f) \in \mathrm{Vect}(S)^G_{/\mathcal{O}_G}$, one may functorially attach a map of R-modules $\mathcal{V} \to \mathcal{V}^\vee \otimes_R \mathcal{V}$, $v \mapsto f^\vee \otimes v$ where $f^\vee \in \mathcal{V}^\vee$ corresponds to f under (2.2). Composing with the tautological map $\mathcal{V}^\vee \otimes_R \mathcal{V} \to \mathrm{coend}(\omega^\vee \otimes_R \omega)$, we obtain a morphism $\mathcal{V} \to \mathrm{coend}(\omega^\vee \otimes_R \omega)$.

This process defines a map:

$$\operatorname*{colim}_{(\mathcal{V},f)\in\operatorname{Vect}(S)^G_{/\mathcal{O}_G}} \mathcal{V} \to \operatorname{coend}(\omega^\vee \otimes_R \omega). \tag{2.4}$$

which we shall prove to be bijective.

Indeed, for any $\mathcal{M} \in \operatorname{QCoh}(S)$, a morphism from the coend to $\mathcal{M}$ is an R-linear natural transformation $\mathcal{V} \to \mathcal{V} \otimes_R \mathcal{M}$, $\mathcal{V} \in \operatorname{Vect}(S)^G$. A morphism from the colimit to $\mathcal{M}$ is a compatible system of R-linear maps $\mathcal{V} \to \mathcal{M}$ for each $f : \mathcal{V} \to \mathcal{O}_G$ in $\operatorname{QCoh}(S)^G$. The bijection between them is given by (2.2).

To conclude, we observe that the morphism $\mathrm{L}_{\mathcal{O}_G}$ corresponds to the canonical map $\operatorname{coend}(\omega^\vee \otimes_R \omega) \to \mathcal{O}_G$ under the isomorphism (2.4). $\square$

2.1.5. We introduce some notations to be used in the proof of Proposition 2.1.2(c).

Let $\mathcal{M} \in \operatorname{Vect}(S)$. The presheaf on S which sends an affine S-scheme $S' = \operatorname{Spec}(R')$ to the group (resp. monoid) of R'-linear automorphisms (resp. endomorphisms) of $\mathcal{M} \otimes_R R'$ is representable by an affine group scheme $\operatorname{GL}(\mathcal{M}) \to S$ (resp. $\underline{\operatorname{End}}(\mathcal{M}) \to S$).

Linearity of G is equivalent to the condition of admitting a closed immersion of group schemes $G \hookrightarrow \operatorname{GL}(\mathcal{M})$ for some $\mathcal{M} \in \operatorname{Vect}(S)$, because there always exists $\mathcal{M}' \in \operatorname{Vect}(S)$ such that $\mathcal{M} \oplus \mathcal{M}'$ is free.

2.1.6. Given $\mathcal{M} \in \operatorname{Vect}(S)$, the following data are equivalent:

(1) a G-equivariance structure on $\mathcal{M}$;
(2) a morphism of monoid schemes $G \to \underline{\operatorname{End}}(\mathcal{M})$ over S.

Indeed, a G-equivariance structure on $\mathcal{M}$ is encoded by a coaction map $\mathcal{M}^\vee \otimes_R \mathcal{M} \to \mathcal{O}_G$, or a map of R-coalgebras $\operatorname{Sym}_R(\mathcal{M}^\vee \otimes_R \mathcal{M}) \to \mathcal{O}_G$.

Since G is a group, any morphism of monoid schemes $G \to \underline{\operatorname{End}}(\mathcal{M})$ factors through the open subscheme $\operatorname{GL}(\mathcal{M}) \subset \underline{\operatorname{End}}(\mathcal{M})$.

Proof of Proposition 2.1.2(c). We borrow the isomorphism (2.4) from the previous proof. It suffices to show that G is linear if and only if the canonical map corresponding to (2.3):

$$\operatorname{coend}(\omega^\vee \otimes_R \omega) \to \mathcal{O}_G \tag{2.5}$$

is surjective.

If G is linear, then there exists some $\mathcal{V} \in \operatorname{Vect}(S)^G$ such that the coaction map $\mathcal{V}^\vee \otimes_R \mathcal{V} \to \mathcal{O}_G$ induces a surjection $\operatorname{Sym}_R(\mathcal{V}^\vee \otimes_R \mathcal{V}) \to \mathcal{O}_G$. This surjection factors through (2.5), implying that surjectivity of the latter.

Conversely, note that $\operatorname{coend}(\omega^\vee \otimes_R \omega)$ is a quotient of $\bigoplus_{\mathcal{V} \in \operatorname{Vect}(S)^G}(\mathcal{V}^\vee \otimes_R \mathcal{V})$. Since $\operatorname{Vect}(S)^G$ admits finite direct sums and $\mathcal{O}_G$ is a finite type R-algebra, there exists some $\mathcal{V} \in \operatorname{Vect}(S)^G$ such that the image of $\mathcal{V}^\vee \otimes_R \mathcal{V} \to \mathcal{O}_G$ contains a set of generators of $\mathcal{O}_G$. This defines a closed immersion of monoid schemes $G \to \underline{\operatorname{End}}(\mathcal{M})$, so G is linear. $\square$

Remark 2.1.7. For a flat affine group scheme $G \to S$, the strong resolution property has two additional equivalent characterizations:

(1) $\operatorname{Vect}(S)^G$ generates $\operatorname{QCoh}(S)^G$ as an abelian category: any morphism f in $\operatorname{QCoh}(S)^G$ annihilated by $\operatorname{Hom}_G(\mathcal{V}, -)$ for all $\mathcal{V} \in \operatorname{Vect}(S)^G$ is necessarily zero.
(2) $\mathcal{O}_G \in \operatorname{QCoh}(S)^G$ is a filtered colimit of objects which belong to $\operatorname{Vect}(S)^G$. (Such $\mathcal{O}_G$ is known as an "Adams Hopf algebra".)

See [Hov04, §1.4] and [Sch12, §6.1] for a proof of these equivalences.

2.2. (3) $\Rightarrow$ (1)

2.2.1. Suppose that X is an S-scheme equipped with a G-action. Let $\mathrm{QCoh}(X)^G$ denote the category of G-equivariant quasi-coherent sheaves over X, and $\mathrm{Vect}(X)^G \subset \mathrm{QCoh}(X)^G$ the full subcategory of G-equivariant vector bundles.

We say that the pair (G, X) satisfies the *strong resolution property* if for every $\mathcal{F} \in \mathrm{QCoh}(X)^G$, there exists a family of objects $\mathcal{V}_\alpha \in \mathrm{Vect}(X)^G$ (for $\alpha \in A$) together with a G-equivariant surjection $\bigoplus_{\alpha \in A} \mathcal{V}_\alpha \twoheadrightarrow \mathcal{F}$.

In particular, the strong resolution property of G is equivalent to that of the pair (G, S).

2.2.2. For an invertible sheaf $\mathcal{L}$ on X, we use the notion of being S-*ample* as defined in [Sta18, 01VG].

Let $f : X \to S$ denote the structure map. The existence of an S-ample invertible sheaf on X implies that f is quasi-compact and separated ([Sta18, 01VI]). In particular, the functor $f_* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(S)$ is well-defined in this situation.

Lemma 2.2.3. *Suppose that* G *satisfies the strong resolution property. Given any* S-*scheme* X *equipped with a* G-*action which admits a* G-*equivariant,* S-*ample invertible sheaf, the pair* (G, X) *satisfies the strong resolution property.*

Proof. Let $f : X \to S$ denote the structure map. Suppose $\mathcal{F} \in \mathrm{QCoh}(X)^G$. For each integer $k \geq 1$, the canonical morphism $f^* f_*(\mathcal{F} \otimes \mathcal{L}^{\otimes k}) \to \mathcal{F} \otimes \mathcal{L}^{\otimes k}$ is G-equivariant, where $f^* f_*(\mathcal{F} \otimes \mathcal{L}^{\otimes k})$ is equipped with the G-equivariance structure induced from that of $\mathcal{F} \otimes \mathcal{L}^{\otimes k}$.

Since $\mathcal{L}$ is S-ample, the induced map below is surjective ([Sta18, 01Q3]):

$$\bigoplus_{k \geq 0} \mathcal{L}^{\otimes -k} \otimes f^* f_*(\mathcal{F} \otimes \mathcal{L}^{\otimes k}) \twoheadrightarrow \mathcal{F}. \tag{2.6}$$

Because G satisfies the strong resolution property, for each $k \geq 0$, there exists a family $\mathcal{V}_k^{(\alpha)} \in \mathrm{Vect}(S)^G$ (for $\alpha \in A_k$) with a surjection $\bigoplus_{\alpha \in A_k} \mathcal{V}_k^{(\alpha)} \twoheadrightarrow f_*(\mathcal{F} \otimes \mathcal{L}^{\otimes k})$. The composition:

$$\bigoplus_{k \geq 0} \bigoplus_{\alpha \in A_k} \mathcal{L}^{\otimes -k} \otimes f^* \mathcal{V}_k^{(\alpha)} \twoheadrightarrow \bigoplus_{k \geq 0} \mathcal{L}^{\otimes -k} \otimes f^* f_*(\mathcal{F} \otimes \mathcal{L}^{\otimes k}) \twoheadrightarrow \mathcal{F}$$

is the sought-for surjection from a sum of objects in $\mathrm{Vect}(X)^G$. $\square$

Lemma 2.2.4. *Suppose that* G *is of finite presentation and satisfies the strong resolution property. Given a closed immersion* $H \to G$ *of flat affine group schemes such that* $X := G/H$ *satisfies the hypothesis of Lemma 2.2.3,* H *also satisfies the strong resolution property.*

Proof. The pair $(G, G/H)$ satisfies the strong resolution property by Lemma 2.2.3. Since $G \to G/H$ is faithfully flat and of finite presentation, the same holds for $G/H \to S$.

We have a commutative diagram of categories:

$$
\begin{array}{ccc}
\mathrm{Vect}(G/H)^G & \xrightarrow{\cong} & \mathrm{Vect}(S)^H \\
\downarrow & & \downarrow \\
\mathrm{QCoh}(G/H)^G & \xrightarrow{\cong} & \mathrm{QCoh}(S)^H
\end{array}
$$

where the horizontal functors are equivalences (fppf descent) and the vertical functors are fully faithful. The strong resolution property of $(G, G/H)$ thus implies that of (H, S). $\square$

2.2.5. Recall that an affine group scheme $G \to S$ is *reductive* if it is smooth with geometric fibers being connected reductive.

If $G \to S$ is reductive, then for any closed immersion of affine group schemes $G \hookrightarrow \mathrm{GL}_{n,S}$ over S, the quotient $\mathrm{GL}_{n,S}/G$ is representable by an affine S-schemes. This follows from [Alp14, Theorem 9.4.1 & 9.7.5].

Proposition 2.2.6 (Thomason). *Suppose that* $G \to S$ *is reductive. If* G *is linear, then it satisfies the strong resolution property.*

Proof. Lemma 2.2.4 reduces the problem to showing that $GL_{n,S}$ satisfies the strong resolution property.

By Lemma 2.2.3 applied to the morphism $S \to \mathrm{Spec}(\mathbb{Z})$, it suffices to show that $GL_{n,\mathrm{Spec}(\mathbb{Z})}$ satisfies the strong resolution property. Since $\mathbb{Z}$ is a Dedekind domain, any flat affine group scheme over it satisfies the strong resolution property ([Ser68, Proposition 2 & 3]). $\square$

2.3. Additional remarks

2.3.1. Suppose that G is reductive and satisfies the equivalent conditions of Theorem 2.0.1. Then any parabolic subgroup $P \subset G$ as well as the unipotent radical $N_P \subset P$ also satisfy the strong resolution property. Indeed, this follows from Lemma 2.2.4.

Corollary 2.3.2. *If* S *is the spectrum of a normal domain, then any reductive group scheme* $G \to S$, *as well as its parabolic subgroups and their unipotent radicals, satisfy Tannakian reconstruction.*

Proof. Combine Theorem 2.0.1 with [Guo20, Lemma 2.2]. For the statements on subgroups of G, we invoke the implication (1) $\Rightarrow$ (2) of Theorem 2.0.1, which does not require the reductive hypothesis. $\square$

Remark 2.3.3. Wedhorn [Wed04, §5.17] asserts that every flat affine group scheme over a valuation ring satisfies Tannakian reconstruction, but the proof contains a gap in §5.6 of *op.cit.*. This result gives a positive answer for reductive group schemes and their special subgroups.

3. The Tannaka group scheme

We assume that S is an affine connected Noetherian scheme. This hypothesis guarantees that étale coverings of S are locally Noetherian, so their connected components are open.

We study the maximal isotrivial quotient of tori in §3.1. Then we apply it to the radical torus of a reductive group schemes $G \to S$ to determine its Tannaka group scheme.

3.1. Maximal isotrivial quotients

3.1.1. Fix a geometric point $\bar{s} \to S$. Let $\Pi_1(S, \bar{s})$ denote the "pro-groupe fondamental élargi" of [ABD$^+$66, X, §10.6]. It pro-represents the functor sending an abstract group Γ to the set of étale Γ-torsors rigidified along $\bar{s}$.

It follows from [ABD$^+$66, X, Théorème 7.1] that the functor $T \mapsto \mathrm{Hom}(T_{\bar{s}}, \mathbb{G}_{m,\bar{s}})$ defines an equivalence of categories between tori on S and finite free $\mathbb{Z}$-modules equipped with a "continuous" $\Pi_1(S, \bar{s})$-action, i.e. one which factors through a group.

Under this equivalence, a torus T is isotrivial if and only if the corresponding $\Pi_1(S, \bar{s})$-action on $\Lambda := \mathrm{Hom}(T_{\bar{s}}, \mathbb{G}_{m,\bar{s}})$ factors through a finite group.

3.1.2. Let $T \to S$ be a torus with associated $\Pi_1(S, \bar{s})$-module Λ. Denote by $\Lambda^f \subset \Lambda$ the subset of elements whose $\Pi_1(S, \bar{s})$-orbit is finite. Then $\Lambda^f \subset \Lambda$ is a $\mathbb{Z}$-submodule and Λ/Λ^f is torsion-free. In particular, it induces a surjection of tori over S:

$$T \twoheadrightarrow T^f. \tag{3.1}$$

The torus T^f is isotrivial and the morphism (3.1) is the universal morphism from T to an isotrivial torus over S: it is the "maximal isotrivial quotient" of T.

Remark 3.1.3. Applying the same construction to $\check{\Lambda} := \mathrm{Hom}(\mathbb{G}_{m,\bar{s}}, \mathrm{T}_{\bar{s}})$ also defines the "maximal isotrivial subtorus" of T.

Lemma 3.1.4. *Pulling back along* (3.1) *defines an equivalence of categories:*

$$\mathrm{Vect}(\mathrm{S})^{\mathrm{T}^f} \cong \mathrm{Vect}(\mathrm{S})^{\mathrm{T}}. \tag{3.2}$$

Proof. Since (3.1) is surjective, the canonical functor $\mathrm{Vect}(\mathrm{S})^{\mathrm{T}^f} \to \mathrm{Vect}(\mathrm{S})^{\mathrm{T}}$ is fully faithful. It remains to prove essential surjectivity, i.e. the T-action on any object $\mathcal{V} \in \mathrm{Vect}(\mathrm{S})^{\mathrm{T}}$ factors through T^f.

Suppose that the $\Pi_1(\mathrm{S}, \bar{s})$-action on Λ factors through a surjection $\Pi_1(\mathrm{S}, \bar{s}) \twoheadrightarrow \Gamma$ where Γ is a group (rather than a pro-group). We then obtain an étale Γ-torsor $\mathrm{S}_1 \to \mathrm{S}$ rigidified along $\bar{s}$, i.e. equipped with a lift $\bar{s}_1 \to \mathrm{S}_1$ of $\bar{s}$.

The scheme S_1 is connected. Otherwise, we write S_1' for the connected component containing $\bar{s}_1$. It is an étale Γ'-torsor for the subgroup $\Gamma' \subset \Gamma$ preserving S_1'. Furthermore, there is a canonical isomorphism $\mathrm{S}_1' \times^{\Gamma'} \Gamma \cong \mathrm{S}_1$, showing that S_1 is induced along $\Gamma' \subset \Gamma$ which contradicts the surjectivity of $\Pi_1(\mathrm{S}, \bar{s}) \twoheadrightarrow \Gamma$.

By construction, the torus $\mathrm{T}_1 := \mathrm{T} \times_{\mathrm{S}} \mathrm{S}_1$ splits and there is a unique isomorphism:

$$\mathrm{Hom}(\mathrm{T}_1, \mathbb{G}_{m,\mathrm{S}_1}) \cong \Lambda, \tag{3.3}$$

extending the isomorphism over $\bar{s}_1$. Thus, the base change $\mathcal{V}_1$ of $\mathcal{V}$ along $\mathrm{S}_1 \to \mathrm{S}$ acquires a Λ-grading by T_1-weight submodules:

$$\mathcal{V}_1 \cong \bigoplus_{\lambda \in \Lambda} (\mathcal{V}_1)^{\lambda}. \tag{3.4}$$

Since $\mathcal{V}_1$ is finite locally free, $(\mathcal{V}_1)^{\lambda} = 0$ for all but finitely many λ and the rank of $(\mathcal{V}_1)^{\lambda}$ is constant along S_1 by connectedness.

The descent datum of T_1 gives rise to an isomorphism $\mathrm{T}_{1,\bar{s}_1} \cong \mathrm{T}_{1,\gamma(\bar{s}_1)}$ for all $\gamma \in \Gamma$. Under (3.3), this isomorphism passes to the action map $\gamma : \Lambda \to \Lambda$. The descent datum of $\mathcal{V}_1$ as a T_1-representation gives rise to an isomorphism $\mathcal{V}_{1,\bar{s}_1} \cong \mathcal{V}_{1,\gamma(\bar{s}_1)}$ under which the weight-λ submodule of $\mathcal{V}_{1,\bar{s}_1}$ corresponds to the weight-$\gamma(\lambda)$ submodule of $\mathcal{V}_{1,\gamma(\bar{s}_1)}$.

In summary, we find:

$$(\mathcal{V}_1)^{\lambda}_{\bar{s}_1} \neq 0 \Leftrightarrow (\mathcal{V}_1)^{\gamma(\lambda)}_{\gamma(\bar{s}_1)} \neq 0 \Leftrightarrow (\mathcal{V}_1)^{\gamma(\lambda)}_{\bar{s}_1} \neq 0.$$

Thus, if $(\mathcal{V}_1)^{\lambda} \neq 0$, the Γ-orbit of λ is necessarily finite, i.e. $\lambda \in \Lambda^f$.

The above argument shows that the T_1-action on $\mathcal{V}_1$ factors through T_1^f. This implies the same assertion about $\mathcal{V}$ since it is of étale local nature. $\qquad \square$

3.1.5. Let us illustrate this observation with Grothendieck's example of a non-isotrivial torus ([ABD+66, X, §1.6, Exemple 7.3]). We work over an algebraically closed field $k = \bar{k}$ and let $\mathrm{S} := \mathbb{A}^1 \sqcup_{\{0,1\}} \{0\}$ be the nodal cubic.

Since $\Pi_1(\mathrm{S}, \bar{s}) \cong \mathbb{Z}$, its action on $\mathbb{Z}^{\oplus 2}$ by $a \cdot (x, y) = (x + ay, y)$ defines a rank-2 torus T as a self-extension of $\mathbb{G}_m$:

$$1 \to \mathbb{G}_m \to \mathrm{T} \to \mathbb{G}_m \to 1. \tag{3.5}$$

The morphism (3.1) corresponds to the quotient morphism $\mathrm{T} \twoheadrightarrow \mathbb{G}_m$ in (3.5). Lemma 3.1.4 asserts that T-equivariant objects in $\mathrm{Vect}(\mathrm{S})$ are induced from $\mathbb{G}_m$-equivariant ones.

3.2. Identification of $\underline{\mathrm{Aut}}^{\otimes}(\omega)$

3.2.1. Let $\mathrm{G} \to \mathrm{S}$ be a reductive group scheme. Specializing (3.1) to $\mathrm{Rad}(\mathrm{G})$, we obtain a surjection of tori $\mathrm{Rad}(\mathrm{G}) \twoheadrightarrow \mathrm{Rad}(\mathrm{G})^f$.

Denote by G^f the push-out of G along this morphism:

$$G \twoheadrightarrow G^f. \tag{3.6}$$

In other words, G^f is the quotient of G by the kernel T_0 of the map $\mathrm{Rad}(G) \twoheadrightarrow \mathrm{Rad}(G)^f$. Since T_0 is of multiplicative type and contained in the center of G, the quotient G^f is representable by a reductive group scheme ([Con14, Corollary 3.3.5]) whose radical torus is identified with $\mathrm{Rad}(G)^f$.

Lemma 3.2.2. *Pulling back along* (3.6) *defines an equivalence of categories:*

$$\mathrm{Vect}(S)^{G^f} \cong \mathrm{Vect}(S)^G. \tag{3.7}$$

Proof. Since $G \twoheadrightarrow G^f$ is surjective, the functor $\mathrm{Vect}(S)^{G^f} \to \mathrm{Vect}(S)^G$ is fully faithful. It suffices to show essential surjectivity, i.e. the G-action on any $\mathcal{V} \in \mathrm{Vect}(S)^G$ factors through G^f. This statement follows from Lemma 3.1.4. $\square$

Theorem 3.2.3. *Let* S *be an affine connected Noetherian scheme and* $G \to S$ *be a reductive group scheme. There is an isomorphism of affine group schemes under* G:

$$G^f \cong \underline{\mathrm{Aut}}^{\otimes}(\omega). \tag{3.8}$$

Proof. Let ω^f denote the symmetric monoidal functor $\mathrm{Vect}(S)^{G^f} \to \mathrm{Vect}(S)$. The naturality of (2.3) yields a commutative diagram of affine group schemes:

$$
\begin{array}{ccc}
G & \longrightarrow & \underline{\mathrm{Aut}}^{\otimes}(\omega) \\
\downarrow & & \downarrow \\
G^f & \longrightarrow & \underline{\mathrm{Aut}}^{\otimes}(\omega^f)
\end{array}
$$

Lemma 3.2.2 shows that the right vertical arrow is an isomorphism. Theorem 2.0.1 shows that the bottom horizontal arrow is an isomorphism, since $\mathrm{Rad}(G)^f$ is isotrivial. The isomorphism (3.8) thus follows. $\square$

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