HALF-INTEGRAL LEVELS

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To James Tao

Abstract. We construct equivalences among four notions associated to a reductive group scheme G: factorization super central extensions of the loop group of G by $\mathbb{G}_m$ subject to a condition on the commutator, factorization super line bundles on the affine Grassmannian of G, rigidified sections of a quotient of 2-truncated K-theory over the Zariski classifying stack of G, and combinatorial data defined by Brylinski and Deligne in a conjectural extension of their classification theorem.

Contents

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"Is this what it feels like to end?"
"I do not know, for this is not our end."
—Kindred

Introduction

Let $k((t))$ denote the field of formal Laurent series with coefficients in a field k. Tate [Tat68] discovered a remarkable central extension of its group of units $k((t))^\times$ by $k^\times$:

$$1 \to k^\times \to \mathscr{G}_{\mathrm{Tate}} \to k((t))^\times \to 1. \quad (0.1)$$

The preimage of $a \in k((t))^\times$ in $\mathscr{G}_{\mathrm{Tate}}$ consists of nonzero elements of the relative determinant line $\det(ak[[t]] \mid k[[t]])$, where $k[[t]] \subset k((t))$ is the lattice of formal Taylor series.

If we think of $k((t))^\times$ as the $k((t))$-points of the algebraic group $\mathbb{G}_m$, then the following question arises: for a reductive group G, what are the "natural" central extensions of $\mathrm{G}(k((t)))$ by $k^\times$?

Brylinski and Deligne [BD01] parametrized a large class of central extensions of $\mathrm{G}(k((t)))$ by $k^\times$ using K-theory, as follows. Denote by $\underline{\mathrm{K}}_2$ the Zariski sheafification of the second algebraic K-group. Starting with a central extension on the big Zariski site of $\mathrm{Spec}(k)$:

$$1 \to \underline{\mathrm{K}}_2 \to \mathrm{E} \to \mathrm{G} \to 1, \quad (0.2)$$

evaluating at $\mathrm{Spec}(k((t)))$ and pushing out along the tame symbol $\underline{\mathrm{K}}_2(k((t))) \to k^\times$, we find a central extension of $\mathrm{G}(k((t)))$ by $k^\times$. They went on to give a complete classification of central extensions (0.2), valid over any regular base scheme of finite type over a field [BD01, Theorem 7.2]. However, no central extension of $\mathbb{G}_m$ by $\underline{\mathrm{K}}_2$ produces Tate's central extension (0.1). This led Brylinski and Deligne to pose [BD01, Questions 12.13(iii)]:

"For $\mathrm{V} = k[[t]]$*, not all natural central extensions by* $k^\times$ *are captured by 12.8.* [...] *We expect that 'natural' central extensions of* $\mathrm{G}(\mathrm{K})$ *by* $k^\times$ *are attached to data as follows: a Weyl group and Galois group invariant integer-valued symmetric bilinear form* [...]*"*

Our first goal is to find an enlargement of the groupoid of central extensions of G by $\underline{\mathrm{K}}_2$ and prove that it meets the expectation of Brylinski and Deligne. To this end, we introduce a Zariski sheaf of connective spectra $\underline{\mathrm{K}}^{\mathrm{super}}_{[1,2]}$ and establish the following result.

Theorem A (Theorem 2.2.3)**.** *Let* X *be a regular scheme of finite type over a field and* G *be a reductive group* X*-scheme equipped with a maximal torus* T*. The following Picard groupoids are canonically equivalent:*

(1) rigidified sections of $\underline{\mathrm{K}}^{\mathrm{super}}_{[1,2]}$ *over the Zariski classifying stack* BG*;*
(2) triples $(b, \widetilde{\Lambda}, \varphi)$ *defined in* [BD01, Questions 12.13(iii)]*.*

Given a $k((t))$-point of X, it is straightforward to produce from a rigidified section of $\underline{\mathrm{K}}^{\mathrm{super}}_{[1,2]}$ over BG a central extension of $\mathrm{G}(k((t)))$ by $k^\times$. In fact, the result will carry a canonical

$\mathbb{Z}/2$-grading, hence a "super central extension". This includes Tate's central extension in the special case $\mathrm{G} = \mathbb{G}_m$.

The second goal of this article is to prove that this passage from K-theory to super central extensions of the loop group is *reversible* if one remembers an additional piece of structure called "factorization".

As observed by Beilinson and Drinfeld [BD04, Dri06], the construction of $\mathscr{G}_{\mathrm{Tate}}$ globalizes, over any smooth curve X, to a *factorization super central extension* of the formal loop group $\mathscr{L}\mathbb{G}_m$. Intuitively speaking, this additional structure describes the behavior of $\mathscr{G}_{\mathrm{Tate}}$ as one formal loop on X "factorizes" into two. The following result shows that factorization super central extensions of the loop group $\mathscr{L}\mathrm{G}$, subject to a "tame commutator" condition which is automatic in characteristic zero, admit a parametrization parallel to Theorem A.

Theorem B (Theorem 3.4.5)**.** *Let* X *be a smooth curve over a field and* G *be a reductive group* X*-scheme. The following Picard groupoids are canonically equivalent:*

(1) factorization super central extensions of $\mathscr{L}\mathrm{G}$ *by* $\mathbb{G}_m$ *with tame commutator;*
(1') factorization super line bundles over the affine Grassmannian Gr_{G}*;*
(2) triples $(b, \widetilde{\Lambda}_+, \varphi)$ *defined in* [BD01, Questions 12.13(iii)] *up to a "twist"—if* G *is equipped with a maximal torus.*

Upon choosing a ϑ-characteristic, *i.e.* a square root $\omega^{1/2}$ of the canonical line bundle of the smooth curve X, the "twist" mentioned in (2) disappears. The Picard groupoids in Theorem A then become canonically equivalent to those in Theorem B, forming a commutative diagram:

$$\begin{array}{ccc} \mathrm{A}(1) & \xrightarrow{\cong} & \mathrm{B}(1) \\ & & \downarrow{\cong} \\ \downarrow{\cong} & & \mathrm{B}(1') \\ & & \downarrow{\cong} \\ \mathrm{A}(2) & \xrightarrow{\cong} & \mathrm{B}(2) \end{array} \tag{0.3}$$

In fact, the equivalences in (0.3) are the "half-integral" generalizations of a family of equivalences which are valid without the choice of a ϑ-characteristic.

Corollary C (Corollary 3.4.7)**.** *Let* X *be a smooth curve over a field and* G *be a reductive group* X*-scheme. The following Picard groupoids are canonically equivalent:*

(1) central extensions of G *by* $\underline{\mathrm{K}}_2$ *on the big Zariski site of* X*;*
(2) factorization central extensions of $\mathscr{L}\mathrm{G}$ *by* $\mathbb{G}_m$ *with tame commutator;*
(3) factorization line bundles over Gr_{G}*;*
(4) triples $(\mathrm{Q}, \widetilde{\Lambda}, \varphi)$ *in* [BD01, Theorem 7.2]*—if* G *is equipped with a maximal torus.*

This corollary already improves the current state of knowledge. Indeed, an equivalence between the Picard groupoids (1) and (3) was conjectured in Gaitsgory–Lysenko [GL18] and established in [Gai20, TZ21] under the additional assumptions that G is split and a certain integer N_{G} is invertible in the ground field. The equivalence supplied by Corollary C is valid for any reductive group X-scheme.

In the literature on covering groups in the equicharacteristic setting, the existence of factorization (super) line bundles over the affine Grassmannian Gr_{G} with favorable properties is sometimes stated as an assumption, see *e.g.* [Lys16, Lys17] and [Laf18, §14]. The combination of Theorems A and B produces them unconditionally.

It is worth mentioning that Theorem B is nontrivial already for $\mathrm{G} = \mathbb{G}_m$. Indeed, fibers of the affine Grassmannian $\mathrm{Gr}_{\mathbb{G}_m}$ over geometric points of X are highly nonreduced formal schemes. The groupoid of (super) line bundles over $\mathrm{Gr}_{\mathbb{G}_m}$ does not appear to have a clean description, but the equivalence (1') ≅ (2) of Theorem B shows that the factorization ones do. Moreover, the equivalence (1) ≅ (1') shows that factorization (super) line bundles over Gr_{G} have canonical multiplicative structures over $\mathscr{L}\mathrm{G}$. Unless G is simply connected, this assertion is not an obvious consequence of existing results.

From a differential geometric perspective, one could trace the conceptual origin of Corollary C to works on Chern–Simons theory. Indeed, Dijkgraaf and Witten [DW90] first recognized that the quantization parameter, or integral "level", of Chern–Simons theory for a compact Lie group G is best understood as an element of the reduced cohomology group $\mathrm{H}^4_e(\mathrm{BG}, \mathbb{Z})$. Suitably categorified, such an element transgresses to a central extension of the loop group of G by U(1). A recent theorem of Waldorf [Wal17] showed that this transgression procedure is reversible if one remembers the "fusion" structure of the target.

In the algebraic context, $\mathrm{H}^4_e(\mathrm{BG}, \mathbb{Z})$ should be replaced by the reduced weight-2 motivic cohomology group of BG, which classifies central extensions of G by $\underline{\mathrm{K}}_2$ via the isomorphism of [EKLV98] (see also [Gai20, Theorem 6.3.5]):

$$\mathrm{H}^4_e(\mathrm{BG}, \mathbb{Z}_{\mathrm{mot}}(2)) \xrightarrow{\simeq} \mathrm{H}^2_e(\mathrm{BG}, \underline{\mathrm{K}}_2). \tag{0.4}$$

Hence, a central extension of G by $\underline{\mathrm{K}}_2$ can be thought of as the algebraic notion of an *integral level* and Corollary C provides four equivalent descriptions of it.[1] The equivalence (1) ≅ (2) of Corollary C is a direct analogue of Waldorf's theorem.

With this understanding, we propose to encode the algebraic notion of a *half-integral level* by the Picard groupoids in (0.3). In fact, Dijkgraaf and Witten [DW90, §5] already observed that on *spin* manifolds, formally dividing a class in $\mathrm{H}^4_e(\mathrm{BG}, \mathbb{Z})$ by 2 sometimes leads to physically meaningful quantities. To interpret these "half-integral characteristic classes" as rigidified sections of $\underline{\mathrm{K}}^{\mathrm{super}}_{[1,2]}$ over BG, we note that the natural inclusion of abelian groups below has a 2-torsion cokernel:

$$\mathrm{H}^2_e(\mathrm{BG}, \underline{\mathrm{K}}_2) \subset \pi_0\Gamma_e(\mathrm{BG}, \underline{\mathrm{K}}^{\mathrm{super}}_{[1,2]}). \tag{0.5}$$

In the example of Tate's central extension, we have the equality $2 \cdot [\mathrm{Tate}] = [c_1]^2$, where $[c_1]$ denotes the first Chern class of the universal line bundle over $\mathrm{B}\mathbb{G}_m$, so $[c_1]^2$ generates the abelian group $\mathrm{H}^2_e(\mathrm{B}\mathbb{G}_m, \underline{\mathrm{K}}_2)$, while [Tate] is half-integral.

Another example is the "critical level", *i.e.* Beilinson and Drinfeld's Pfaffian [BD91, §4], representing half of $[c_2]$ of the adjoint bundle over BG. It is half-integral precisely when the half sum of positive roots $\check{\rho}$ is not an integral weight. Indeed, one of the applications of Theorem A is that it gives a new construction of the Pfaffian line bundle on the moduli stack of G-bundles on a spin curve (see §2.5).

Half-integral levels in our sense give rise to super conformal blocks on spin curves, as predicted by [DW90], although we do not attempt to fully develop this notion here.

Let us now explain the structure of this article and comment on the proofs.

Structure of the article. This article is divided into two parts which can be read independently. The first part proves Theorem A and the second part proves Theorem B.

[1] This algebraic notion is naturally associated to the *chiral* Wess–Zumino–Witten (WZW) model. We also mention that Henriques [Hen17] proposed a definition of integral levels for the chiral WZW model via vertex algebras, while our notion is more directly related to chiral algebras, see [BD04, Roz21].

In §1, we define $\underline{\mathrm{K}}^{\mathrm{super}}_{[1,2]}$ using a small but essential amount of homotopy theory. Namely, it is set to be the cofiber of a morphism of Zariski sheaves of connective spectra:

$$\mathrm{Sq} : \mathrm{B}\underline{\mathrm{K}}_1 \to \underline{\mathrm{K}}_{[1,2]}. \tag{0.6}$$

Here, $\underline{\mathrm{K}}_1$ and $\underline{\mathrm{K}}_{[1,2]}$ are the Zariski sheafified truncations of the K-theory spectrum. We also explain how to integrate sections of $\underline{\mathrm{K}}^{\mathrm{super}}_{[1,2]}$ over a global spin curve.

In §2, we prove Theorem A. The proof combines [BD01, Theorem 7.2] with our description of $\underline{\mathrm{K}}_{[1,2]}$ obtained in §1.

In §3, we formulate Theorem B. To define the notion of "tame commutator", we make essential use of the Contou-Carrère symbol over the Ran space, as constructed in Campbell–Hayash [CH21]. One of the phenomena we observe here is that the condition of having "tame commutator" is automatic in characteristic zero. This fact turns out to be equivalent to a new universality statement for the Contou-Carrère symbol.

Corollary D (Corollary 3.3.9)**.** *Let* X *be a smooth curve over a field k with* $\mathrm{char}(k) = 0$. *Then any pairing $\mathscr{L}\mathbb{G}_m \otimes \mathscr{L}\mathbb{G}_m \to \mathbb{G}_m$ compatible with factorization is an integral power of the Contou-Carrère symbol.*

We deduce this corollary from a surprising theorem of Tao [Tao21a], which asserts that the presheaf $\mathrm{Gr}_{\mathbb{G}_m}$ over the Ran space is *reduced* in a suitable sense, provided $\mathrm{char}(k) = 0$. The assertion of Corollary D is false if $\mathrm{char}(k) > 0$. We do not use it in the proof of Theorem B, which is valid in arbitrary characteristics.

In §4, we prove Theorem B. Our strategy is to first construct functors among the Picard groupoids in Theorem B:

$$(1) \to (1') \to (2). \tag{0.7}$$

Our previous work [TZ21] shows that the second functor is fully faithful. Here, we prove that the composition (0.7) is an equivalence by exploiting the group structure inherent in (1). In our approach, each of the equivalences (1) ≅ (2), (1') ≅ (2) is established using special cases of the other in iteration, so we do not obtain one without the other.

Finally, we mention a shortcoming of this article: the top horizontal functor appearing in (0.3) is defined *ad hoc* as the composition of the other functors. It should have a conceptually transparent description as a "transgression" along the space of formal loops:

$$\int_{(\mathring{\mathrm{D}},\omega^{1/2})} : \Gamma_e(\mathrm{BG}, \underline{\mathrm{K}}^{\mathrm{super}}_{[1,2]}) \to \mathrm{Hom}(\mathscr{L}\mathrm{G}, \mathsf{Pic}^{\mathrm{super}}), \tag{0.8}$$

as in the differential geometric context, but we are unable to find such a description. One difficulty seems to be that we do not understand the behavior of Zariski-sheafified K-groups over singular spaces such as $\mathscr{L}\mathrm{G}$. An attempt at defining (0.8) as a "transgression" in the integral case, *i.e.* for sections of $\underline{\mathrm{K}}_2[2]$ over BG, was made in Kapranov–Vasserot [KV07], but it relies on [KV07, Proposition 4.2.1] which is false as stated. A different strategy was carried out in Gaitsgory [Gai20], but it requires the hypothesis that N_{G} be invertible in the ground field, which we wish to avoid.

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Part 1. K-theory

1. $\underline{\mathrm{K}}^{\mathrm{super}}_{[1,2]}$

The main goal of this section is to introduce the Zariski sheaf of connective spectra $\underline{\mathrm{K}}^{\mathrm{super}}_{[1,2]}$. The first section §1.1 reviews necessary notions concerning algebraic K-theory. In §1.2, we give a "hands-on" description of the truncation $\underline{\mathrm{K}}_{[1,2]}$. Using this description, we are able to define $\underline{\mathrm{K}}^{\mathrm{super}}_{[1,2]}$ in §1.3. The material of §1.4 is not needed in the sequal: its goal is to show that sections of $\underline{\mathrm{K}}^{\mathrm{super}}_{[1,2]}$ can be integrated over a global spin curve relative to a regular base scheme S to yield a super line bundle over S.

1.1. Connective K-theory.

1.1.1. Let Spc denote the ∞-category of spaces. It is a symmetric monoidal ∞-category under the Cartesian product.

Write $\mathrm{Mon}_{\mathbb{E}_\infty}(\mathrm{Spc})$ for the ∞-category of $\mathbb{E}_\infty$-monoids in Spc. It contains a full subcategory $\mathrm{Grp}_{\mathbb{E}_\infty}(\mathrm{Spc})$ consisting of grouplike $\mathbb{E}_\infty$-monoids. The forgetful functor $\mathrm{Grp}_{\mathbb{E}_\infty}(\mathrm{Spc}) \to \mathrm{Mon}_{\mathbb{E}_\infty}(\mathrm{Spc})$ admits a left adjoint, called *group completion*:

$$\Omega\mathsf{B} : \mathrm{Mon}_{\mathbb{E}_\infty}(\mathrm{Spc}) \to \mathrm{Grp}_{\mathbb{E}_\infty}(\mathrm{Spc}). \tag{1.1}$$

Let Sptr denote the ∞-category of spectra. We use homotopical grading and denote by $\mathrm{Sptr}_{\geq 0}$ the full subcategory of connective spectra.

There is a canonical equivalence of ∞-categories [Lur17, Remark 5.2.6.26]:

$$\mathrm{Grp}_{\mathbb{E}_\infty}(\mathrm{Spc}) \cong \mathrm{Sptr}_{\geq 0}. \tag{1.2}$$

We shall also use without explicit mention the equivalence between Picard groupoids and 1-truncated connective spectra.

1.1.2. Let R be a commutative ring. Denote by Vect(R) the category of finitely generated projective R-modules and $\mathrm{Vect}(\mathrm{R})^{\cong}$ its maximal subgroupoid. The operation of direct sum equips $\mathrm{Vect}(\mathrm{R})^{\cong}$ with a symmetric monoidal structure. Its image under (1.1) is by definition the *connective* K-*theory* K(R) of R.

We shall view K(R) either as a grouplike $\mathbb{E}_\infty$-monoid or as a connective spectrum, using the canonical equivalence (1.2).

Note that the unit of the adjunction between (1.1) and the forgetful functor supplies a morphism of $\mathbb{E}_\infty$-monoids:

$$\mathrm{Vect}(\mathrm{R})^{\cong} \to \mathrm{K}(\mathrm{R}), \quad \mathscr{E} \mapsto [\mathscr{E}]. \tag{1.3}$$

1.1.3. We equip $\mathrm{Mon}_{\mathbb{E}_\infty}(\mathrm{Spc})$ and $\mathrm{Grp}_{\mathbb{E}_\infty}(\mathrm{Spc})$ with the canonical symmetric monoidal structure of [GGN15, Theorem 5.1]. With respect to these symmetric monoidal structures, (1.1) is symmetric monoidal. Hence it lifts to a functor of $\mathbb{E}_\infty$-monoids:

$$\Omega\mathsf{B} : \mathrm{Mon}_{\mathbb{E}_\infty}(\mathrm{Mon}_{\mathbb{E}_\infty}(\mathrm{Spc})) \to \mathrm{Mon}_{\mathbb{E}_\infty}(\mathrm{Grp}_{\mathbb{E}_\infty}(\mathrm{Spc})). \tag{1.4}$$

The right adjoint of (1.1), being lax symmetric monoidal, also lifts to a functor of $\mathbb{E}_\infty$-monoids and supplies the right adjoint of (1.4), see [GGN15, Lemma 3.6].

The operation of tensor product upgrades $\mathrm{Vect}(\mathrm{R})^{\cong}$ into an $\mathbb{E}_\infty$-monoid in $\mathrm{Mon}_{\mathbb{E}_\infty}(\mathrm{Spc})$. Thus K(R) acquires an $\mathbb{E}_\infty$-monoid structure in $\mathrm{Grp}_{\mathbb{E}_\infty}(\mathrm{Spc})$ (*i.e.* K(R) is a connective $\mathbb{E}_\infty$-spectrum) such that the unit (1.3) is symmetric monoidal.

Remark 1.1.4. Informally, the symmetric monoidal structure on (1.3) says that for each pair of objects $\mathscr{E}_1, \mathscr{E}_2 \in \mathrm{Vect}(\mathrm{R})^{\cong}$, $[\mathscr{E}_1 \otimes \mathscr{E}_2]$ is canonically equivalent to $[\mathscr{E}_1] \cdot [\mathscr{E}_2]$, together with the homotopy coherence data.

1.1.5. For any integer a, we write $\mathrm{K}_{\geq a}(\mathrm{R})$ (resp. $\mathrm{K}_{\leq a}(\mathrm{R})$) for the truncation $\tau_{\geq a}\mathrm{K}(\mathrm{R})$ (resp. $\tau_{\leq a}\mathrm{K}(\mathrm{R})$). For a pair of integers $a \leq b$, we write $\mathrm{K}_{[a,b]} := \tau_{\geq a}\tau_{\leq b}\mathrm{K}(\mathrm{R})$. We also use $\mathrm{K}_a(\mathrm{R})$ to denote $\Omega^a\mathrm{K}_{[a,a]}(\mathrm{R}) \cong \pi_a\mathrm{K}(\mathrm{R})$.

The association $\mathrm{S} = \mathrm{Spec}(\mathrm{R}) \mapsto \mathrm{K}(\mathrm{R})$ defines a presheaf K of connective $\mathbb{E}_\infty$-spectra on the category of affine schemes. Let $\underline{\mathrm{K}}$ denote its sheafification in the Zariski topology.

Zariski sheafification of the truncated presheaves above define $\underline{\mathrm{K}}_{\geq a}$, $\underline{\mathrm{K}}_{\leq a}$, $\underline{\mathrm{K}}_{[a,b]}$, and $\underline{\mathrm{K}}_a$. Since sheafification is t-exact, the forgetful functor from presheaves of spectra to sheaves of spectra is left t-exact. Hence $\underline{\mathrm{K}}_{\leq a}$ is a-truncated as a presheaf of spectra, *i.e.* its value at any R has vanishing homotopy groups above degree a.

Remark 1.1.6. For example, the map sending $\mathscr{E} \in \mathrm{Vect}(\mathrm{R})^{\cong}$ to its determinant line bundle $\det(\mathscr{E})$ induces an isomorphism of sheaves of Picard groupoids:

$$\underline{\mathrm{K}}_{[0,1]} \xrightarrow{\cong} \mathsf{Pic}^{\mathbb{Z}},$$

where $\mathsf{Pic}^{\mathbb{Z}}$ sends R to the Picard groupoid of $\mathbb{Z}$-graded line bundles on Spec(R), see [BS17, Proposition 12.18].

1.2. The sheaf $\underline{\mathrm{K}}_{[1,2]}$.

1.2.1. The goal of this subsection is to give an explicit description of $\underline{\mathrm{K}}_{[1,2]}$.

More precisely, we consider the fiber sequence defined by truncation:

$$\mathsf{B}^2\mathrm{K}_2(\mathrm{R}) \to \mathrm{K}_{[1,2]}(\mathrm{R}) \to \mathsf{B}\mathrm{K}_1(\mathrm{R})$$

for each ring R, which induces a fiber sequence of Zariski sheaves of connective spectra:

$$\mathsf{B}^2\underline{\mathrm{K}}_2 \to \underline{\mathrm{K}}_{[1,2]} \to \mathsf{B}\underline{\mathrm{K}}_1. \tag{1.5}$$

Our description will be that of the fiber sequence (1.5).

1.2.2. Denote by $\mathsf{Pic}(\mathrm{R}) \subset \mathrm{Vect}(\mathrm{R})^{\cong}$ the full subcategory of line bundles over Spec(R).

The map (1.3) induces a morphism of the underlying *pointed spaces* $\mathsf{Pic}(\mathrm{R}) \to \mathrm{K}(\mathrm{R})$ sending $\mathscr{L}$ to $[\mathscr{L}] - [\mathscr{O}]$. Thus, we obtain a morphism of Zariski sheaves of pointed spaces, without changing the same notation:

$$\mathsf{Pic} \to \underline{\mathrm{K}}, \quad \mathscr{L} \mapsto [\mathscr{L}] - [\mathscr{O}]. \tag{1.6}$$

Since the class of $[\mathscr{L}] - [\mathscr{O}]$ in $\mathrm{K}_0(\mathrm{R})$ vanishes Zariski locally on Spec(R), the morphism (1.6) factors through $\underline{\mathrm{K}}_{\geq 1}$ and we may compose it with the truncation map $\underline{\mathrm{K}}_{\geq 1} \to \underline{\mathrm{K}}_{[1,2]}$ to obtain a map of Zariski sheaves of pointed spaces:

$$s : \mathsf{Pic} \to \underline{\mathrm{K}}_{[1,2]}. \tag{1.7}$$

The description of $\underline{\mathrm{K}}_{[0,1]}$ via determinant (Remark 1.1.6) shows that (1.7) is a section of (1.5) on the underlying sheaves of pointed spaces, *i.e.* the composition of (1.7) with the truncation map $\underline{\mathrm{K}}_{[1,2]} \to \mathsf{B}\underline{\mathrm{K}}_1$ is the canonical isomorphism $\mathsf{Pic} \xrightarrow{\cong} \mathsf{B}\underline{\mathrm{K}}_1$.

1.2.3. Let $\mathscr{C}$ be a site and A_1, A_2 be sheaves of abelian groups over $\mathscr{C}$.

Consider the following two groupoids:

(1) the groupoid of extensions A of $\mathsf{B}(\mathrm{A}_1)$ by $\mathsf{B}^2(\mathrm{A}_2)$ as sheaves of connective spectra, equipped with a section $s : \mathsf{B}(\mathrm{A}_1) \to \mathrm{A}$ of the underlying sheaves of pointed spaces;
(2) the (discrete) groupoid of anti-symmetric pairings $\mathrm{A}_1 \otimes \mathrm{A}_1 \to \mathrm{A}_2$.

Let us construct a functor from (1) to (2):

$$\left\{\begin{array}{ccccc} & & & & \mathsf{B}(\mathrm{A}_1) \\ & & & \swarrow^{\text{pointed}} & \downarrow \text{id} \\ \mathsf{B}^2(\mathrm{A}_2) & \longrightarrow & \mathrm{A} & \longrightarrow & \mathsf{B}(\mathrm{A}_1)\end{array}\right\} \to \left\{\begin{array}{c}\text{anti-symmetric} \\ \mathrm{A}_1 \otimes \mathrm{A}_1 \to \mathrm{A}_2\end{array}\right\}. \tag{1.8}$$

Indeed, the section s defines a "cocycle" morphism of sheaves of spaces:

$$\mathsf{B}(\mathrm{A}_1) \times \mathsf{B}(\mathrm{A}_1) \to \mathsf{B}^2(\mathrm{A}_2), \quad (x, y) \mapsto s(x+y) - s(x) - s(y). \tag{1.9}$$

The morphism (1.9) is *bi-rigidified* in the following sense: it is equipped with trivializations along $\mathsf{B}(\mathrm{A}_1) \times e$ and $e \times \mathsf{B}(\mathrm{A}_1)$, which are isomorphic over $e \times e$.

To such a morphism, we may apply the loop space functor in the first, then the second factor, to obtain a map:

$$\mathrm{A}_1 \times \mathrm{A}_1 \to \mathrm{A}_2. \tag{1.10}$$

Lemma 1.2.4. *The map* (1.10) *is bilinear and anti-symmetric. The resulting functor* (1.8) *is an equivalence of groupoids.*

1.2.5. The proof of Lemma 1.2.4 proceeds by first giving an alternative definition of the functor (1.8) which is evidently an equivalence, and then showing that it coincides with the construction above.

First, we observe that groupoid (1) is equivalent to the groupoid (1') of extensions A' of A_1 by $\mathsf{B}(\mathrm{A}_2)$ as sheaves of connective spectra, equipped with an $\mathbb{E}_1$-monoidal section $\mathrm{A}_1 \to \mathrm{A}'$. The equivalence is given by the functors Ω and B.

Put differently, the groupoid (1') consists of symmetric monoidal structures on the sheaf of associative monoids $\mathsf{B}(\mathrm{A}_2) \times \mathrm{A}_1$ such that the inclusion and projection functors:

$$\mathsf{B}(\mathrm{A}_2) \subset \mathsf{B}(\mathrm{A}_2) \times \mathrm{A}_1 \to \mathrm{A}_1$$

are symmetric monoidal.

Such symmetric monoidal structures are in turn given by commutativity constraints on $\mathsf{B}(\mathrm{A}_2) \times \mathrm{A}_1$ vanishing on $\mathsf{B}(\mathrm{A}_2)$, which are anti-symmetric bilinear pairings:

$$\mathrm{A}_1 \otimes \mathrm{A}_1 \to \mathrm{A}_2. \tag{1.11}$$

Indeed, the commutativity constraint for two objects $x, y \in \mathsf{B}(\mathrm{A}_2) \times \mathrm{A}_1$ is an isomorphism $x \otimes y \xrightarrow{\simeq} y \otimes x$ which depends only on the classes of x, y in A_1. Such an isomorphism is the multiplication by a unique element of A_2. The hexagon and inverse axioms translate to the bilinearity and anti-symmetry of the resulting map $\mathrm{A}_1 \times \mathrm{A}_1 \to \mathrm{A}_2$.

The procedure above establishes an equivalence between the groupoid (1) and such pairings. Hence, it remains to prove that the pairing (1.11) extracted from any object of the groupoid (1) concides with (1.10). This follows from the observation in topology below.

1.2.6. Let M, N be $\mathbb{E}_1$-monoids in Spc and let $s : \mathrm{M} \to \mathrm{N}$ be a morphism of pointed spaces. Then we may construct two maps of spaces:

$$\Omega(\mathrm{M}) \times \Omega(\mathrm{M}) \underset{s_2}{\overset{s_1}{\rightrightarrows}} \Omega^2(\mathrm{N}). \tag{1.12}$$

The map s_1 is given by applying Ω to the first, then the second factor, to the bi-rigidifed map of spaces:

$$\mathrm{M} \times \mathrm{M} \to \mathrm{N}, \quad (x, y) \mapsto s(x+y) - s(x) - s(y). \tag{1.13}$$

The map s_2 uses the $\mathbb{E}_1$-monoidal morphism $\Omega_s : \Omega(\mathrm{M}) \to \Omega(\mathrm{N})$ induced from s and sends two loops $a, b \in \Omega(\mathrm{M})$ to the following loop in $\Omega(\mathrm{N})$:

$$\begin{aligned} \mathbf{1} &\xrightarrow{\simeq} \Omega_s(a+b) - (\Omega_s(a) + \Omega_s(b)) \\ &\xrightarrow{\simeq} \Omega_s(b+a) - (\Omega_s(b) + \Omega_s(a)) \xrightarrow{\simeq} \mathbf{1}. \end{aligned} \tag{1.14}$$

where the first and last isomorphisms are defined by the $\mathbb{E}_1$-monoid structure on Ω_s and the middle one is defined by the braidings[2] on $\Omega(\mathrm{M})$ and $\Omega(\mathrm{N})$.

Claim: there is a homotopy equivalence $s_1 \xrightarrow{\simeq} s_2$.

To prove the claim, we choose as our model for "spaces" topological spaces having the homotopy type of a CW complex, see [Lur18, 012Z] for its equivalence with the standard model using Kan complexes.

The monoidal operation on $\Omega(\mathrm{M})$ can be viewed as concatenation of loops and the braiding is given as follows. For two loops $a, b \in \Omega(\mathrm{M})$ we find a morphism:

$$[0,1] \times [0,1] \to \mathrm{M}, \quad (t_1, t_2) \mapsto a(t_1) + b(t_2), \tag{1.15}$$

where the sum is the $\mathbb{E}_1$-monoid product on M. Then (1.15) can be viewed as a homotopy from its restriction to $([0,1] \times \{0\}) \cup (\{1\} \times [0,1])$ to its restriction to $(\{0\} \times [0,1]) \cup ([0,1] \times \{1\})$, exhibiting the braiding $a + b \xrightarrow{\simeq} b + a$ in $\Omega(\mathrm{M})$.

The same description holds for the braiding in $\Omega(\mathrm{N})$.

Now we come to the morphism s_1. It carries a, b to the element of $\Omega^2(\mathrm{N})$ which is represented by a map $\mathrm{S}^2 \to \mathrm{N}$ fitting into the following diagram:

$$\begin{array}{ccc} [0,1] \times [0,1] & \xrightarrow{(a,b)} & \mathrm{M} \times \mathrm{M} \\ \downarrow & & \downarrow{\scriptstyle (1.13)} \\ \mathrm{S}^2 & \longrightarrow & \mathrm{N} \end{array} \tag{1.16}$$

where the left vertical map collapses the outer edges of the square.

Reading (1.16) as a homotopy from its restriction to $([0,1] \times \{0\}) \cup (\{1\} \times [0,1])$ to its restriction to $(\{0\} \times [0,1]) \cup ([0,1] \times \{1\})$ (both equivalent to the trivial loop in N), we see that it is precisely the loop (1.14) defined by the braidings on $\Omega(\mathrm{M})$ and $\Omega(\mathrm{N})$.

The proof of the claim, and thus Lemma 1.2.4, is concluded.

1.2.7. We shall now use the equivalence of Lemma 1.2.4 to classify the fiber sequence (1.5) together with the distinguished section s (1.7).

Namely, its image under the functor (1.8) is an anti-symmetric form:

$$\underline{\mathrm{K}}_1 \otimes \underline{\mathrm{K}}_1 \to \underline{\mathrm{K}}_2. \tag{1.17}$$

Proposition 1.2.8. *The map* (1.17) *equals the product pairing* $x, y \mapsto \{x, y\}$.

Proof. Let $\mathscr{L}_1$, $\mathscr{L}_2$ be sections of $\mathsf{B}(\underline{\mathrm{K}}_1) \cong \mathsf{B}\mathbb{G}_m$. The image of $(\mathscr{L}_1, \mathscr{L}_2)$ under the "cocycle" morphism, *i.e.* the special case of (1.9) for $\mathrm{A}_1 = \underline{\mathrm{K}}_1$, $\mathrm{A}_2 = \underline{\mathrm{K}}_2$:

$$\mathsf{B}(\underline{\mathrm{K}}_1) \times \mathsf{B}(\underline{\mathrm{K}}_1) \to \mathsf{B}^2(\underline{\mathrm{K}}_2) \tag{1.18}$$

is the section in $\mathsf{B}^2(\underline{\mathrm{K}}_2)$ obtained from:

$$([\mathscr{L}_1 \otimes \mathscr{L}_2] - [\mathscr{O}]) - ([\mathscr{L}_1] - [\mathscr{O}]) - ([\mathscr{L}_2] - [\mathscr{O}]) \tag{1.19}$$

under the truncation map $\underline{\mathrm{K}}_{\geq 2} \to \mathsf{B}^2(\underline{\mathrm{K}}_2)$.

[2]We use the term "braiding" to refer to the isomorphism $a \otimes b \xrightarrow{\simeq} b \otimes a$ in an $\mathbb{E}_2$-monoid and the term "commutativity constraint" to refer to the same isomorphism in an $\mathbb{E}_\infty$-monoid.

Using the fact that (1.3) is symmetric monoidal, the section (1.19) is equivalent to $([\mathscr{L}_1]-[\mathscr{O}])\cdot([\mathscr{L}_2]-[\mathscr{O}])$, where $\cdot$ denotes multiplication on $\underline{\mathrm{K}}$.

The map $\mathsf{Pic}\to\underline{\mathrm{K}}_{\geq 1}$ induced from $\mathscr{L}\mapsto[\mathscr{L}]-[\mathscr{O}]$ coincides with the map $\mathsf{B}(\underline{\mathrm{K}}_1)\to\underline{\mathrm{K}}_{\geq 1}$ defined by truncation. Thus, (1.18) renders the following diagram commutative:

$$\begin{array}{ccc} \mathsf{B}(\underline{\mathrm{K}}_1)\times\mathsf{B}(\underline{\mathrm{K}}_1) & \xrightarrow{(1.18)} & \mathsf{B}^2(\underline{\mathrm{K}}_2) \\ \downarrow & & \downarrow \\ \underline{\mathrm{K}}_{\geq 1}\otimes\underline{\mathrm{K}}_{\geq 1} & \xrightarrow{\cdot} & \underline{\mathrm{K}}_{\geq 2} \end{array}$$

Here, the bottom horizontal arrow is the multiplicative structure on $\underline{\mathrm{K}}$ and the vertical maps are defined by truncations. However, the only bi-rigidified morphism $\mathsf{B}(\underline{\mathrm{K}}_1)\times\mathsf{B}(\underline{\mathrm{K}}_1)\to\mathsf{B}^2(\underline{\mathrm{K}}_2)$ rendering this diagram commutative is the product pairing. □

Remark 1.2.9. Let us view the map defined by the product pairing $x,y\mapsto\{x,y\}$ as a morphism of Zariski sheaves of connective spectra:

$$\mathsf{Pic}\otimes\mathsf{Pic}\to\mathsf{B}^2(\underline{\mathrm{K}}_2),\quad(\mathscr{L}_1,\mathscr{L}_2)\mapsto\{\mathscr{L}_1,\mathscr{L}_2\}. \tag{1.20}$$

Then Proposition 1.2.8 exihibits a canonical isomorphism of sections of $\underline{\mathrm{K}}_{[1,2]}$:

$$s(\mathscr{L}_1\otimes\mathscr{L}_2)-s(\mathscr{L}_1)-s(\mathscr{L}_2)\cong\{\mathscr{L}_1,\mathscr{L}_2\}, \tag{1.21}$$

where $s:\mathsf{Pic}\to\underline{\mathrm{K}}_{[1,2]}$ is the morphism (1.7) of sheaves of pointed spaces. One may therefore view s as a "quadratic refinement" of the product pairing (1.20).

1.2.10. Combining Proposition 1.2.8 with the alternative construction of the pairing given in §1.2.5, we obtain an explicit description of the loop space of (1.5):

$$\mathsf{B}\underline{\mathrm{K}}_2\to\Omega(\underline{\mathrm{K}}_{[1,2]})\to\underline{\mathrm{K}}_1. \tag{1.22}$$

Namely, it splits as sheaves of $\mathbb{E}_1$-monoids, given by (1.7). The commutativity constraint on $\Omega(\underline{\mathrm{K}}_{[1,2]})$ is described by the anti-symmetric pairing:

$$\underline{\mathrm{K}}_1\otimes\underline{\mathrm{K}}_1\to\underline{\mathrm{K}}_2,\quad x,y\mapsto\{x,y\}.$$

1.2.11. Let R be a ring and $\mathscr{M}$ be a line bundle over $\mathrm{S}:=\mathrm{Spec}(\mathrm{R})$. We shall temporarily work over the big Zariski site of Spec(R).

Multiplication by the object $[\mathscr{M}]\in\mathrm{K}(\mathrm{R})$ induces a morphism of sheaves of connective spectra $\cdot[\mathscr{M}]:\underline{\mathrm{K}}\to\underline{\mathrm{K}}$, hence a morphism:

$$\cdot[\mathscr{M}]:\underline{\mathrm{K}}_{[1,2]}\to\underline{\mathrm{K}}_{[1,2]}. \tag{1.23}$$

Since the image of $[\mathscr{M}]$ in $\Gamma(\mathrm{Spec}(\mathrm{R}),\underline{\mathrm{K}}_0)$ is the multiplicative unit, multiplication by $[\mathscr{M}]$ induces the identity map on the homotopy sheaves $\underline{\mathrm{K}}_n$ for each $n\geq 0$. In particular, we obtain an automorphism of the triangles (1.5):

$$\begin{array}{ccccc} \mathsf{B}^2\underline{\mathrm{K}}_2 & \longrightarrow & \underline{\mathrm{K}}_{[1,2]} & \longrightarrow & \mathsf{B}\underline{\mathrm{K}}_1 \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \cdot[\mathscr{M}]} & & \downarrow{\scriptstyle \mathrm{id}} \\ \mathsf{B}^2\underline{\mathrm{K}}_2 & \longrightarrow & \underline{\mathrm{K}}_{[1,2]} & \longrightarrow & \mathsf{B}\underline{\mathrm{K}}_1 \end{array} \tag{1.24}$$

Claim: (1.23) is the sum of the identity on $\underline{\mathrm{K}}_{[1,2]}$ with the shearing map $\mathsf{B}\underline{\mathrm{K}}_1\cong\mathsf{Pic}\to\mathsf{B}^2\underline{\mathrm{K}}_2$ defined by $\mathscr{L}\mapsto\{\mathscr{L},\mathscr{M}\}$ (in the notation (1.20)).

To see this, we may lift a section $\mathscr{L}$ of $\mathsf{B}\underline{\mathrm{K}}_1$ to $s(\mathscr{L})=[\mathscr{L}]-[\mathscr{O}]$ of $\underline{\mathrm{K}}_{[1,2]}$. The section $s(\mathscr{L})\cdot[\mathscr{M}]\cong[\mathscr{L}\otimes\mathscr{M}]-[\mathscr{M}]$ is then the sum of $s(\mathscr{L})$ with $\{\mathscr{L},\mathscr{M}\}$ by (1.21).

1.3. The sheaf $\underline{\mathrm{K}}^{\mathrm{super}}_{[1,2]}$.

1.3.1. Let $\mathsf{Pic}^{\mathrm{super}}$ denote the Zariski sheaf of super (*i.e.* $\mathbb{Z}/2$-graded) line bundles. As a sheaf of connective spectra, it coincides with the cofiber of the map:

$$\underline{\mathbb{Z}} \to \mathsf{Pic}^{\mathbb{Z}}, \quad n \mapsto (\mathscr{O}, 2n). \tag{1.25}$$

We could suggestively denote $\mathsf{Pic}^{\mathrm{super}}$ by $\underline{\mathrm{K}}^{\mathrm{super}}_{[0,1]}$, viewing (1.25) as a morphism $\underline{\mathrm{K}}_0 \to \underline{\mathrm{K}}_{[0,1]}$ lifting the squaring map on $\underline{\mathrm{K}}_0$.

The goal of this subsection is to introduce the Zariski sheaf $\underline{\mathrm{K}}^{\mathrm{super}}_{[1,2]}$ of connective spectra, defined as the cofiber of a map lifting the squaring map on $\mathsf{B}\underline{\mathrm{K}}_1$:

$$\mathrm{Sq} : \mathsf{B}\underline{\mathrm{K}}_1 \to \underline{\mathrm{K}}_{[1,2]}. \tag{1.26}$$

1.3.2. Let us first define (1.26) as a morphism of sheaves of *pointed spaces*. To do so, we interpret $\mathsf{B}\underline{\mathrm{K}}_1$ as Pic and define (1.26) by the formula:

$$\mathsf{Pic} \to \underline{\mathrm{K}}_{[1,2]}, \quad \mathscr{L} \mapsto [\mathscr{L}] - [\mathscr{L}^{-1}]. \tag{1.27}$$

More precisely, the formula $\mathscr{L} \mapsto [\mathscr{L}] - [\mathscr{L}^{-1}]$ defines a map $\mathsf{Pic}(\mathrm{R}) \to \mathrm{K}(\mathrm{R})$, which induces (1.27) upon Zariski sheafification and truncation as in §1.2.2.

We argue that the structure of a morphism of connective spectra on (1.27) is unique, if it exists. Indeed, since (1.27) has 1-connective source and target, it is equivalent to a morphism of sheaves of $\mathbb{E}_1$-monoids:

$$f : \mathbb{G}_m \to \Omega(\underline{\mathrm{K}}_{[1,2]}). \tag{1.28}$$

Since $\Omega(\underline{\mathrm{K}}_{[1,2]})$ is 1-truncated, an $\mathbb{E}_\infty$-monoid structure on (1.28) is equivalent to the *condition* that it preserves the commutativity constraint.

In particular, the following assertion involves no additional structure.

Proposition 1.3.3. *The morphism of sheaves of pointed spaces* (1.27) *lifts to a morphism of sheaves of connective spectra.*

1.3.4. The proof of Proposition 1.3.3 proceeds by explicitly identifying the morphism (1.28) using the description of $\Omega(\underline{\mathrm{K}}_{[1,2]})$ in §1.2.10.

Namely, under the $\mathbb{E}_1$-monoidal splitting $\Omega(\underline{\mathrm{K}}_{[1,2]}) \cong \mathsf{B}(\underline{\mathrm{K}}_2) \times \underline{\mathrm{K}}_1$, (1.28) corresponds to two $\mathbb{E}_1$-monoidal morphisms:

$$f_1 : \mathbb{G}_m \to \underline{\mathrm{K}}_1,$$
$$f_2 : \mathbb{G}_m \to \mathsf{B}(\underline{\mathrm{K}}_2).$$

Lemma 1.3.5. *The following statements hold:*

(1) f_1 is the squaring map $x \mapsto x^2$;
(2) f_2 is trivial as a morphism of sheaves of pointed spaces, *and its $\mathbb{E}_1$-monoid structure is defined by the automorphism of the trivial $\underline{\mathrm{K}}_2$-torsor:*

$$f_2(x) \otimes f_2(y) \xrightarrow{\simeq} f_2(xy), \quad 1 \mapsto 2 \cdot \{x, y\},$$

for each $x, y \in \mathbb{G}_m$.

Proof. As the section (1.7) lifts the identity map on $\mathsf{Pic} \cong \mathsf{B}(\underline{\mathrm{K}}_1)$, statement (1) follows from the isomorphism $\mathscr{L} \otimes (\mathscr{L}^{-1})^{-1} \cong \mathscr{L}^{\otimes 2}$.

For statement (2), we first apply the isomorphism (1.21) to the pairs of sections $\mathscr{L}, \mathscr{L}^{-1} \in \mathsf{Pic}$ and $\mathscr{L}, \mathscr{L} \in \mathsf{Pic}$ to obtain isomorphisms in $\underline{\mathrm{K}}_{[1,2]}$:

$$\begin{aligned} s(\mathscr{L}) + s(\mathscr{L}^{-1}) &\cong -\{\mathscr{L}, \mathscr{L}^{-1}\}, \\ 2 \cdot s(\mathscr{L}) &\cong s(\mathscr{L}^2) - \{\mathscr{L}, \mathscr{L}\}, \end{aligned}$$

where s denotes the section (1.7).

Their difference yields an isomorphism in $\underline{\mathrm{K}}_{[1,2]}$:

$$[\mathscr{L}] - [\mathscr{L}^{-1}] \cong s(\mathscr{L}^2) - 2 \cdot \{\mathscr{L}, \mathscr{L}\}. \tag{1.29}$$

In particular, this shows that f_2 is given by (-2) times the loop space of the self-pairing:

$$\mathsf{Pic} \to \mathsf{B}^2(\underline{\mathrm{K}}_2), \quad \mathscr{L} \mapsto \{\mathscr{L}, \mathscr{L}\}. \tag{1.30}$$

According to [PR11, Theorem 2.5], the loop space of (1.30) is the map $\mathbb{G}_m \to \mathsf{B}(\underline{\mathrm{K}}_2)$ which is trivial as a morphism of sheaves of pointed spaces, with $\mathbb{E}_1$-monoid structure given, for any $x, y \in \mathbb{G}_m$, by the automorphism $1 \mapsto -\{x, y\}$ of the trivial $\underline{\mathrm{K}}_2$-torsor. The desired conclusion follows. □

Proof of Proposition 1.3.3. It suffices to prove that (1.28) preserves the commutativity constraint. In other words, given $x, y \in \mathbb{G}_m$, we must show that the following diagram of sections of $\Omega(\underline{\mathrm{K}}_{[1,2]})$ commutes:

$$\begin{array}{ccc} f(x) \otimes f(y) & \xrightarrow{\simeq} & f(xy) \\ \downarrow{\scriptstyle c_{f(x),f(y)}} & & \downarrow{\scriptstyle f(c_{x,y})} \\ f(y) \otimes f(x) & \xrightarrow{\simeq} & f(yx) \end{array} \tag{1.31}$$

Here, the horizontal morphisms are given by the $\mathbb{E}_1$-monoid structure of f and the vertical morphisms are the commutativity constraints of $\Omega(\underline{\mathrm{K}}_{[1,2]})$, respectively $\mathbb{G}_m$ (identity).

By Lemma 1.3.5, the commutativity of (1.31) is equivalent to the following equality of sections of $\underline{\mathrm{K}}_2$:

$$2 \cdot \{x, y\} = \{x^2, y^2\} + 2 \cdot \{y, x\}.$$

This follows at once from the bilinearity and anti-symmetry of the pairing. □

Remark 1.3.6. From the proof of Proposition 1.3.3, we see that if we replace (1.27) by the "obvious" lift of the squaring map $\mathscr{L} \mapsto 2 \cdot s(\mathscr{L})$, it would *not* define a morphism of sheaves of connective spectra.

1.3.7. Having constructed (1.27), thus Sq (1.26), as a morphism of sheaves of connective spectra, we define $\underline{\mathrm{K}}^{\mathrm{super}}_{[1,2]}$ to be the cofiber of Sq.

The following diagram summarizes four cofiber sequences of Zariski sheaves of connective spectra relevant for us:

$$\begin{array}{ccccc} & & \mathsf{B}(\underline{\mathrm{K}}_1) & \xrightarrow{\mathrm{id}} & \mathsf{B}(\underline{\mathrm{K}}_1) \\ & & \downarrow{\scriptstyle \mathrm{Sq}} & & \downarrow{\scriptstyle \cdot 2} \\ \mathsf{B}^2(\underline{\mathrm{K}}_2) & \longrightarrow & \underline{\mathrm{K}}_{[1,2]} & \longrightarrow & \mathsf{B}(\underline{\mathrm{K}}_1) \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow \\ \mathsf{B}^2(\underline{\mathrm{K}}_2) & \longrightarrow & \underline{\mathrm{K}}^{\mathrm{super}}_{[1,2]} & \longrightarrow & \mathsf{B}(\underline{\mathrm{K}}_1)/2 \end{array} \tag{1.32}$$

1.4. Integration on curves.

1.4.1. Given a quasi-compact and quasi-separated scheme S, we write $\mathbf{K}(\mathrm{S})$ for the *non-connective* K-theory spectrum of the stable ∞-category Perf(S) [BGT13, §9]. The association $\mathrm{S} \mapsto \mathbf{K}(\mathrm{S})$ is a Zariski sheaf of spectra [TT90, Theorem 8.1].

If S is regular, then the restriction of $\mathbf{K}$ to the small Zariski site of S takes values in connective spectra, so $\mathbf{K}(\mathrm{S})$ coincides with $\Gamma(\mathrm{S}, \underline{\mathrm{K}})$ [TT90, Proposition 6.8].

1.4.2. Let S be a regular affine scheme of finite type over a field. This assumption guarantees that each Zariski sheaf $\underline{\mathrm{K}}_n$ over S has cohomological amplitude $\leq n$ by the Gersten resolution.

Let $p : \mathrm{X}_\mathrm{S} \to \mathrm{S}$ be a smooth, proper morphism of relative dimension 1 with connected geometric fibers.

The functor $\mathrm{Perf}(\mathrm{X}_\mathrm{S}) \to \mathrm{Perf}(\mathrm{S})$, $\mathscr{E} \mapsto \mathrm{R}p_*\mathscr{E}$ induces a morphism of spectra $\mathbf{K}(\mathrm{X}_\mathrm{S}) \to \mathbf{K}(\mathrm{S})$. By regularity, this amounts to a morphism of connective spectra:

$$\Gamma(\mathrm{X}_\mathrm{S}, \underline{\mathrm{K}}) \to \Gamma(\mathrm{S}, \underline{\mathrm{K}}). \tag{1.33}$$

Lemma 1.4.3. *For $n = 0, 1$, the morphism* (1.33) *fits into a commutative diagram:*

$$\begin{array}{ccc} \Gamma(\mathrm{X}_\mathrm{S}, \underline{\mathrm{K}}) & \xrightarrow{(1.33)} & \Gamma(\mathrm{S}, \underline{\mathrm{K}}) \\ \downarrow & & \downarrow \\ \Gamma(\mathrm{X}_\mathrm{S}, \underline{\mathrm{K}}_{\leq n+1}) & \to & \Gamma(\mathrm{S}, \underline{\mathrm{K}}_{\leq n}) \end{array} \tag{1.34}$$

where the vertical arrows are defined by truncation on $\underline{\mathrm{K}}$.

Proof. We treat the case $n = 1$, as the case $n = 0$ is similar but simpler.

For $n = 1$, it suffices to trivialize the composition of maps of connective spectra:

$$\Gamma(\mathrm{X}_\mathrm{S}, \underline{\mathrm{K}}_{\geq 3}) \longrightarrow \Gamma(\mathrm{X}_\mathrm{S}, \underline{\mathrm{K}}) \xrightarrow{(1.33)} \Gamma(\mathrm{S}, \underline{\mathrm{K}}) \longrightarrow \Gamma(\mathrm{S}, \underline{\mathrm{K}}_{\leq 1}), \tag{1.35}$$

where the last arrow is defined by truncation on $\underline{\mathrm{K}}$.

Since $\underline{\mathrm{K}}_{\leq 1} \cong \mathsf{Pic}^{\mathbb{Z}}$ (Remark 1.1.6) and S is regular, a section of $\Gamma(\mathrm{S}, \underline{\mathrm{K}}_{\leq 1})$ is trivialized once it is trivialized away from codimension ≥ 2. Hence we may replace S by the spectrum of a discrete valuation ring R with field of fractions F.

In this case, X_S is Noetherian of Krull dimension 2, so for each $i \geq 0$, the complex $\Gamma(\mathrm{X}_\mathrm{S}, \underline{\mathrm{K}}_i[i])$ is concentrated in cohomological degrees $\leq -i + 2$. Triviality of (1.35) thus amounts to the *condition* that its induced map on H^{-1} below vanishes:

$$\mathrm{H}^2(\mathrm{X}_\mathrm{S}, \underline{\mathrm{K}}_3) \to \mathrm{H}^0(\mathrm{S}, \underline{\mathrm{K}}_1) \cong \mathrm{R}^\times. \tag{1.36}$$

However, the formation of (1.35) is of Zariski local nature on S, so (1.36) fits into the commutative diagram:

$$\begin{array}{ccc} \mathrm{H}^2(\mathrm{X}_\mathrm{S}, \underline{\mathrm{K}}_3) & \to & \mathrm{R}^\times \\ \downarrow & & \downarrow \\ \mathrm{H}^2(\mathrm{X}_\mathrm{F}, \underline{\mathrm{K}}_3) & \to & \mathrm{F}^\times \end{array}$$

Here, $\mathrm{X}_\mathrm{F} := \mathrm{X} \times_\mathrm{S} \mathrm{Spec}(\mathrm{F})$ has Krull dimension 1, so $\mathrm{H}^2(\mathrm{X}_\mathrm{F}, \underline{\mathrm{K}}_3) = 0$ and (1.36) vanishes. □

1.4.4. We may now define a morphism:

$$\int_{\mathrm{X}_\mathrm{S}} : \Gamma(\mathrm{X}_\mathrm{S}, \underline{\mathrm{K}}_{[1,2]}) \to \Gamma(\mathrm{S}, \mathsf{Pic}^{\mathbb{Z}}), \tag{1.37}$$

to be the composition:

$$\Gamma(\mathrm{X_S}, \underline{\mathrm{K}}_{[1,2]}) \to \Gamma(\mathrm{X_S}, \underline{\mathrm{K}}_{\le 2}) \to \Gamma(\mathrm{S}, \underline{\mathrm{K}}_{\le 1}) \cong \Gamma(\mathrm{S}, \mathsf{Pic}^{\mathbb{Z}}).$$

where the first morphism comes from the inclusion $\underline{\mathrm{K}}_{[1,2]} \to \underline{\mathrm{K}}_{\le 2}$, and the second morphism is the bottom arrow of (1.34) for $n = 1$.

Comparing the cases $n = 1$ and $n = 0$ in (1.34) shows that (1.37) induces a morphism of fiber sequences:

$$\begin{array}{ccccc} \Gamma(\mathrm{X_S}, \mathsf{B}^2\underline{\mathrm{K}}_2) & \to & \Gamma(\mathrm{X_S}, \underline{\mathrm{K}}_{[1,2]}) & \to & \Gamma(\mathrm{X_S}, \mathsf{B}\underline{\mathrm{K}}_1) \\ \downarrow{\scriptstyle \int_{\mathrm{X_S}}} & & \downarrow{\scriptstyle \int_{\mathrm{X_S}}} & & \downarrow{\scriptstyle \int_{\mathrm{X_S}}} \\ \Gamma(\mathrm{S}, \mathsf{Pic}) & \longrightarrow & \Gamma(\mathrm{S}, \mathsf{Pic}^{\mathbb{Z}}) & \longrightarrow & \Gamma(\mathrm{S}, \underline{\mathbb{Z}}) \end{array} \tag{1.38}$$

Here, the rightmost vertical arrow has the following explict description: it associates to a line bundle over $\mathrm{X_S}$ its degree, viewed as a locally constant function over S.

Remark 1.4.5. The first vertical functor in (1.38) is constructed by Gaitsgory in [Gai20, §2.4] using the Gersten resolution of $\underline{\mathrm{K}}_2$.

1.4.6. Let us now define the "super" variant of (1.37), which requires a *spin structure* over $\mathrm{X_S}$. From now on, we fix a square root $\omega^{1/2}$ of the relative canonical bundle $\omega_{\mathrm{X_S/S}}$.

Define the morphism:

$$\int_{(\mathrm{X_S}, \omega^{1/2})} : \Gamma(\mathrm{X_S}, \underline{\mathrm{K}}_{[1,2]}) \to \Gamma(\mathrm{S}, \mathsf{Pic}^{\mathbb{Z}}) \tag{1.39}$$

to be the composition of (1.37) with the multiplication $\cdot[\omega^{1/2}] : \underline{\mathrm{K}}_{[1,2]} \to \underline{\mathrm{K}}_{[1,2]}$ (*i.e.* the morphism (1.23) for $\mathscr{M} := \omega^{1/2}$).

The following observation shows that the $\omega^{1/2}$-twisted integration morphism (1.39) intertwines the squaring maps (1.25) and (1.26).

Lemma 1.4.7. *The following diagram is canonically commutative:*

$$\begin{array}{ccc} \Gamma(\mathrm{X_S}, \mathsf{B}\underline{\mathrm{K}}_1) & \xrightarrow{\int_{\mathrm{X_S}}} & \Gamma(\mathrm{S}, \underline{\mathbb{Z}}) \\ \downarrow{\scriptstyle \mathrm{Sq}} & & \downarrow{\scriptstyle n \mapsto (\mathscr{O}, 2n)} \\ \Gamma(\mathrm{X_S}, \underline{\mathrm{K}}_{[1,2]}) & \xrightarrow{\int_{(\mathrm{X_S}, \omega^{1/2})}} & \Gamma(\mathrm{S}, \mathsf{Pic}^{\mathbb{Z}}) \end{array} \tag{1.40}$$

Proof. By construction, the lower circuit of (1.40) sends a line bundle $\mathscr{L}$ over $\mathrm{X_S}$ to the $\mathbb{Z}$-graded line bundle:

$$\det(\mathrm{R}p_*(\mathscr{L} \otimes \omega^{1/2})) \otimes \det(\mathrm{R}p_*(\mathscr{L}^{-1} \otimes \omega^{1/2}))^{-1},$$

This is the trivial line bundle by Grothendieck–Serre duality. It is placed in degree $2\deg(\mathscr{L})$ by the Riemann–Roch formula. □

1.4.8. Since $\mathrm{X_S}$ is regular, the Zariski cohomology group $\mathrm{H}^2(\mathrm{X_S}, \mathbb{G}_m)$ vanishes. Thus, taking cofibers of the vertical arrows in (1.40) yields a morphism:

$$\int_{(\mathrm{X_S}, \omega^{1/2})} : \Gamma(\mathrm{X_S}, \underline{\mathrm{K}}^{\mathrm{super}}_{[1,2]}) \to \Gamma(\mathrm{S}, \mathsf{Pic}^{\mathrm{super}}). \tag{1.41}$$

The morphisms of fiber sequences (1.24) and (1.38) induce a morphism of fiber sequences:

$$\begin{array}{ccccc} \Gamma(X_S, \mathsf{B}^2\underline{K}_2) & \to & \Gamma(X_S, \underline{K}^{\mathrm{super}}_{[1,2]}) & \to & \Gamma(X_S, \mathsf{B}(\underline{K}_1/2)) \\ \big\downarrow{\scriptstyle\int_{X_S}} & & \big\downarrow{\scriptstyle\int_{(X_S,\omega^{1/2})}} & & \big\downarrow{\scriptstyle\int_{X_S}} \\ \Gamma(S, \mathsf{Pic}) & \longrightarrow & \Gamma(S, \mathsf{Pic}^{\mathrm{super}}) & \longrightarrow & \Gamma(S, \underline{\mathbb{Z}}/2) \end{array} \tag{1.42}$$

Here, the term $\Gamma(X_S, \mathsf{B}(\underline{K}_1/2))$ is identified with the cofiber of the multiplication by 2 map on $\Gamma(X_S, \mathsf{B}\underline{K}_1)$, and the rightmost vertical arrow has the following description: it sends a line bundle over X_S to its degree mod 2.

1.4.9. We now explain how sections of $\underline{K}^{\mathrm{super}}_{[1,2]}$ over the Zariski classifying stack of a split reductive group scheme define super conformal blocks, at least in the vacuum case.

Let $\mathscr{M}^{\mathrm{spin}}$ denote the moduli stack of spin curves. Namely, an S-point of $\mathscr{M}^{\mathrm{spin}}$ consists of a morphism $p : X_S \to S$ of smooth, proper morphism of relative dimension 1 with connected geometric fibers together with a square root $\omega^{1/2}$ of the relative canonical bundle.

Given an affine group scheme G, denote by $\mathsf{B}G$ the stack classifying *Zariski* locally trivial G-torsors.[3]

Denote by Bun_G the stack over $\mathscr{M}^{\mathrm{spin}}$ whose S-points are triples $(X_S, \omega^{1/2}, P)$, where $(X_S, \omega^{1/2})$ is an S-point of $\mathscr{M}^{\mathrm{spin}}$ and P is a G-bundle over X_S.

If G is *split* reductive, we shall define a functor:

$$\Gamma(\mathsf{B}G, \underline{K}^{\mathrm{super}}_{[1,2]}) \to \Gamma(\mathrm{Bun}_G, \mathsf{Pic}^{\mathrm{super}}). \tag{1.43}$$

Given a section κ of $\underline{K}^{\mathrm{super}}_{[1,2]}$ over $\mathsf{B}G$, we define the $\mathbb{Z}/2$-graded quasi-coherent sheaf $\mathbb{V}_\kappa$ over $\mathscr{M}^{\mathrm{spin}}$ to be the pushforward along $\mathrm{Bun}_G \to \mathscr{M}^{\mathrm{spin}}$ of the image of κ along (1.43).

Construction of (1.43). Consider an S-point $(X_S, \omega^{1/2}, P)$ of Bun_G where S is regular and affine. Étale locally over S, we may assume that P is Zariski locally trivial [DS95, Theorem 2], so it defines a morphism $P : X_S \to \mathsf{B}G$.

Pulling back along P and applying (1.41) yields a functor:

$$\Gamma(\mathsf{B}G, \underline{K}^{\mathrm{super}}_{[1,2]}) \to \Gamma(S, \mathsf{Pic}^{\mathrm{super}}). \tag{1.44}$$

Since $\mathrm{Bun}_G \to \mathrm{Spec}(\mathbb{Z})$ is smooth, a super line bundle over Bun_G is equivalent to a compatible system of super line bundles over S, for all regular affine schemes S over Bun_G. Using functoriality of (1.44) in S, we obtain (1.43). □

Remark 1.4.10. Let us work over an algebraically closed field k.

If G is split simple and simply connected with a split maximal torus $T \subset G$, the Picard groupoid of sections of $\mathsf{B}^2\underline{K}_2$ over $\mathsf{B}G$ rigidified along the base point $e : \mathrm{Spec}(k) \to \mathsf{B}G$ is discrete and isomorphic to the abelian group of Weyl-invariant quadratic forms on the cocharacter lattice Λ of T [BD01, Theorem 4.7].

This abelian group has a canonical generator: the Weyl-invariant quadratic form Q with $Q(\alpha) = 1$ at any short coroot α. To each integer κ, the quadratic form $\kappa \cdot Q$ thus defines a section of $\mathsf{B}^2\underline{K}_2$ over $\mathsf{B}G$.

The quasi-coherent sheaf $\mathbb{V}_\kappa$ defined by this section, via the construction of §1.4.9, is identified with the space of (vacuum) conformal blocks at level κ in the usual sense, see [BL94]. They are known to be finite locally free if $\mathrm{char}(k) = 0$ [TUY89].

We have not undertaken a serious investigation of $\mathbb{V}_\kappa$ in the generality of §1.4.9.

[3]This is in accordance with the notation B used elsewhere in this section, but our BG is different from the usual stack classifying étale or fppf locally trivial G-torsors.

Remark 1.4.11. Let us note an analogue of (1.42) for surfaces. Suppose that X is a proper smooth surface over an algebraically closed field k. Consider the composition:

$$\int_{\mathrm{X}} : \Gamma(\mathrm{X}, \mathsf{B}^2\underline{\mathrm{K}}_2) \to \mathrm{H}^2(\mathrm{X}, \underline{\mathrm{K}}_2) \cong \mathrm{CH}^2(\mathrm{X}) \xrightarrow{\deg} \mathbb{Z}. \tag{1.45}$$

Suppose that the dualizing sheaf $\omega_{\mathrm{X}/k}$ admits a square root $\omega^{1/2}$. *Claim*: the morphism (1.45) canonically extends to a morphism $\Gamma(\mathrm{X}, \underline{\mathrm{K}}^{\mathrm{super}}_{[1,2]}) \to \mathbb{Z}$.

This extension will be defined as an analogue of the morphism (1.41) for surfaces. Namely, we note that (1.37) has an analogue for surfaces: the map $\Gamma(\mathrm{X}, \underline{\mathrm{K}}_{[1,2]}) \to \mathbb{Z}$ induced from $[\mathscr{E}] \mapsto \chi(\mathrm{R}\Gamma(\mathrm{X}, \mathscr{E}))$. To see that it factors through $\Gamma(\mathrm{X}, \underline{\mathrm{K}}^{\mathrm{super}}_{[1,2]})$ when $\omega^{1/2}$ exists, we appeal to the equality:

$$\chi(\mathrm{R}\Gamma(\mathrm{X}, \mathscr{L} \otimes \omega^{1/2})) - \chi(\mathrm{R}\Gamma(\mathrm{X}, \mathscr{L}^{-1} \otimes \omega^{1/2})) = 0, \tag{1.46}$$

for every line bundle $\mathscr{L}$ over X, which follows from the Riemann–Roch formula:

$$\chi(\mathrm{R}\Gamma(\mathrm{X}, \mathscr{L} \otimes \omega^{1/2})) = \chi(\mathrm{R}\Gamma(\mathrm{X}, \mathscr{O})) + \frac{1}{2}(\mathscr{L} \cdot \mathscr{L} - \omega^{1/2} \cdot \omega^{1/2}),$$

where $\cdot$ denotes the intersection pairing, *i.e.* the composition of (1.20) with (1.45).

2. Brylinski–Deligne classification

In this section, we classify rigidified sections of $\underline{\mathrm{K}}^{\mathrm{super}}_{[1,2]}$ over the Zariski classifying stack $\mathsf{B}\mathrm{G}$ of a reductive group scheme G over a base scheme S, assumed regular and of finite type over a field. The main result is Theorem 2.2.3.

We begin in §2.1 with a classification of rigidified sections of $\underline{\mathrm{K}}_{[1,2]}$ over $\mathsf{B}\mathrm{G}$ (Proposition 2.1.8). Using tools developed in §1.2, we reduce this result to the Brylinski–Deligne theorem [BD01, Theorem 7.2]. In §2.2, we state the main result. The next subsection §2.3 is a technical interlude classifying central extensions of G by $\mathbb{G}_m$ over an arbitrary base scheme. The results of §2.1 and §2.3 are combined in §2.4 to prove Theorem 2.2.3.

2.1. Classification: $\underline{\mathrm{K}}_{[1,2]}$.

2.1.1. Let S be a regular scheme of finite type over a field.

Let $\mathrm{G} \to \mathrm{S}$ be a reductive group scheme equipped with a maximal torus $\mathrm{T} \subset \mathrm{G}$. Cocharacters of T form an étale sheaf of abelian groups Λ over S.

Denote by G_{sc} the simply connected form of G. The preimage of T in G_{sc} is a maximal torus $\mathrm{T}_{\mathrm{sc}} \subset \mathrm{G}_{\mathrm{sc}}$, whose sheaf of cocharacters is denoted by Λ_{sc}. The algebraic fundamental group $\pi_1\mathrm{G}$ may then be realized as $\Lambda/\Lambda_{\mathrm{sc}}$.

2.1.2. Denote by $\mathsf{B}\mathrm{G}$ the stack of *Zariski* locally trivial G-torsor. Denote by $e : \mathrm{S} \to \mathsf{B}\mathrm{G}$ the unit section. For a Zariski sheaf $\mathscr{F}$ of connective spectra, we write $\Gamma_e(\mathsf{B}\mathrm{G}, \mathscr{F})$ for the fiber of the morphism:

$$e^* : \Gamma(\mathsf{B}\mathrm{G}, \mathscr{F}) \to \Gamma(\mathrm{S}, \mathscr{F}).$$

We also denote by $\underline{\Gamma}_e(\mathsf{B}\mathrm{G}, \mathscr{F})$ the presheaf over S whose section over an S_1-scheme is $\Gamma_e(\mathsf{B}\mathrm{G} \times_{\mathrm{S}} \mathrm{S}_1, \mathscr{F})$. It is a sheaf in the Zariski topology.

In this subsection, we describe $\underline{\Gamma}_e(\mathsf{B}\mathrm{G}, \underline{\mathrm{K}}_{[1,2]})$ in terms of the combinatorics of G.

2.1.3. We first recall Brylinski and Deligne's description of $\underline{\Gamma}_e(\mathsf{B}\mathrm{G}, \mathsf{B}^2\underline{\mathrm{K}}_2)$. Indeed, [BD01, Theorem 7.2] constructs a canonical equivalence of sheaves of Picard groupoids:

$$\underline{\Gamma}_e(\mathsf{B}\mathrm{G}, \mathsf{B}^2\underline{\mathrm{K}}_2) \xrightarrow{\simeq} \vartheta_{\mathrm{G}}(\Lambda), \tag{2.1}$$

where sections of $\vartheta_{\mathrm{G}}(\Lambda)$ are triples $(\mathrm{Q}, \tilde{\Lambda}, \varphi)$ defined below:

(1) Q is a Weyl-invariant integral quadratic form on Λ;
(2) $\tilde{\Lambda}$ is a central extension of Λ by $\mathbb{G}_m$, whose commutator pairing $\Lambda\otimes\Lambda\to\mathbb{G}_m$ equals $\lambda_1,\lambda_2\mapsto(-1)^{b(\lambda_1,\lambda_2)}$ for $b(\lambda_1,\lambda_2):=\mathrm{Q}(\lambda_1+\lambda_2)-\mathrm{Q}(\lambda_1)-\mathrm{Q}(\lambda_2)$;
(3) φ is an isomorphism between the restriction of $\tilde{\Lambda}$ to Λ_{sc} and the central extension induced from Q_{sc} (in the sense of Remark 2.1.4), the restriction of Q to Λ_{sc}.

The Picard groupoid structure on $\vartheta_{\mathrm{G}}(\Lambda)$ is defined by sum in Q and Baer sum in $\tilde{\Lambda}$.

Remark 2.1.4. To be more explicit, [BD01, §3] first shows that $\underline{\Gamma}_e(\mathsf{BT},\mathsf{B}^2\underline{\mathrm{K}}_2)$ is canonically equivalent to the sheaf of Picard groupoids $\vartheta(\Lambda)$ whose sections are pairs $(\mathrm{Q},\tilde{\Lambda})$, where Q is an integral quadratic form on Λ, and $\tilde{\Lambda}$ is as in (2).

Then [BD01, §4] shows that $\underline{\Gamma}_e(\mathsf{BG}_{\mathrm{sc}},\mathsf{B}^2\underline{\mathrm{K}}_2)$ is the sheaf of discrete groupoids whose sections are Weyl-invariant integral quadratic forms on Λ_{sc}. Restriction along $\mathrm{T}_{\mathrm{sc}}\subset\mathrm{G}_{\mathrm{sc}}$ and applying the description of $\underline{\Gamma}_e(\mathsf{BT}_{\mathrm{sc}},\mathsf{B}^2\underline{\mathrm{K}}_2)$, we obtain a functor from Weyl-invariant quadratic forms on Λ_{sc} to $\vartheta(\Lambda_{\mathrm{sc}})$:

$$\mathrm{Quad}(\Lambda_{\mathrm{sc}})^{\mathrm{W}}\to\vartheta(\Lambda_{\mathrm{sc}}). \tag{2.2}$$

In particular, any $\mathrm{Q}_{\mathrm{sc}}\in\mathrm{Quad}(\Lambda_{\mathrm{sc}})^{\mathrm{W}}$ induces a central extension of Λ_{sc} by $\mathbb{G}_m$.

The functor from $\underline{\Gamma}_e(\mathsf{BG},\mathsf{B}^2\underline{\mathrm{K}}_2)$ to $\vartheta_{\mathrm{G}}(\Lambda)$ is given as follows. The pair $(\mathrm{Q},\tilde{\Lambda})$ is defined by its restriction along $\mathrm{T}\subset\mathrm{G}$, and the isomorphism φ comes from functoriality with respect to the commutative diagram:

$$\begin{array}{ccc}\mathrm{T}_{\mathrm{sc}} & \subset & \mathrm{G}_{\mathrm{sc}}\\ \downarrow & & \downarrow\\ \mathrm{T} & \subset & \mathrm{G}\end{array}$$

Remark 2.1.5. Both sides of (2.1) are étale sheaves over S. Indeed, it is clear that $\vartheta_{\mathrm{G}}(\Lambda)$ satisfies étale descent. The étale descent of $\underline{\Gamma}_e(\mathsf{BG},\mathsf{B}^2\underline{\mathrm{K}}_2)$ is established in [BD01, §2], logically prior to proving that (2.1) is an equivalence.

2.1.6. Let us define an enlargement $\vartheta_{\mathrm{G}}^{\mathbb{Z}}(\Lambda)$ of $\vartheta_{\mathrm{G}}(\Lambda)$. Namely, a section of $\vartheta_{\mathrm{G}}^{\mathbb{Z}}(\Lambda)$ is a quadruple $(\mathrm{Q},\tilde{\Lambda},\varphi,x)$ where $(\mathrm{Q},\tilde{\Lambda},\varphi)$ is a section of $\vartheta_{\mathrm{G}}(\Lambda)$ and:

(4) $x:\Lambda\to\mathbb{Z}$ is a character vanishing on Λ_{sc} (*i.e.* a character of $\pi_1\mathrm{G}$).

Therefore, as a sheaf of *pointed spaces*, $\vartheta_{\mathrm{G}}^{\mathbb{Z}}(\Lambda)$ is the product $\vartheta_{\mathrm{G}}(\Lambda)\times\underline{\mathrm{Hom}}(\pi_1\mathrm{G},\mathbb{Z})$. As a sheaf of *Picard groupoids*, we demand that it fits into a fiber sequence:

$$\vartheta_{\mathrm{G}}(\Lambda)\to\vartheta_{\mathrm{G}}^{\mathbb{Z}}(\Lambda)\to\underline{\mathrm{Hom}}(\pi_1\mathrm{G},\mathbb{Z}). \tag{2.3}$$

Specifying the Picard groupoid structure on $\vartheta_{\mathrm{G}}^{\mathbb{Z}}(\Lambda)$ thus amounts to specifying a *symmetric cocycle*, *i.e.* a morphism of Picard groupoids:

$$\underline{\mathrm{Hom}}(\pi_1\mathrm{G},\mathbb{Z})\otimes\underline{\mathrm{Hom}}(\pi_1\mathrm{G},\mathbb{Z})\to\vartheta_{\mathrm{G}}(\Lambda), \tag{2.4}$$

together with a null-homotopy of its precomposition with the anti-symmetrizer.

Construction of (2.4). Given sections x_1,x_2 of $\underline{\mathrm{Hom}}(\pi_1\mathrm{G},\mathbb{Z})$, the morphism (2.4) assigns to $x_1\otimes x_2$ the triple $(\mathrm{Q},\tilde{\Lambda},\varphi)$, where $\mathrm{Q}(\lambda):=x_1(\lambda)x_2(\lambda)$, $\tilde{\Lambda}$ is the central extension defined by the cocycle $\lambda_1,\lambda_2\mapsto(-1)^{x_1(\lambda_1)x_2(\lambda_2)}$, and φ is the identity automorphism of the trivial central extension of Λ_{sc} by $\mathbb{G}_m$.

In order to construct a null-homotopy of the image of $x_1\otimes x_2-x_2\otimes x_1$, we need to trivialize the central extension of Λ by $\mathbb{G}_m$ defined by the cocycle:

$$\lambda_1,\lambda_2\mapsto(-1)^{x_1(\lambda_1)x_2(\lambda_2)-x_2(\lambda_1)x_1(\lambda_2)}. \tag{2.5}$$

In other words, we need to find a map $q : \Lambda \to \mathbb{G}_m$ such that $q(\lambda_1 + \lambda_2)q(\lambda_1)^{-1}q(\lambda_2)^{-1}$ coincides with (2.5). The desired map is set to be $q(\lambda) := (-1)^{x_1(\lambda)x_2(\lambda)}$. □

Remark 2.1.7. By associating to each $\lambda \in \Lambda$ its fiber $\mathscr{L}^\lambda \subset \tilde{\Lambda}$ viewed as a $\mathbb{G}_m$-torsor, the central extension $\tilde{\Lambda}$ in a section $(\mathrm{Q}, \tilde{\Lambda}, \varphi)$ of $\vartheta_{\mathrm{G}}(\Lambda)$ can be viewed as a monoidal morphism $\Lambda \to \mathsf{Pic}$ which preserves the commutativity constraint up to the factor $(-1)^b$.

Likewise, an object $(\mathrm{Q}, \tilde{\Lambda}, \varphi, x)$ of $\vartheta_{\mathrm{G}}^{\mathbb{Z}}(\Lambda)$ defines a monoidal morphism:

$$\Lambda \to \mathsf{Pic}^{\mathbb{Z}}, \quad \lambda \mapsto (\mathscr{L}^\lambda, x(\lambda)),$$

which preserves the commutativity constraint up to the factor $(-1)^{\tilde{b}}$ (viewed as a $\mathbb{G}_m$-valued bilinear form on Λ), where $\tilde{b}$ is defined by:

$$\lambda_1, \lambda_2 \mapsto b(\lambda_1, \lambda_2) + x(\lambda_1)x(\lambda_2). \tag{2.6}$$

Proposition 2.1.8. *There is a canonical equivalence of sheaves of Picard groupoids:*

$$\underline{\Gamma}_e(\mathsf{BG}, \underline{\mathrm{K}}_{[1,2]}) \xrightarrow{\simeq} \vartheta_{\mathrm{G}}^{\mathbb{Z}}(\Lambda). \tag{2.7}$$

It is related to the Brylinski–Deligne equivalence by a commutative diagram:

$$\begin{array}{ccc} \underline{\Gamma}_e(\mathsf{BG}, \mathsf{B}^2\underline{\mathrm{K}}_2) & \subset & \underline{\Gamma}_e(\mathsf{BG}, \underline{\mathrm{K}}_{[1,2]}) \\ \downarrow{\scriptstyle (2.1)} & & \downarrow{\scriptstyle (2.7)} \\ \vartheta_{\mathrm{G}}(\Lambda) & \subset & \vartheta_{\mathrm{G}}^{\mathbb{Z}}(\Lambda) \end{array}$$

Proof. As a presheaf of pointed spaces, $\underline{\Gamma}_e(\mathsf{BG}, \underline{\mathrm{K}}_{[1,2]})$ is the direct product $\underline{\Gamma}_e(\mathsf{BG}, \mathsf{B}^2\underline{\mathrm{K}}_2) \times \underline{\Gamma}_e(\mathsf{BG}, \mathsf{B}\underline{\mathrm{K}}_1)$ thanks to the section (1.7). In particular, $\underline{\Gamma}_e(\mathsf{BG}, \underline{\mathrm{K}}_{[1,2]})$ satisfies étale descent, see Remark 2.1.5.

The desired functor (2.7) is defined to be the Brylinski–Deligne equivalence (2.1) on the first factor and the canonical isomorphism:

$$\underline{\Gamma}_e(\mathsf{BG}, \mathsf{B}\underline{\mathrm{K}}_1) \xrightarrow{\simeq} \underline{\mathrm{Hom}}(\mathrm{G}, \underline{\mathrm{K}}_1) \xrightarrow{\simeq} \underline{\mathrm{Hom}}(\pi_1\mathrm{G}, \mathbb{Z}) \tag{2.8}$$

on the second factor.

It thus remains to lift this functor to one between Picard groupoids. We appeal to the description of $\underline{\mathrm{K}}_{[1,2]}$ using the symmetric cocycle $\mathsf{B}\underline{\mathrm{K}}_1 \otimes \mathsf{B}\underline{\mathrm{K}}_1 \to \mathsf{B}^2\underline{\mathrm{K}}_2$ associated to the anti-symmetric pairing $\{\cdot,\cdot\} : \underline{\mathrm{K}}_1 \otimes \underline{\mathrm{K}}_1 \to \underline{\mathrm{K}}_2$ (Proposition 1.2.8). Indeed, it suffices to construct an isomorphism between the $\underline{\Gamma}_e(\mathsf{BG}, \mathsf{B}^2\underline{\mathrm{K}}_2)$-valued pairing it induces on $\underline{\Gamma}_e(\mathsf{BG}, \mathsf{B}\underline{\mathrm{K}}_1)$ and the pairing (2.4):

$$\begin{array}{ccc} \underline{\Gamma}_e(\mathsf{BG}, \mathsf{B}\underline{\mathrm{K}}_1) \otimes \underline{\Gamma}_e(\mathsf{BG}, \mathsf{B}\underline{\mathrm{K}}_1) & \xrightarrow{\{\cdot,\cdot\}} & \underline{\Gamma}_e(\mathsf{BG}, \mathsf{B}^2\underline{\mathrm{K}}_2) \\ \downarrow{\scriptstyle (2.8)} & & \downarrow{\scriptstyle (2.1)} \\ \underline{\mathrm{Hom}}(\pi_1\mathrm{G}, \mathbb{Z}) \otimes \underline{\mathrm{Hom}}(\pi_1\mathrm{G}, \mathbb{Z}) & \xrightarrow{(2.4)} & \vartheta_{\mathrm{G}}(\Lambda) \end{array} \tag{2.9}$$

compatibly with null homotopies of their pre-composition with the anti-symmetrizer.

By definition of $\vartheta_{\mathrm{G}}(\Lambda)$, it suffices to treat the case $\mathrm{G} = \mathrm{T}$ as long as the isomorphism we construct is functorial in T.

In this case, any pair of characters x_1, x_2 of T defines under the top horizontal arrow of (2.9) the central extension:

$$1 \to \underline{\mathrm{K}}_2 \to \mathrm{E} \to \mathrm{T} \to 1$$

corresponding to the cocycle $\mathrm{T} \otimes \mathrm{T} \to \underline{\mathrm{K}}_2$, $(t_1, t_2) \mapsto \{x_1(t_1), x_2(t_2)\}$. The null-homotopy the central extension defined by the cocycle $(t_1, t_2) \mapsto \{x_1(t_1), x_2(t_2)\} - \{x_2(t_1), x_1(t_2)\}$ is exhibited by the map $\mathrm{T} \to \underline{\mathrm{K}}_2$, $t \mapsto \{x_1(t), x_2(t)\}$. These data correspond to the description of (2.4) (for G = T) under the equivalence of [BD01, Theorem 3.16]. □

2.2. Classification: $\underline{\mathrm{K}}^{\mathrm{super}}_{[1,2]}$.

2.2.1. Let us define another enlargement $\vartheta^{\mathrm{super}}_{\mathrm{G}}(\Lambda)$ of $\vartheta_{\mathrm{G}}(\Lambda)$ which fits into a fiber sequence of sheaves of Picard groupoids over S:

$$\vartheta_{\mathrm{G}}(\Lambda) \to \vartheta^{\mathrm{super}}_{\mathrm{G}}(\Lambda) \to \underline{\mathrm{Hom}}(\pi_1\mathrm{G}, \mathbb{Z}/2). \tag{2.10}$$

Namely, an object of $\vartheta^{\mathrm{super}}_{\mathrm{G}}(\Lambda)$ is a triple $(b, \tilde{\Lambda}, \varphi)$ where:

1. b is a Weyl-invariant integral symmetric bilinear form on Λ, such that $b(\lambda, \lambda) \in 2\mathbb{Z}$ for any $\lambda \in \Lambda_{\mathrm{sc}}$;
2. $\tilde{\Lambda}$ is a central extension of Λ by $\mathbb{G}_m$, whose commutator pairing equals $\lambda_1, \lambda_2 \mapsto (-1)^{b(\lambda_1,\lambda_2)+\epsilon(\lambda_1)\epsilon(\lambda_2)}$, where $\epsilon(\lambda) := b(\lambda, \lambda) \bmod 2$;
3. φ is an isomorphism between the restriction of $\tilde{\Lambda}$ to Λ_{sc} and the central extension induced by Q_{sc} as in §2.1.3.

To define the Picard groupoid structure on $\vartheta^{\mathrm{super}}_{\mathrm{G}}(\Lambda)$, it is more natural to interpret $\tilde{\Lambda}$ as a monoidal morphism (see Remark 2.1.7):

$$\Lambda \to \mathsf{Pic}^{\mathrm{super}}, \quad \lambda \mapsto (\mathscr{L}^{\lambda}, \epsilon(\lambda)), \tag{2.11}$$

which preserves the commutativity constraint up to the bilinear form $(-1)^b$. The Picard groupoid structure on $\vartheta^{\mathrm{super}}_{\mathrm{G}}(\Lambda)$ is induced from sum in b and the Picard groupoid structure of $\mathsf{Pic}^{\mathrm{super}}$. In particular, it is *not* strictly commutative in general.

Let us construct the fiber sequence (2.10). The inclusion of $\vartheta_{\mathrm{G}}(\Lambda)$ in $\vartheta^{\mathrm{super}}_{\mathrm{G}}(\Lambda)$ sends $(\mathrm{Q}, \tilde{\Lambda}, \varphi)$ to the triple $(b, \tilde{\Lambda}, \varphi)$, where $b(\lambda_1, \lambda_2) := \mathrm{Q}(\lambda_1 + \lambda_2) - \mathrm{Q}(\lambda_1) - \mathrm{Q}(\lambda_2)$. The second map $\vartheta^{\mathrm{super}}_{\mathrm{G}}(\Lambda) \to \mathrm{Hom}(\pi_1\mathrm{G}, \mathbb{Z}/2)$ assigns to $(b, \tilde{\Lambda}, \varphi)$ the homomorphism ϵ as in (2). Note that ϵ vanishes if and only if b comes from a quadratic form.

Remark 2.2.2. The sheaf of Picard groupoids $\vartheta^{\mathrm{super}}_{\mathrm{G}}(\Lambda)$ is introduced in [BD01, Questions 12.13(iii)]. For G = T a torus, a variant of it has also appeared in [BD04, §3.10], where its sections are called *ϑ-data*.

Theorem 2.2.3. *There is a canonical equivalence of sheaves of Picard groupoids:*

$$\underline{\Gamma}_e(\mathsf{BG}, \underline{\mathrm{K}}^{\mathrm{super}}_{[1,2]}) \xrightarrow{\simeq} \vartheta^{\mathrm{super}}_{\mathrm{G}}(\Lambda). \tag{2.12}$$

It is related to the Brylinski–Deligne equivalence by a commutative diagram:

$$\begin{array}{ccc} \underline{\Gamma}_e(\mathsf{BG}, \mathsf{B}^2\underline{\mathrm{K}}_2) & \subset & \underline{\Gamma}_e(\mathsf{BG}, \underline{\mathrm{K}}^{\mathrm{super}}_{[1,2]}) \\ \big\downarrow{\scriptstyle (2.1)} & & \big\downarrow{\scriptstyle (2.12)} \\ \vartheta_{\mathrm{G}}(\Lambda) & \subset & \vartheta^{\mathrm{super}}_{\mathrm{G}}(\Lambda) \end{array} \tag{2.13}$$

2.2.4. The proof of Theorem 2.2.3 will occupy the remainder of this section. For now, we shall formulate a compatibility statement between the isomorphisms (2.7) and (2.12) (which will in fact be used to define (2.12).)

To do so, we need to construct two morphisms of sheaves of Picard groupoids:

$$\mathrm{Hom}(\pi_1\mathrm{G}, \mathbb{Z}) \to \vartheta^{\mathbb{Z}}_{\mathrm{G}}(\Lambda) \tag{2.14}$$

$$\vartheta^{\mathbb{Z}}_{\mathrm{G}}(\Lambda) \to \vartheta^{\mathrm{super}}_{\mathrm{G}}(\Lambda). \tag{2.15}$$

The morphism (2.14) sends a character $x : \pi_1 \mathrm{G} \to \mathbb{Z}$ to the quadruple $(\mathrm{Q}, \tilde{\Lambda}, \varphi, 2x)$ where $\mathrm{Q}(\lambda) \coloneqq -2x(\lambda)^2$, $\tilde{\Lambda}$ is the trivial central extension, and φ is the identity automorphism of the trivial central extension.

The morphism (2.15) is the identity on the subgroupoid $\vartheta_{\mathrm{G}}(\Lambda)$. To any character x in the additional factor $\mathrm{Hom}(\pi_1\mathrm{G}, \mathbb{Z})$, it assigns the triple $(b, \tilde{\Lambda}, \varphi)$ where $b(\lambda_1, \lambda_2) \coloneqq x(\lambda_1)x(\lambda_2)$, $\tilde{\Lambda}$ is the trivial central extension (*i.e.* the morphism (2.11) is given by $\lambda \mapsto (\mathscr{O}, \epsilon(\lambda))$), and φ is the identity automorphism of the trivial central extension.

Lemma 2.2.5. *The maps* (2.14), (2.15) *thus defined are morphisms of Picard groupoids, and fit into a fiber sequence of such:*

$$\mathrm{Hom}(\pi_1\mathrm{G}, \mathbb{Z}) \to \vartheta_{\mathrm{G}}^{\mathbb{Z}}(\Lambda) \to \vartheta_{\mathrm{G}}^{\mathrm{super}}(\Lambda). \tag{2.16}$$

Proof. We only verify that (2.16) is indeed a fiber sequence. Let $(\mathrm{Q}, \tilde{\Lambda}, \varphi, x)$ be an object in the fiber of (2.15). Thus the induced symmetric form:

$$\lambda_1, \lambda_2 \mapsto \mathrm{Q}(\lambda_1 + \lambda_2) - \mathrm{Q}(\lambda_1) - \mathrm{Q}(\lambda_2) + x(\lambda_1)x(\lambda_2)$$

must vanish. Setting $\lambda_1 = \lambda_2$, this implies that $x(\lambda) \in 2\mathbb{Z}$ for all $\lambda \in \Lambda$, so we may write $x = 2y$ for a character $y : \pi_1\mathrm{G} \to \mathbb{Z}$ and there holds $\mathrm{Q}(\lambda) = -2y(\lambda)^2$. The fact that $(\mathrm{Q}, \tilde{\Lambda}, \varphi, x)$ lies in the fiber also supplies us with a trivialization of $\widetilde{\Lambda}$ compatible with φ. This yields an isomorphism between $(\mathrm{Q}, \tilde{\Lambda}, \varphi, x)$ and the image of y under (2.14). ☐

2.2.6. The compatibility statement asserts that (2.16) coincides with the cofiber sequence defining $\underline{\mathrm{K}}_{[1,2]}^{\mathrm{super}}$ (see §1.3.7) evaluated at BG:

$$\begin{array}{ccccc} \underline{\Gamma}_e(\mathsf{BG}, \mathsf{B}\underline{\mathrm{K}}_1) & \xrightarrow{\mathrm{Sq}} & \underline{\Gamma}_e(\mathsf{BG}, \underline{\mathrm{K}}_{[1,2]}) & \to & \underline{\Gamma}_e(\mathsf{BG}, \underline{\mathrm{K}}_{[1,2]}^{\mathrm{super}}) \\ \downarrow \cong & & \downarrow (2.7) & & \downarrow (2.12) \\ \mathrm{Hom}(\pi_1\mathrm{G}, \mathbb{Z}) & \xrightarrow{(2.14)} & \vartheta_{\mathrm{G}}^{\mathbb{Z}}(\Lambda) & \xrightarrow{(2.15)} & \vartheta_{\mathrm{G}}^{\mathrm{super}}(\Lambda) \end{array} \tag{2.17}$$

The following statement can be verified without any knowledge of (2.12).

Lemma 2.2.7. *The left square in* (2.17) *is canonically commutative.*

Proof. We use the expression (1.29) of the map Sq as the difference:

$$\mathsf{B}\underline{\mathrm{K}}_1 \to \underline{\mathrm{K}}_{[1,2]}, \quad \mathscr{L} \mapsto s(\mathscr{L}^2) - 2\{\mathscr{L}, \mathscr{L}\}. \tag{2.18}$$

Under the equivalence (2.7), s corresponds to the natural inclusion of $\mathrm{Hom}(\pi_1\mathrm{G}, \mathbb{Z})$ in $\vartheta_{\mathrm{G}}^{\mathbb{Z}}(\Lambda)$, while the map $\mathscr{L} \mapsto \{\mathscr{L}, \mathscr{L}\}$ corresponds to the restriction of (2.4) along the diagonal copy of $\mathrm{Hom}(\pi_1\mathrm{G}, \mathbb{Z})$ (established in the proof of Proposition 2.1.8). The map induced from (2.18) upon taking $\underline{\Gamma}_e(\mathsf{BG}, \cdot)$ is thus readily computed to be (2.14). ☐

2.3. Central extensions by $\underline{\mathrm{K}}_1$.

2.3.1. In this subsection, we let S be an arbitrary base scheme and $\mathrm{G} \to \mathrm{S}$ be a reductive group scheme. Our goal is to classify central extensions of G by $\underline{\mathrm{K}}_1 \cong \mathbb{G}_m$.

When S is the spectrum of a field, this classification is obtained by Weissman [Wei11, Theorem 1.11]. We give a self-contained proof valid over any base scheme.

2.3.2. For a reductive group scheme $\mathrm{H} \to \mathrm{S}$, we write $\mathrm{Rad}(\mathrm{H})$ for the radical of H as defined in [ABD+66, XXII, Définition 4.3.6]. Namely, it is the maximal torus of the center of H. The formation of $\mathrm{Rad}(\mathrm{H})$ is stable under base change and recovers the classical notion (maximal connected normal solvable subgroup) over a geometric point of S.

Given a central extension of a reductive group scheme H by $\mathbb{G}_m$ (or any torus):

$$1 \to \mathbb{G}_m \to \widetilde{\mathrm{H}} \to \mathrm{H} \to 1, \tag{2.19}$$

we first observe that $\widetilde{\mathrm{H}}$ is representable by a reductive group scheme. Indeed, one checks directly that $\widetilde{\mathrm{H}} \to \mathrm{S}$ is smooth and its geometric fibers have vanishing unipotent radicals.

2.3.3. By functoriality of the algebraic fundamental group, we obtain a functor from the Picard groupoid of central extension of G by $\mathbb{G}_m$ to that of extensions of $\pi_1\mathrm{G}$ by $\mathbb{Z}$ as sheaves of abelian groups:

$$\mathrm{Hom}_{\mathbb{E}_1}(\mathrm{G}, \mathsf{B}\mathbb{G}_m) \to \mathrm{Hom}_{\mathbb{Z}}(\pi_1\mathrm{G}, \mathsf{B}\mathbb{Z}). \tag{2.20}$$

Proposition 2.3.4. *The functor* (2.20) *is an equivalence.*

2.3.5. The Picard groupoids in (2.20) are of étale local nature on S, so we may assume the existence of a maximal torus $\mathrm{T} \subset \mathrm{G}$ in the proof of Proposition 2.3.4.

Since the G-conjugation extends along the map $\mathrm{G}_{\mathrm{sc}} \to \mathrm{G}$, the quotient stack $\mathrm{G}/\mathrm{G}_{\mathrm{sc}}$ has a monoidal structure. As such, we have isomorphisms of monoidal stacks:

$$\mathrm{G}/\mathrm{G}_{\mathrm{sc}} \xleftarrow{\simeq} \mathrm{T}/\mathrm{T}_{\mathrm{sc}} \xrightarrow{\simeq} \pi_1(\mathrm{G}) \otimes \mathbb{G}_m. \tag{2.21}$$

Here, the tensor product is understood in the derived sense and sheafified in the fppf, or equivalently the étale topology.

Proof of Proposition 2.3.4. In view of the isomorphisms (2.21), it suffices to prove that the following two forgetful functors are equivalences:

$$\mathrm{Hom}_{\mathbb{Z}}(\mathrm{T}/\mathrm{T}_{\mathrm{sc}}, \mathsf{B}\mathbb{G}_m) \to \mathrm{Hom}_{\mathbb{E}_1}(\mathrm{T}/\mathrm{T}_{\mathrm{sc}}, \mathsf{B}\mathbb{G}_m), \tag{2.22}$$

$$\mathrm{Hom}_{\mathbb{E}_1}(\mathrm{G}/\mathrm{G}_{\mathrm{sc}}, \mathsf{B}\mathbb{G}_m) \to \mathrm{Hom}_{\mathbb{E}_1}(\mathrm{G}, \mathsf{B}\mathbb{G}_m). \tag{2.23}$$

Indeed, the left-hand-side of (2.22) is identified with $\mathrm{Hom}_{\mathbb{Z}}(\pi_1\mathrm{G}, \mathsf{B}\mathbb{Z})$ by the vanishing of $\underline{\mathrm{Ext}}^1(-, \mathbb{G}_m)$ on the category of fppf sheaves of abelian groups.

Given a central extension of a reductive group scheme H by $\mathbb{G}_m$ as in (2.19), we have a short exact sequence of tori:

$$1 \to \mathbb{G}_m \to \mathrm{Rad}(\widetilde{\mathrm{H}}) \to \mathrm{Rad}(\mathrm{H}) \to 1. \tag{2.24}$$

Indeed, the fact that $\mathrm{Rad}(\widetilde{\mathrm{H}}) \to \mathrm{Rad}(\mathrm{H})$ is surjective can be checked on geometric fibers. Moreover, $\mathrm{Rad}(\widetilde{\mathrm{H}})$ contains $\mathbb{G}_m$ since the latter is central, so the inclusion of the kernel $\mathrm{Rad}(\widetilde{\mathrm{H}}) \cap \mathbb{G}_m$ inside $\mathbb{G}_m$ is an isomorphism.

We make two observations:

(1) If H is a torus, then so is $\widetilde{\mathrm{H}}$. This is because the map $\mathrm{Rad}(\widetilde{\mathrm{H}}) \to \widetilde{\mathrm{H}}$ is an isomorphism by comparing (2.19) with (2.24).

(2) If H is semisimple, then we find an isomorphism $\mathbb{G}_m \xrightarrow{\simeq} \mathrm{Rad}(\widetilde{\mathrm{H}})$.

To prove that (2.22) is an equivalence, it suffices to show that any central extension of a torus by $\mathbb{G}_m$ is commutative. This follows from observation (1).

To prove that (2.23) is an equivalence, we first write the left-hand-side as the groupoid of central extensions:

$$1 \to \mathbb{G}_m \to \widetilde{\mathrm{G}} \to \mathrm{G} \to 1, \tag{2.25}$$

equipped with a $\widetilde{\mathrm{G}}$-equivariant splitting over G_{sc} for the adjoint action. Our task is to show that such a splitting exists uniquely.

To construct such a splitting, we may assume that G is simply connected in (2.25). Let $\widetilde{\mathrm{G}}_{\mathrm{der}} \subset \widetilde{\mathrm{G}}$ denote its derived subgroup. We claim that the composition:

$$\widetilde{\mathrm{G}}_{\mathrm{der}} \subset \widetilde{\mathrm{G}} \to \mathrm{G} \tag{2.26}$$

is an isomorphism.

It suffices to prove that (2.26) is a central isogeny, *i.e.* it is finite, flat, and surjective, with kernel contained in the center of $\widetilde{\mathrm{G}}_{\mathrm{der}}$. The statement on the kernel is clear. The fact that (2.26) is finite, finite, and surjective may be established smooth locally, so we base change along $\widetilde{\mathrm{G}} \to \mathrm{G}$, where (2.26) becomes the multiplication map:

$$\widetilde{\mathrm{G}}_{\mathrm{der}} \times \mathbb{G}_m \to \widetilde{\mathrm{G}}.$$

However, by observation (2), this morphism is identified with the isogeny $\widetilde{\mathrm{G}}_{\mathrm{der}} \times \mathrm{Rad}(\widetilde{\mathrm{G}}) \to \widetilde{\mathrm{G}}$ of [ABD+66, XXII, 6.2.3].

The isomorphism (2.26) for G simply connected equips (2.25) with a section over G_{sc}. It is unique since any two sections differ by a character $\mathrm{G}_{\mathrm{sc}} \to \mathbb{G}_m$ which is necessarily trivial. To see that this section is $\widetilde{\mathrm{G}}$-equivariant, it suffices to observe that the diagram:

$$\begin{array}{ccc} \widetilde{\mathrm{G}} \times_{\mathrm{G}} \mathrm{G}_{\mathrm{sc}} & \to & \mathrm{G}_{\mathrm{sc}} \\ \downarrow & & \downarrow \\ \widetilde{\mathrm{G}} & \longrightarrow & \mathrm{G} \end{array}$$

is $\widetilde{\mathrm{G}}$-equivariant, and any automorphism of $\widetilde{\mathrm{G}} \times_{\mathrm{G}} \mathrm{G}_{\mathrm{sc}}$ preserves its derived subgroup. □

2.4. Proof of Theorem 2.2.3.

2.4.1. We return to the set-up of §2.1.1. In particular, the base scheme S is assumed to be regular and of finite type over a field. In this subsection, we construct the equivalence (2.12) and thereby prove Theorem 2.2.3.

We shall construct this equivalence in two stages: we first do it when $\pi_1\mathrm{G}$ is torsion-free and satisfies a Galois cohomological condition. This step uses Proposition 2.1.8 and Proposition 2.3.4. We then bootstrap the general case from this one, using the flasque resolution over general base due to González-Avilés [GA13].

The fact that we have to play with Galois cohomology is because we do not know *a priori* that $\underline{\Gamma}_e(\mathsf{BG}, \underline{\mathrm{K}}^{\mathrm{super}}_{[1,2]})$ satisfies étale descent.

2.4.2. Note that our hypothesis on S guarantees that every S-tori is isotrivial, *i.e.* split by a finite étale cover [ABD+66, X, Théorème 5.16]. In particular, it makes sense for an S-torus to be quasi-trivial, see [CTS87, Definition 1.2].

Lemma 2.4.3. *If* $\pi_1\mathrm{G}$ *is the sheaf of cocharacters of a quasi-trivial torus, then both rows in* (2.17) *are cofiber sequences of Zariski sheaves.*

Proof. This assertion amounts to the Zariski local surjectivity of the two horizontal morphisms appearing in (2.17):

$$f_1 : \underline{\Gamma}_e(\mathsf{BG}, \underline{\mathrm{K}}_{[1,2]}) \to \underline{\Gamma}_e(\mathsf{BG}, \underline{\mathrm{K}}^{\mathrm{super}}_{[1,2]}),$$
$$f_2 : \vartheta^{\mathbb{Z}}_{\mathrm{G}}(\Lambda) \to \vartheta^{\mathrm{super}}_{\mathrm{G}}(\Lambda).$$

For f_2, we note that comparing (2.3) with (2.10) leads to a Cartesian square:

$$\begin{array}{ccc} \vartheta^{\mathbb{Z}}_{\mathrm{G}}(\Lambda) & \twoheadrightarrow & \underline{\mathrm{Hom}}(\pi_1\mathrm{G}, \mathbb{Z}) \\ \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle \text{mod } 2} \\ \vartheta^{\mathrm{super}}_{\mathrm{G}}(\Lambda) & \to & \underline{\mathrm{Hom}}(\pi_1\mathrm{G}, \mathbb{Z}/2) \end{array} \qquad (2.27)$$

Claim: the "mod 2" morphism is surjective in the Zariski topology.

Indeed, Zariski locally on S, we may find a finite Galois cover $S_1 \to S$ which splits $\pi_1 G$. Denote by Γ the Galois group of S_1/S and M the $\mathbb{Z}$-linear dual of the Γ-module associated to $\pi_1 G$ at a geometric point of S. Then the problem amounts to the surjectivity of $M^\Gamma \to (M/2)^\Gamma$, which follows from $H^1(\Gamma, M) = 0$ by quasi-triviality.

It follows that f_2 is also surjective in the Zariski topology.

For f_1, the canonical maps in (1.32) induce a Cartesian square:

$$\begin{array}{ccc} \underline{\Gamma}_e(\mathsf{BG}, \underline{K}_{[1,2]}) & \twoheadrightarrow & \underline{\Gamma}_e(\mathsf{BG}, \mathrm{B}\underline{K}_1) \\ \downarrow f_1 & & \downarrow \text{mod } 2 \\ \underline{\Gamma}_e(\mathsf{BG}, \underline{K}^{\text{super}}_{[1,2]}) & \to & \underline{\Gamma}_e(\mathsf{BG}, \mathrm{B}\underline{K}_1/2) \end{array} \tag{2.28}$$

Proposition 2.3.4 implies that the "mod 2" morphism is identified with the one appearing in (2.27). In particular, it is also surjective in the Zariski topology given the hypothesis on $\pi_1 G$. The same thus holds for f_1. □

2.4.4. Suppose that $\pi_1 G$ is the sheaf of cocharacters of a quasi-trivial torus. By Lemma 2.2.7 and Lemma 2.4.3, we may define a morphism fitting into (2.17):

$$\underline{\Gamma}_e(\mathsf{BG}, \underline{K}^{\text{super}}_{[1,2]}) \to \vartheta^{\text{super}}_{G}(\Lambda). \tag{2.29}$$

It is an equivalence by Proposition 2.1.8.

2.4.5. For general G, we first introduce an auxiliary sheaf of Picard groupoids $\widetilde{\vartheta}^{\text{super}}_{G}(\Lambda)$, defined to be the fiber product:

$$\begin{array}{ccc} \widetilde{\vartheta}^{\text{super}}_{G}(\Lambda) & \to & \mathrm{Quad}(\Lambda_{\text{sc}})^{W} \\ \downarrow & & \downarrow (2.2) \\ \vartheta^{\text{super}}(\Lambda) & \longrightarrow & \vartheta^{\text{super}}(\Lambda_{\text{sc}}) \end{array} \tag{2.30}$$

where the bottom horizontal map is defined by functoriality with respect to $\Lambda_{\text{sc}} \to \Lambda$. Concretely, a section of $\widetilde{\vartheta}^{\text{super}}_{G}(\Lambda)$ is a triple $(b, \widetilde{\Lambda}, \varphi)$ as in $\vartheta^{\text{super}}_{G}(\Lambda)$, but the Weyl-invariance on b is relaxed: it is only required to be Weyl-invariant over Λ_{sc}.

Restrictions along $T \subset G$, $G_{\text{sc}} \to G$ and applying the functor (2.29) to T, G_{sc}, and T_{sc} produces a functor:

$$\underline{\Gamma}_e(\mathsf{BG}, \underline{K}^{\text{super}}_{[1,2]}) \to \widetilde{\vartheta}^{\text{super}}_{G}(\Lambda). \tag{2.31}$$

2.4.6. Suppose that we have a central extension of reductive group S-schemes:

$$1 \to T_1 \to \widetilde{G} \to G \to 1, \tag{2.32}$$

where T_1 is a torus with sheaf of cocharacters Λ_1. Denote by $\widetilde{T}$ the preimage of T in $\widetilde{G}$. It is a maximal torus with sheaf of cocharacters $\widetilde{\Lambda}$.

By functoriality, we find two morphisms of presheaves of Picard groupoids:

$$\underline{\Gamma}_e(\mathsf{BG}, \underline{K}^{\text{super}}_{[1,2]}) \to \lim_{[n]} \underline{\Gamma}_e(\mathsf{B}(\widetilde{G} \times T_1^{\times n}), \underline{K}^{\text{super}}_{[1,2]}), \tag{2.33}$$

$$\vartheta^{\text{super}}_{G}(\Lambda) \to \lim_{[n]} \vartheta^{\text{super}}_{\widetilde{G} \times T_1^{\times n}}(\widetilde{\Lambda} \oplus \Lambda_1^{\oplus n}), \tag{2.34}$$

where the limits are taken over the simplicial category.

Lemma 2.4.7. *The following statements hold:*

(1) *the functor* (2.33) *is an equivalence;*
(2) *the functor* (2.34) *is fully faithful.*

Proof. Statement (1) follows from the fact that the induced map on Zariski classifying stacks $\mathsf{B}\widetilde{\mathrm{G}} \to \mathsf{B}\mathrm{G}$ is surjective in the Zariski topology.

To prove statement (2), we fit $\vartheta_{\mathrm{G}}^{\mathrm{super}}(\Lambda)$ into a fiber sequence of étale sheaves of Picard groupoids over S:

$$\underline{\mathrm{Hom}}_{\mathbb{Z}}(\Lambda, \mathsf{B}\mathbb{G}_m) \to \vartheta_{\mathrm{G}}^{\mathrm{super}}(\Lambda) \to \Gamma^2(\check{\Lambda})^{\mathrm{W}}. \tag{2.35}$$

Here, $\check{\Lambda}$ is the dual of Λ, so $\Gamma^2(\check{\Lambda})^{\mathrm{W}}$ is the abelian group of Weyl-invariant symmetric bilinear forms on Λ. The second map in (2.35) sends a triple $(b, \widetilde{\Lambda}, \varphi)$ to b, so its fiber is precisely the Picard groupoid of symmetric monoidal morphisms $\Lambda \to \mathsf{B}\mathbb{G}_m$.

The fully faithfulness will follow, if we know that the two outer terms in (2.35) satisfy descent along $\widetilde{\Lambda} \to \Lambda$. For $\underline{\mathrm{Hom}}_{\mathbb{Z}}(\Lambda, \mathsf{B}\mathbb{G}_m)$, this is because Λ is identified with $\mathrm{colim}_{[n]}(\widetilde{\Lambda} \oplus \Lambda_1^{\oplus n})$. For $\Gamma^2(\check{\Lambda})^{\mathrm{W}}$, this is the elementary observation that a symmetric bilinear form on $\widetilde{\Lambda}$ descends to Λ if its restrictions to $\widetilde{\Lambda} \oplus \Lambda_1$ along the action and projection maps coincide. □

Remark 2.4.8. In fact, the functor (2.34) is also an equivalence. This will be established in the course of the proof of Theorem 2.2.3 below.

Proof of Theorem 2.2.3. The case where $\pi_1\mathrm{G}$ is the sheaf of cocharacters of a quasi-trivial torus is already treated in §2.4.4.

For general G, it remains to prove that the functor (2.31) factors through an equivalence onto the full subgroupoid $\vartheta_{\mathrm{G}}^{\mathrm{super}}(\Lambda)$.

To do so, we choose a central extension (2.32) with the additional property that $\pi_1\widetilde{\mathrm{G}}$ is the sheaf of cocharacters of a quasi-trivial torus. Such central extensions exist, thanks to [GA13, Proposition 3.2].

Combining the equivalences for $\widetilde{\mathrm{G}} \times \mathrm{T}_1^{\times n}$ and Lemma 2.4.7, we obtain the following (solid) functors among Zariski sheaves of Picard groupoids:

$$\begin{array}{ccc} \underline{\Gamma}_e(\mathsf{B}\mathrm{G}, \underline{\mathrm{K}}_{[1,2]}^{\mathrm{super}}) & \xrightarrow{\simeq} & \lim_{[n]} \underline{\Gamma}_e(\mathsf{B}(\widetilde{\mathrm{G}} \times \mathrm{T}_1^{\times n}), \underline{\mathrm{K}}_{[1,2]}^{\mathrm{super}}) \\ \downarrow & & \downarrow \cong \\ \vartheta_{\mathrm{G}}^{\mathrm{super}}(\Lambda) & \subset & \lim_{[n]} \vartheta_{\widetilde{\mathrm{G}} \times \mathrm{T}_1^{\times n}}^{\mathrm{super}}(\widetilde{\Lambda} \oplus \Lambda_1^{\oplus n}) \end{array} \tag{2.36}$$

Note that a symmetric bilinear form on Λ is Weyl-invariant if and only if its restriction to $\widetilde{\Lambda}$ is. Hence, the functor (2.31) factors through the full subgroupoid $\vartheta_{\mathrm{G}}^{\mathrm{super}}(\Lambda)$, supplying the dashed arrow in (2.36). It follows that all functors in (2.36) are equivalences. □

Corollary 2.4.9. *Let* G *be a reductive group* S*-scheme. The Zariski sheaf of Picard groupoids* $\underline{\Gamma}_e(\mathsf{B}\mathrm{G}, \underline{\mathrm{K}}_{[1,2]}^{\mathrm{super}})$ *over* S *satisfies étale descent.*

Proof. Working Zariski locally over S, we may assume that G admits a maximal torus T [ABD+66, XIV, Corollaire 3.20]. Let Λ denote its sheaf of cocharacters.

Theorem 2.2.3 then implies that $\underline{\Gamma}_e(\mathsf{B}\mathrm{G}, \underline{\mathrm{K}}_{[1,2]}^{\mathrm{super}})$ is equivalent to $\vartheta_{\mathrm{G}}^{\mathrm{super}}(\Lambda)$, which clearly satisfies étale descent. □

2.4.10. We finish our study of $\underline{\Gamma}_e(\mathsf{B}\mathrm{G}, \underline{\mathrm{K}}_{[1,2]}^{\mathrm{super}})$ by computing its homotopy sheaves. Let us assume that $\mathrm{T} \subset \mathrm{G}$ is a fixed maximal torus with sheaf of cocharacters Λ.

From the Cartesian square (2.30), we obtain a long exact sequence of étale sheaves of abelian groups:

$$\begin{aligned} 1 \to \pi_1\widetilde{\vartheta}_{\mathrm{G}}^{\mathrm{super}}(\Lambda) \to \pi_1\vartheta^{\mathrm{super}}(\Lambda) \to \pi_1\vartheta^{\mathrm{super}}(\Lambda_{\mathrm{sc}}) \\ \to \pi_0\widetilde{\vartheta}_{\mathrm{G}}^{\mathrm{super}}(\Lambda) \to \pi_0\vartheta^{\mathrm{super}}(\Lambda) \oplus \mathrm{Quad}(\Lambda_{\mathrm{sc}})^{\mathrm{W}} \to \pi_0\vartheta^{\mathrm{super}}(\Lambda_{\mathrm{sc}}). \end{aligned}$$

The homotopy sheaves of $\vartheta^{\mathrm{super}}(\Lambda)$ are easily computed: $\pi_1\vartheta^{\mathrm{super}}(\Lambda)$ is isomorphic to $\underline{\mathrm{Hom}}(\Lambda,\mathbb{G}_m)$, and $\pi_0\vartheta^{\mathrm{super}}(\Lambda)$ is isomorphic to the sheaf of symmetric bilinear forms on Λ. Therefore, $\pi_1\widetilde{\vartheta}^{\mathrm{super}}_{\mathrm{G}}(\Lambda)$ is isomorphic to $\underline{\mathrm{Hom}}(\pi_1\mathrm{G},\mathbb{G}_m)$, and $\pi_0\widetilde{\vartheta}^{\mathrm{super}}_{\mathrm{G}}(\Lambda)$ is isomorphic to the sheaf of symmetric bilinear forms on Λ whose restriction to Λ_{sc} comes from a Weyl-invariant quadratic form.

By definition of the full subgroupoid $\vartheta^{\mathrm{super}}_{\mathrm{G}}(\Lambda)\subset\widetilde{\vartheta}^{\mathrm{super}}_{\mathrm{G}}(\Lambda)$, we see that $b\in\pi_0\widetilde{\vartheta}^{\mathrm{super}}_{\mathrm{G}}(\Lambda)$ belongs to $\pi_0\vartheta^{\mathrm{super}}_{\mathrm{G}}(\Lambda)$ if and only if it is Weyl-invariant.

Writing $\Gamma^2(\check{\Lambda})^{\mathrm{W}}_{\mathrm{sc}}$ for the abelian sheaf of Weyl-invariant symmetric bilinear forms on Λ whose restriction to Λ_{sc} comes from a quadratic form, we obtain:

$$\pi_i\underline{\Gamma}_e(\mathrm{BG},\underline{\mathrm{K}}^{\mathrm{super}}_{[1,2]})\cong\begin{cases}\Gamma^2(\check{\Lambda})^{\mathrm{W}}_{\mathrm{sc}} & i=0,\\ \underline{\mathrm{Hom}}(\pi_1\mathrm{G},\mathbb{G}_m) & i=1,\\ 0 & i\geq 2.\end{cases}$$

2.5. Examples.

2.5.1. Recall the canonical identification between $\underline{\mathrm{K}}_{[0,1]}$ and $\mathsf{Pic}^{\mathbb{Z}}$ (Remark 1.1.6). It induces an isomorphism $\mathsf{B}\underline{\mathrm{K}}_1\cong\mathsf{B}\mathbb{G}_m$, whose inverse can be viewed as a rigidified section of $\mathsf{B}\underline{\mathrm{K}}_1$ over $\mathsf{B}\mathbb{G}_m$ and we denote it by c_1. It represents the K-theoretic first Chern class.

Using the cup product and the pairing $\underline{\mathrm{K}}_1\otimes\underline{\mathrm{K}}_1\to\underline{\mathrm{K}}_2$, $x,y\mapsto\{x,y\}$, we obtain a rigidified section $c_1\cup c_1$ of $\mathsf{B}^2\underline{\mathrm{K}}_2$ over $\mathsf{B}\mathbb{G}_m$. Its value at an R-point $\mathscr{L}$ is given by the pairing $\{\mathscr{L},\mathscr{L}\}$ (*cf.* Remark 1.2.9). The central extension of $\mathbb{G}_m$ by $\underline{\mathrm{K}}_2$ corresponding to $c_1\cup c_1$ is given by $\underline{\mathrm{K}}_2\times\mathbb{G}_m$ with $x,y\mapsto\{x,y\}$ as cocycle. The quadratic form associated to $c_1\cup c_1$ (*cf.* Remark 2.1.4) takes value 1 at the identity cocharacter of $\mathbb{G}_m$.

2.5.2. *The Tate section.* We have a rigidified section of $\underline{\mathrm{K}}_{[1,2]}$ over $\mathsf{B}\mathbb{G}_m$, sending $\mathscr{L}$ to $[\mathscr{L}]-[\mathscr{O}]$. We shall write Tate for the induced rigidified section of $\underline{\mathrm{K}}^{\mathrm{super}}_{[1,2]}$.

Let us show that as a rigidified section of $\underline{\mathrm{K}}^{\mathrm{super}}_{[1,2]}$, $c_1\cup c_1$ is twice the Tate extension.

Lemma 2.5.3. *There is a canonical isomorphism of rigidified sections of* $\underline{\mathrm{K}}^{\mathrm{super}}_{[1,2]}$ *over* $\mathsf{B}\mathbb{G}_m$*:*

$$2\cdot\mathrm{Tate}\cong c_1\cup c_1.$$

Proof. We need to establish an isomorphism in $\underline{\mathrm{K}}^{\mathrm{super}}_{[1,2]}$:

$$2\cdot([\mathscr{L}]-[\mathscr{O}])\cong\{\mathscr{L},\mathscr{L}\}$$

natural in the line bundle $\mathscr{L}$. Note that $[\mathscr{L}]-[\mathscr{O}]$ is denoted by $s(\mathscr{L})$ in §1.2.2 and the proof of Lemma 1.3.5 yields isomorphisms in $\underline{\mathrm{K}}_{[1,2]}$:

$$\begin{aligned}[\mathscr{L}]-[\mathscr{L}^{-1}]&\cong s(\mathscr{L}^2)-2\cdot\{\mathscr{L},\mathscr{L}\}\\ &\cong 2\cdot s(\mathscr{L})-\{\mathscr{L},\mathscr{L}\}.\end{aligned}$$

By definition, the section $[\mathscr{L}]-[\mathscr{L}^{-1}]$ vanishes in $\underline{\mathrm{K}}_{[1,2]}$, which gives rise to the desired isomorphism $2\cdot s(\mathscr{L})\cong\{\mathscr{L},\mathscr{L}\}$. □

Remark 2.5.4. The rigidified section Tate : $\mathsf{B}\mathbb{G}_m\to\underline{\mathrm{K}}^{\mathrm{super}}_{[1,2]}$ has the following quadratic multiplicative structure:

$$\mathrm{Tate}(\mathscr{L}^n)\cong n^2\cdot\mathrm{Tate}(\mathscr{L}).$$

Indeed, inducting on n using the relation (1.21) yields an isomorphism between $\mathrm{Tate}(\mathscr{L}^n)$ and $n\cdot\mathrm{Tate}(\mathscr{L})+\binom{n}{2}\{\mathscr{L},\mathscr{L}\}$, but the latter is $n^2\cdot\mathrm{Tate}(\mathscr{L})$ by Lemma 2.5.3.

2.5.5. Consider the group scheme SL_2 equipped with the diagonal maximal torus $\mathbb{G}_m \subset \mathrm{SL}_2$. Write V for the universal rank-2 vector bundle over $\mathsf{B}\mathrm{SL}_2$. The section $[\mathrm{V}] - [\mathscr{O}^{\oplus 2}]$ of $\underline{\mathrm{K}}$ over $\mathsf{B}\mathrm{SL}_2$ factors through $\underline{\mathrm{K}}_{\geq 2}$, so it induces a section of $\mathsf{B}^2\underline{\mathrm{K}}_2$ over $\mathsf{B}\mathrm{SL}_2$.

The pullback of $[\mathrm{V}] - [\mathscr{O}^{\oplus 2}]$ to $\mathsf{B}\mathbb{G}_m$ is the rigidified section sending $\mathscr{L}$ to $[\mathscr{L} \oplus \mathscr{L}^{-1}] - [\mathscr{O}^{\oplus 2}]$. The proof of Lemma 1.3.5 yields isomorphisms:

$$\begin{aligned} [\mathscr{L} \oplus \mathscr{L}^{-1}] - [\mathscr{O}^{\oplus 2}] &\cong ([\mathscr{L}] - [\mathscr{O}]) + ([\mathscr{L}^{-1}] - [\mathscr{O}]) \\ &\cong \{\mathscr{L}, \mathscr{L}\}. \end{aligned} \tag{2.37}$$

In other words, the pullback of $[\mathrm{V}] - [\mathscr{O}^{\oplus 2}]$ to $\mathsf{B}\mathbb{G}_m$ is canonically isomorphic to $c_1 \cup c_1$. In particular, it is isomorphic to twice the Tate section (Lemma 2.5.3).

Remark 2.5.6. It follows from §2.5.5 that the rigidified section $[\mathrm{V}] - [\mathscr{O}^{\oplus 2}] : \mathsf{B}\mathrm{SL}_2 \to \mathsf{B}^2\underline{\mathrm{K}}_2$ is classified by the Weyl-invariant quadratic form whose value at a coroot is 1.

More generally, for any integer $n \geq 2$, we may consider the universal rank-n vector bundle V over $\mathsf{B}\mathrm{SL}_n$. The section $[\mathrm{V}] - [\mathscr{O}^{\oplus n}]$ of $\mathsf{B}^2\underline{\mathrm{K}}_2$ over $\mathsf{B}\mathrm{SL}_n$ is also classified by the Weyl-invariant quadratic form whose value at a coroot is 1, with respect to the diagonal maximal torus. This follows by choosing a subgroup $\mathrm{SL}_2 \subset \mathrm{SL}_n$ corresponding to a simple coroot and reducing to the case of SL_2.

2.5.7. *Pfaffian and all that.* Let G be a split reductive group with fixed split maximal torus and Borel subgroup $\mathrm{T} \subset \mathrm{B} \subset \mathrm{G}$ and a pinning. Let Ad denote the adjoint bundle over $\mathsf{B}\mathrm{G}$, *i.e.* the vector bundle associated to the adjoint representation $\mathfrak{g}$.

We shall construct a "half" of $[\mathrm{Ad}] - [\mathscr{O}^{\dim \mathfrak{g}}]$ as a rigidified section of $\underline{\mathrm{K}}^{\mathrm{super}}_{[1,2]}$.

Proposition 2.5.8. *In the context of §2.5.7, there is a canonical rigidified section* Pf *of* $\underline{\mathrm{K}}^{\mathrm{super}}_{[1,2]}$ *over* $\mathsf{B}\mathrm{G}$ *equipped with an isomorphism:*

$$2 \cdot \mathrm{Pf} \cong [\mathrm{Ad}] - [\mathscr{O}^{\dim \mathfrak{g}}]. \tag{2.38}$$

2.5.9. We shall use the classification theorem (Theorem 2.2.3) to construct Pf and the isomorphism (2.38).

Denote by Λ the cocharacter lattice of T and Λ_{sc} that of the induced maximal torus T_{sc} of the simply connected form G_{sc} of G. The choice of B endows Λ_{sc} with a basis consisting of simple coroots $\alpha \in \Delta$. The choice of a pinning induces a canonical extension of each α to a subgroup of G_{sc} isomorphic to SL_2:

$$\begin{array}{ccc} \mathbb{G}_m & \subset & \mathrm{SL}_2 \\ \downarrow{\scriptstyle \alpha} & & \downarrow \\ \mathrm{T}_{\mathrm{sc}} & \subset & \mathrm{G}_{\mathrm{sc}} \end{array} \tag{2.39}$$

2.5.10. In the presence of a pinning, Theorem 2.2.3 can be reformulated as classifying rigidified sections of $\underline{\mathrm{K}}^{\mathrm{super}}_{[1,2]}$ over $\mathsf{B}\mathrm{G}$ by pairs $(f, \{\varphi_\alpha\}_{\alpha \in \Delta})$, where:

(1) f is a rigidified section of $\underline{\mathrm{K}}^{\mathrm{super}}_{[1,2]}$ over $\mathsf{B}\mathrm{T}$, whose associated symmetric bilinear form b is Weyl-invariant and its restriction to Λ_{sc} comes from a quadratic form Q_{sc};
(2) for each $\alpha \in \Delta$, φ_α is an isomorphism between the restriction of f along $\alpha : \mathsf{B}\mathbb{G}_m \to \mathsf{B}\mathrm{T}$ and the $\mathrm{Q}_{\mathrm{sc}}(\alpha)$-multiple of the $c_1 \cup c_1$.

Indeed, f accounts for the data $(b, \widetilde{\Lambda})$ of §2.2.1. To see that φ of *loc.cit.* is equivalent to $\{\varphi_\alpha\}_{\alpha \in \Delta}$, we argue as follows: φ is an isomorphism of two central extensions of Λ_{sc} by $\mathbb{G}_m$ with equal commutators, so it is uniquely determined over the basis Δ. On the other hand, the rigidified section $\mathsf{B}\mathrm{G}_{\mathrm{sc}} \to \mathsf{B}^2\underline{\mathrm{K}}_2$ classified by Q_{sc} restricts to the $\mathrm{Q}_{\mathrm{sc}}(\alpha)$-multiple of $c_1 \cup c_1$ along $\alpha : \mathsf{B}\mathbb{G}_m \to \mathsf{B}\mathrm{T}_{\mathrm{sc}} \to \mathsf{B}\mathrm{G}_{\mathrm{sc}}$, in view of §2.5.5 and (2.39).

2.5.11. Let us turn to the construction of Pf.

Proof of Proposition 2.5.8. The symmetric bilinear form associated to the rigidified section $[\mathrm{Ad}] - [\mathscr{O}^{\dim \mathfrak{g}}] : \mathsf{BG} \to \mathsf{B}^2\underline{\mathrm{K}}_2$ is the Killing form (*cf.* Remark 2.5.6):

$$\lambda_1, \lambda_2 \mapsto \sum_{\check{\beta} \in \Phi} \langle \check{\beta}, \lambda_1 \rangle \langle \check{\beta}, \lambda_2 \rangle,$$

where Φ is the set of roots of G. The choice of B expresses Φ as the union of positive roots Φ_+ and negative roots Φ_-. In particular, the Killing form is twice the symmetric bilinear form b defined by:

$$b(\lambda_1, \lambda_2) := \sum_{\check{\beta} \in \Phi_+} \langle \check{\beta}, \lambda_1 \rangle \langle \check{\beta}, \lambda_2 \rangle.$$

In particular, we have $b(\lambda, \lambda) = \langle 2\check{\rho}, \lambda \rangle \mod 2$, for $2\check{\rho} := \sum_{\check{\beta} \in \Phi_+} \check{\beta}$, so $b(\alpha, \alpha)$ is even for all $\alpha \in \Delta$. This means that the restriction of b to Λ_{sc} comes from a quadratic form Q_{sc}.

Next, the restriction of $[\mathrm{Ad}] - [\mathscr{O}^{\dim \mathfrak{g}}]$ to BT is the section:

$$\sum_{\check{\beta} \in \Phi_+} ([\mathscr{L}^{\check{\beta}}] - [\mathscr{O}]) + \sum_{\check{\beta} \in \Phi_+} ([\mathscr{L}^{-\check{\beta}}] - [\mathscr{O}]) \cong 2 \cdot \sum_{\check{\beta} \in \Phi_+} ([\mathscr{L}^{\check{\beta}}] - [\mathscr{O}]),$$

where $\mathscr{L}^{\check{\beta}}$ denotes the line bundle over BT defined by the root $\check{\beta}$, and we applied the isomorphism $[\mathscr{L}^{\check{\beta}}] \cong [\mathscr{L}^{-\check{\beta}}]$ in $\underline{\mathrm{K}}^{\mathrm{super}}_{[1,2]}$. In particular, the restriction of $[\mathrm{Ad}] - [\mathscr{O}^{\dim \mathfrak{g}}]$ to BT is twice the rigidified section:

$$\sum_{\check{\beta} \in \Phi_+} ([\mathscr{L}^{\check{\beta}}] - [\mathscr{O}]). \tag{2.40}$$

Note that the symmetric bilinear form associated to (2.40) is b. We shall argue that the restriction of (2.40) along each $\alpha : \mathsf{B}\mathbb{G}_m \to \mathsf{BT}$ is the $\mathrm{Q}_{\mathrm{sc}}(\alpha)$-multiple of $c_1 \cup c_1$. This would furnish the construction of Pf in view of §2.5.10.

Indeed, the restriction of (2.40) along α is the rigidified section:

$$\begin{aligned} \sum_{\check{\beta} \in \Phi_+} ([\mathscr{L}^{\langle \check{\beta}, \alpha \rangle}] - [\mathscr{O}]) &\cong \sum_{\check{\beta} \in \Phi_+} \langle \check{\beta}, \alpha \rangle^2 \cdot \mathrm{Tate} \\ &\cong 2 \cdot \mathrm{Q}_{\mathrm{sc}}(\alpha) \cdot \mathrm{Tate} \cong \mathrm{Q}_{\mathrm{sc}}(\alpha) \cdot (c_1 \cup c_1) \end{aligned}$$

where the first isomorphism follows from the quadratic structure of the Tate section (Remark 2.5.4) and the last isomorphism follows from Lemma 2.5.3.

The isomorphism $2 \cdot \mathrm{Pf} \cong [\mathrm{Ad}] - (\mathscr{O}^{\dim \mathfrak{g}})$ results directly from the construction of Pf, so we omit the details. ☐

Remark 2.5.12. The image of Pf in $\underline{\mathrm{Hom}}(\pi_1 \mathrm{G}, \mathbb{Z}/2)$ under (2.10) is the character:

$$\lambda \mapsto \langle 2\check{\rho}, \lambda \rangle \mod 2. \tag{2.41}$$

Thus, Pf comes from a rigidified section of $\mathsf{B}^2\underline{\mathrm{K}}_2$ if and only if $\check{\rho}$ is integral.

The same fact also shows that Pf generally does *not* come from a rigidified section of $\underline{\mathrm{K}}_{[1,2]}$, *i.e.* it is genuinely "half-integral". Indeed, if it did, then (2.41) would have to lift to character $\pi_1 \mathrm{G} \to \mathbb{Z}$, but this is generally not the case.

2.5.13. Let us combine Proposition 2.5.8 and the integration functor of §1.4.9 to construct the Pfaffian line bundle on the moduli stack of G-bundles over a spin curve.

More precisely, let $p : \mathrm{X}_{\mathrm{S}} \to \mathrm{S}$ be a smooth, proper morphism of relative dimension 1 with connected geometric fibers together with a square root $\omega^{1/2}$ of the relative canonical bundle.

Denote by Bun_G the moduli stack of G-bundles over X_S. The rigidified relative canonical bundle of $\mathrm{Bun}_G \to S$ is the line bundle:

$$\mathscr{L}_{\det} := \det(Rp_*\mathfrak{g}_P) \otimes (\det(Rp_*\mathfrak{g}_{P^0}))^{-1}, \tag{2.42}$$

where $p : X_S \times_S \mathrm{Bun}_G \to \mathrm{Bun}_G$ is the projection and $\mathfrak{g}_P$ (resp. $\mathfrak{g}_{P_0}$) is the adjoint bundle of the universal G-bundle P (resp. trivial G-bundle P^0).

2.5.14. The commutative diagram (1.42) yields the following commutative diagram via the construction of §1.4.9:

$$\begin{array}{ccc} \Gamma(\mathsf{BG}, \mathsf{B}^2\underline{\mathrm{K}}_2) & \longrightarrow & \Gamma(\mathsf{BG}, \underline{\mathrm{K}}^{\mathrm{super}}_{[1,2]}) \\ \big\downarrow {\scriptstyle \int_{X_S}} & & \big\downarrow {\scriptstyle \int_{(X_S,\omega^{1/2})}} \\ \Gamma(\mathrm{Bun}_G, \mathsf{Pic}) & \to & \Gamma(\mathrm{Bun}_G, \mathsf{Pic}^{\mathrm{super}}) \end{array}$$

Here, the horizontal morphisms are the tautological inclusions.

Note that $\mathscr{L}_{\det}$ is the image of $[\mathrm{Ad}] - [\mathscr{O}^{\dim \mathfrak{g}}]$ under the left vertical functor. By Proposition 2.5.8, the image of $[\mathrm{Ad}] - [\mathscr{O}^{\dim \mathfrak{g}}]$ in $\Gamma(\mathsf{BG}, \underline{\mathrm{K}}^{\mathrm{super}}_{[1,2]})$ is twice the Pfaffian section Pf. In particular, $\mathscr{L}_{\det}$ admits a square root as a *super* line bundle, given by:

$$\mathscr{L}_{\mathrm{Pf}} := \int_{(X_S,\omega^{1/2})} \mathrm{Pf}.$$

Remark 2.5.15. A square root of $\mathscr{L}_{\det}$ has been constructed in [BD04, §4] using a different method. One feature of our construction is that it yields a purely group-theoretic object Pf, while the spin curve $(X_S, \omega^{1/2})$ only appears in the integration functor.

Part 2. Loop groups

3. Statements

The goal of this section is to state the classification of factorization super central extensions of $\mathscr{L}G$: Theorem 3.4.5. The first two subsections §3.1, §3.2 review the notions of factorization structure and loop groups. In §3.3, we use the Contou-Carrère symbol to define the notion of "tame commutator" and study its basic properties. In §3.4, we state the classification theorem of factorization super central extensions of $\mathscr{L}G$ and briefly indicate the structure of its proof.

We work over a ground field k. Let X be a smooth curve over k.

3.1. Factorization.

3.1.1. Denote by Ran the presheaf whose S-points are nonempty finite subsets of $\mathrm{Maps}(S, X)$. We shall write an S-point of Ran as $x^I = (x^i)_{i\in I}$, where I is a nonempty finite set.

Given an S-point x^I of Ran, we denote by Γ_{x^I} the sum of the graphs $\Gamma_{x^i} \subset S \times X$ over $i \in I$ as effective Cartier divisors. Let D_{x^I} be the completion of $S \times X$ along Γ_{x^I} and $\overset{\circ}{D}_{x^I}$ be its open subscheme $D_{x^I} \backslash \Gamma_{x^I}$.

Two S-points x^I, x^J of Ran are called *disjoint* if $\Gamma_{x^I} \cap \Gamma_{x^J} = \varnothing$. Denote by $x^I \sqcup x^J$ the S-point of Ran given by their union.

Remark 3.1.2. For each nonempty finite set I, there is a tautological map $X^I \to \mathrm{Ran}$, sending an S-point x^I of X^I to the associated finite subset of $\mathrm{Maps}(S, X)$. The presheaf Ran is identified with the colimit of presheaves:

$$\operatorname*{colim}_{I}(X^I) \xrightarrow{\simeq} \mathrm{Ran},$$

indexed by the category of nonempty finite sets with surjections.

3.1.3. Let $\mathscr{Y}$ be a presheaf over Ran. Given an S-point x^{I} of Ran, we write $\mathscr{Y}_{x^{\mathrm{I}}}$ for the base change of $\mathscr{Y}$ along x^{I}.

The presheaf $\mathscr{Y}$ is called *factorization* when we are supplied with a functorial system (in S) of isomorphisms for all disjoint pairs of S-points $(x^{\mathrm{I}}, x^{\mathrm{J}})$ of Ran:

$$\varphi_{x^{\mathrm{I}},x^{\mathrm{J}}} : \mathscr{Y}_{x^{\mathrm{I}} \sqcup x^{\mathrm{J}}} \xrightarrow{\simeq} \mathscr{Y}_{x^{\mathrm{I}}} \times_{\mathrm{S}} \mathscr{Y}_{x^{\mathrm{J}}}, \tag{3.1}$$

satisfying the analogues of commutativity and associativity conditions. Namely, the following diagram commutes:

$$\begin{array}{ccc} \mathscr{Y}_{x^{\mathrm{I}} \sqcup x^{\mathrm{J}}} & \xrightarrow{\varphi_{x^{\mathrm{I}},x^{\mathrm{J}}}} & \mathscr{Y}_{x^{\mathrm{I}}} \times_{\mathrm{S}} \mathscr{Y}_{x^{\mathrm{J}}} \\ \downarrow{\cong} & & \downarrow{\cong} \\ \mathscr{Y}_{x^{\mathrm{J}} \sqcup x^{\mathrm{I}}} & \xrightarrow{\varphi_{x^{\mathrm{J}},x^{\mathrm{I}}}} & \mathscr{Y}_{x^{\mathrm{J}}} \times_{\mathrm{S}} \mathscr{Y}_{x^{\mathrm{I}}} \end{array} \tag{3.2}$$

where the left vertical arrow comes from the equality $x^{\mathrm{I}} \sqcup x^{\mathrm{J}} = x^{\mathrm{J}} \sqcup x^{\mathrm{I}}$ as S-points of Ran and the right vertical arrow is the map swapping the two factors; the following diagram commutes for pairwise disjoint S-points $(x^{\mathrm{I}}, x^{\mathrm{J}}, x^{\mathrm{K}})$ of Ran:

$$\begin{array}{ccccc} & & \mathscr{Y}_{x^{\mathrm{I}} \sqcup x^{\mathrm{J}} \sqcup x^{\mathrm{K}}} & & \\ & \swarrow^{\varphi_{x^{\mathrm{I}},x^{\mathrm{J}} \sqcup x^{\mathrm{K}}}} & & \searrow^{\varphi_{x^{\mathrm{I}} \sqcup x^{\mathrm{J}},x^{\mathrm{K}}}} & \\ \mathscr{Y}_{x^{\mathrm{I}}} \times_{\mathrm{S}} \mathscr{Y}_{x^{\mathrm{J}} \sqcup x^{\mathrm{K}}} & & & & \mathscr{Y}_{x^{\mathrm{I}} \sqcup x^{\mathrm{J}}} \times_{\mathrm{S}} \mathscr{Y}_{x^{\mathrm{K}}} \\ & \searrow_{\mathrm{id} \times \varphi_{x^{\mathrm{J}},x^{\mathrm{K}}}} & & \swarrow_{\varphi_{x^{\mathrm{I}},x^{\mathrm{J}}} \times \mathrm{id}} & \\ & & \mathscr{Y}_{x^{\mathrm{I}}} \times_{\mathrm{S}} \mathscr{Y}_{x^{\mathrm{J}}} \times_{\mathrm{S}} \mathscr{Y}_{x^{\mathrm{K}}} & & \end{array} \tag{3.3}$$

3.1.4. Let $\mathscr{Y}$ be a factorization presheaf such that $\mathscr{Y}_{x^{\mathrm{I}}}$ satisfies fppf descent for each S-point x^{I} of Ran. (We do not impose fppf descent on $\mathscr{Y}$ because Ran itself does not satisfy étale descent, see [GL19, Warning 2.4.4].)

A *factorization super line bundle* over $\mathscr{Y}$ is a super line bundle $\mathscr{L}$ over $\mathscr{Y}$ equipped with functorial isomorphisms for all disjoint pairs of S-points $(x^{\mathrm{I}}, x^{\mathrm{J}})$ of Ran with respect to (3.1):

$$(\varphi_{x^{\mathrm{I}},x^{\mathrm{J}}})^*(\mathscr{L}_{x^{\mathrm{I}}} \boxtimes \mathscr{L}_{x^{\mathrm{J}}}) \xrightarrow{\simeq} \mathscr{L}_{x^{\mathrm{I}} \sqcup x^{\mathrm{J}}}, \tag{3.4}$$

which are compatible with (3.2) and (3.3).

Let us spell out the compatibility with (3.2). Denote by $\mathrm{exch} : \mathscr{Y}_{x^{\mathrm{I}}} \times_{\mathrm{S}} \mathscr{Y}_{x^{\mathrm{J}}} \to \mathscr{Y}_{x^{\mathrm{J}}} \times_{\mathrm{S}} \mathscr{Y}_{x^{\mathrm{I}}}$ the map which exchanges the coordinates. The commutativity constraint of the Picard groupoid of super line bundles yields an isomorphism:

$$\mathrm{exch}^*(\mathscr{L}_{x^{\mathrm{J}}} \boxtimes \mathscr{L}_{x^{\mathrm{I}}}) \xrightarrow{\simeq} \mathscr{L}_{x^{\mathrm{I}}} \boxtimes \mathscr{L}_{x^{\mathrm{J}}}. \tag{3.5}$$

The compatbility states that the image of (3.5) under $(\varphi_{x^{\mathrm{I}},x^{\mathrm{J}}})^*$, viewed as an isomorphism $(\varphi_{x^{\mathrm{J}},x^{\mathrm{I}}})^*(\mathscr{L}_{x^{\mathrm{J}}} \boxtimes \mathscr{L}_{x^{\mathrm{I}}}) \xrightarrow{\simeq} (\varphi_{x^{\mathrm{I}},x^{\mathrm{J}}})^*(\mathscr{L}_{x^{\mathrm{I}}} \boxtimes \mathscr{L}_{x^{\mathrm{J}}})$ by the commutativity of (3.2), intertwines the isomorphisms (3.4) attached to $(x^{\mathrm{I}}, x^{\mathrm{J}})$, respectively $(x^{\mathrm{J}}, x^{\mathrm{I}})$.

3.1.5. Let $\mathscr{H}$ be a group factorization presheaf such that $\mathscr{H}_{x^{\mathrm{I}}}$ satisfies fppf descent for each S-point x^{I} of Ran.

A multiplicative super line bundle $\mathscr{L}$ over $\mathscr{H}$ is called *factorization* if it is equipped with a factorization structure which commutes with the multiplicative structure, *i.e.* (3.4) is an isomorphism of multiplicative line bundles over $\mathscr{H}_{x^{\mathrm{I}} \sqcup x^{\mathrm{J}}} \cong \mathscr{H}_{x^{\mathrm{I}}} \times_{\mathrm{S}} \mathscr{H}_{x^{\mathrm{J}}}$.

Note that a multiplicative factorization super line bundle over $\mathscr{H}$ is equivalent to a super central extension of group presheaves over Ran:

$$1 \to \mathbb{G}_{m,\mathrm{Ran}} \to \widetilde{\mathscr{H}} \to \mathscr{H} \to 1, \tag{3.6}$$

equipped with a functorial homomorphism $\widetilde{\varphi}_{x^{\mathrm{I}},x^{\mathrm{J}}}$ lifting the factorization isomorphism $\varphi_{x^{\mathrm{I}},x^{\mathrm{J}}}$ of $\mathscr{H}$ for each disjoint pair of S-points $(x^{\mathrm{I}},x^{\mathrm{J}})$ of Ran:

$$\begin{array}{ccccccccc} 1 & \to & \mathbb{G}_{m,\mathrm{S}}\times_{\mathrm{S}}\mathbb{G}_{m,\mathrm{S}} & \to & \widetilde{\mathscr{H}}_{x^{\mathrm{I}}}\times_{\mathrm{S}}\widetilde{\mathscr{H}}_{x^{\mathrm{J}}} & \to & \mathscr{H}_{x^{\mathrm{I}}}\times_{\mathrm{S}}\mathscr{H}_{x^{\mathrm{J}}} & \to & 1 \\ & & \downarrow{\scriptstyle (a,b)\mapsto ab} & & \downarrow{\scriptstyle \widetilde{\varphi}_{x^{\mathrm{I}},x^{\mathrm{J}}}} & & \downarrow{\scriptstyle \varphi_{x^{\mathrm{I}},x^{\mathrm{J}}}} & & \\ 1 & \longrightarrow & \mathbb{G}_{m,\mathrm{S}} & \longrightarrow & \widetilde{\mathscr{H}}_{x^{\mathrm{I}}\sqcup x^{\mathrm{J}}} & \longrightarrow & \mathscr{H}_{x^{\mathrm{I}}\sqcup x^{\mathrm{J}}} & \longrightarrow & 1 \end{array} \tag{3.7}$$

which satisfies commutativity and associativity. The data (3.6), (3.7) subject to these conditions are called a *factorization super central extension* of $\mathscr{H}$ by $\mathbb{G}_{m,\mathrm{Ran}}$. They form a Picard groupoid to be denoted by:

$$\mathrm{Hom}_{\mathrm{fact}}(\mathscr{H},\mathsf{Pic}^{\mathrm{super}}).$$

(We interpret them as homomorphisms $\mathscr{H}\to\mathsf{Pic}^{\mathrm{super}}$ compatible with factorization.)

Let us again be explicit about commutativity: (3.6) being a super central extension, each S-point $(x^{\mathrm{I}},h^{\mathrm{I}})$ of $\widetilde{\mathscr{H}}$ carries a grading, viewed as a locally constant section of $\underline{\mathbb{Z}}/2$ over S. Commutativity refers to the equality:

$$\widetilde{\varphi}_{x^{\mathrm{I}},x^{\mathrm{J}}}(h^{\mathrm{I}},h^{\mathrm{J}})=(-1)^{\epsilon^{\mathrm{I}}\epsilon^{\mathrm{J}}}\widetilde{\varphi}_{x^{\mathrm{J}},x^{\mathrm{I}}}(h^{\mathrm{J}},h^{\mathrm{I}}),$$

whenever h^{I} (resp. h^{J}) has grading ϵ^{I} (resp. ϵ^{J}).

3.2. Loop groups.

3.2.1. Let $\mathrm{Y}\to\mathrm{X}$ be an affine morphism of finite type.

Denote by $\mathscr{L}\mathrm{Y}$ (resp. $\mathscr{L}^+\mathrm{Y}$) the presheaf whose S-points are pairs $(x^{\mathrm{I}},y^{\mathrm{I}})$ where x^{I} is an S-point of Ran and y^{I} is an X-morphism $\mathring{\mathrm{D}}_{x^{\mathrm{I}}}\to\mathrm{Y}$ (resp. $\mathrm{D}_{x^{\mathrm{I}}}\to\mathrm{Y}$). Note that $\mathscr{L}^+\mathrm{Y}$ is a closed subpresheaf of $\mathscr{L}\mathrm{Y}$ and the structural morphism $\mathscr{L}\mathrm{Y}\to\mathrm{Ran}$ (resp. $\mathscr{L}^+\mathrm{Y}\to\mathrm{Ran}$) is indschematic (resp. schematic), see [KV04, 2.4-2.5].

Furthermore, $\mathscr{L}\mathrm{Y}$ admits a canonical factorization structure. Indeed, for any disjoint pair of S-points $(x^{\mathrm{I}},x^{\mathrm{J}})$ of Ran, there is a functorial isomorphism:

$$\mathscr{L}_{x^{\mathrm{I}}\sqcup x^{\mathrm{J}}}\mathrm{Y}\xrightarrow{\simeq}\mathscr{L}_{x^{\mathrm{I}}}\mathrm{Y}\times_{\mathrm{S}}\mathscr{L}_{x^{\mathrm{J}}}\mathrm{Y},$$

induced from $\mathring{\mathrm{D}}_{x^{\mathrm{I}}\sqcup x^{\mathrm{J}}}\cong\mathring{\mathrm{D}}_{x^{\mathrm{I}}}\sqcup\mathring{\mathrm{D}}_{x^{\mathrm{J}}}$, which is clearly commutative and associative. Analogously, $\mathscr{L}^+\mathrm{Y}$ also admits a canonical factorization structure.

Since the association $\mathrm{Y}\mapsto\mathscr{L}\mathrm{Y}$ (resp. $\mathscr{L}^+\mathrm{Y}$) preserves limits, it carries an affine group X-scheme G of finite type to a factorization group presheaf $\mathscr{L}\mathrm{G}$ (resp. $\mathscr{L}^+\mathrm{G}$) over Ran.

3.2.2. Let G be a smooth group X-scheme with connected geometric fibers. We also introduce the *affine Grassmannian* Gr_{G} as the presheaf whose S-points are triples $(x^{\mathrm{I}},\mathrm{P},\alpha)$, where x^{I} is an S-point of X, P is a G-torsor over $\mathrm{S}\times\mathrm{X}$, and α is a trivialization of P over $\mathrm{S}\times\mathrm{X}\backslash\Gamma_{x^{\mathrm{I}}}$. Then $\mathrm{Gr}_{\mathrm{G}}\to\mathrm{Ran}$ is ind-schematic of ind-finite type; it is ind-proper when G is reductive [Zhu17, Theorem 3.1.3].

The factorization structure on Gr_{G} is defined by Beauville–Laszlo gluing [Zhu17, Theorem 3.2.1] and the canonical map $\mathscr{L}\mathrm{G}\to\mathrm{Gr}_{\mathrm{G}}$ realizes the latter as the quotient $\mathscr{L}\mathrm{G}/\mathscr{L}^+\mathrm{G}$ in the étale topology [Zhu17, Proposition 3.1.9].

3.2.3. For later purposes, we give a convenient description of $\mathscr{L}^+\mathrm{G}\to\mathrm{Ran}$ as an inverse limit of smooth affine group schemes relative to Ran.

Consider an S-point x^{I} of Ran. The morphism $\Gamma_{x^{\mathrm{I}}}\to\mathrm{S}$ is finite locally free. Denote by $\mathrm{R}_{\Gamma}\mathrm{G}$ the Weil restriction along $\Gamma_{x^{\mathrm{I}}}\to\mathrm{S}$ of G (pulled back along $\Gamma_{x^{\mathrm{I}}}\subset\mathrm{S}\times\mathrm{X}\twoheadrightarrow\mathrm{X}$.) Then

$\mathrm{R}_\Gamma \mathrm{G}$ is representable by a smooth affine group S-scheme [BLR90, §7.6]. The evaluation map defines a short exact sequence:

$$1 \to \mathscr{L}^{\geq 1}_{x^{\mathrm{I}}}\mathrm{G} \to \mathscr{L}^{+}_{x^{\mathrm{I}}}\mathrm{G} \to \mathrm{R}_\Gamma \mathrm{G} \to 1. \tag{3.8}$$

More generally, we let $\Gamma^{(n)}_{x^{\mathrm{I}}}$ (for $n \geq 0$) denote the nth order infinitesimal neighborhood of the closed immersion $\Gamma_{x^{\mathrm{I}}} \subset \mathrm{S}\times\mathrm{X}$. Then $\Gamma^{(n)}_{x^{\mathrm{I}}} \to \mathrm{S}$ is finite locally free: writing $\mathscr{I}$ for the ideal sheaf defining $\Gamma_{x^{\mathrm{I}}}$, we see that each $\mathscr{I}^n/\mathscr{I}^{n+1}$ is locally isomorphic to $\mathscr{O}_{\mathrm{S}\times\mathrm{X}}/\mathscr{I}$ as an $\mathscr{O}_{\mathrm{S}}$-module. Let $\mathrm{R}_{\Gamma^{(n)}}\mathrm{G}$ be the Weil restrction along $\Gamma^{(n)}_{x^{\mathrm{I}}} \to \mathrm{S}$, which is again representable by a smooth affine group S-scheme. This gives us a limit presentation:

$$\mathscr{L}^{+}_{x^{\mathrm{I}}}\mathrm{G} \xrightarrow{\simeq} \lim_n \mathrm{R}_{\Gamma^{(n)}}\mathrm{G}.$$

Under the Tannakian formalism, the formula $\xi \mapsto 1+\xi$ defines an isormorphism between the vector group S-scheme $\mathfrak{g}\otimes(\mathscr{I}^{n+1}/\mathscr{I}^{n+2})$ and the kernel of the evaluation map $\mathrm{R}_{\Gamma^{(n+1)}}\mathrm{G} \to \mathrm{R}_{\Gamma^{(n)}}\mathrm{G}$. In particular, the group scheme $\mathscr{L}^{\geq 1}_{x^{\mathrm{I}}}\mathrm{G}$ in (3.8) is an (infinite) iterated extension of vector group S-schemes.

3.3. Contou-Carrère.

3.3.1. For each integer $n \geq 1$, we shall define *Tate central extension* as a factorization super central extension:

$$1 \to \mathbb{G}_{m,\mathrm{Ran}} \to \widetilde{\mathrm{GL}}_n \to \mathscr{L}\mathrm{GL}_n \to 1. \tag{3.9}$$

Viewing $\widetilde{\mathrm{GL}}_n$ as a super line bundle over $\mathscr{L}\mathrm{GL}_n$, its fiber at an S-point $(x^{\mathrm{I}}, a^{\mathrm{I}})$ of $\mathscr{L}\mathrm{GL}_n$ is the super $\mathscr{O}_{\mathrm{S}}$-module:

$$\det(a^{\mathrm{I}}\mathscr{O}^{\oplus n}_{\mathrm{D}_{x^{\mathrm{I}}}} \mid \mathscr{O}^{\oplus n}_{\mathrm{D}_{x^{\mathrm{I}}}}) \text{ with grading } \mathrm{rank}(a^{\mathrm{I}}\mathscr{O}^{\oplus n}_{\mathrm{D}_{x^{\mathrm{I}}}} \mid \mathscr{O}^{\oplus n}_{\mathrm{D}_{x^{\mathrm{I}}}}) \bmod 2,$$

where $\det(\mathrm{L}_1 \mid \mathrm{L}_2)$ denotes the relative determinant of two lattices L_1, L_2 in the Tate $\mathscr{O}_{\mathrm{S}}$-module $\mathscr{O}^{\oplus n}_{\mathring{\mathrm{D}}_{x^{\mathrm{I}}}}$ and $\mathrm{rank}(\mathrm{L}_1 \mid \mathrm{L}_2)$ denotes their relative rank. (See [Dri06] or [CH21, §3] for the definition of these notions).

The multiplicative structure of (3.9) is defined by the canonical isomorphism:

$$\det(a^{\mathrm{I}}b^{\mathrm{I}}\mathscr{O}^{\oplus n}_{\mathrm{D}_{x^{\mathrm{I}}}} \mid \mathscr{O}^{\oplus n}_{\mathrm{D}_{x^{\mathrm{I}}}}) \cong \det(a^{\mathrm{I}}\mathscr{O}^{\oplus n}_{\mathrm{D}_{x^{\mathrm{I}}}} \mid \mathscr{O}^{\oplus n}_{\mathrm{D}_{x^{\mathrm{I}}}}) \otimes \det(b^{\mathrm{I}}\mathscr{O}^{\oplus n}_{\mathrm{D}_{x^{\mathrm{I}}}} \mid \mathscr{O}^{\oplus n}_{\mathrm{D}_{x^{\mathrm{I}}}}),$$

for any S-points $(x^{\mathrm{I}}, a^{\mathrm{I}})$ and $(x^{\mathrm{I}}, b^{\mathrm{I}})$ of $\mathscr{L}\mathrm{GL}_n$. The factorization isomorphism arises from the $\mathbb{Z}/2$-graded multiplicativity of determinants with respect to direct sums:

$$\det(a^{\mathrm{I}}\mathscr{O}^{\oplus n}_{\mathrm{D}_{x^{\mathrm{I}}}} \oplus b^{\mathrm{J}}\mathscr{O}^{\oplus n}_{\mathrm{D}_{x^{\mathrm{J}}}} \mid \mathscr{O}^{\oplus n}_{\mathrm{D}_{x^{\mathrm{I}}}} \oplus \mathscr{O}^{\oplus n}_{\mathrm{D}_{x^{\mathrm{J}}}}) \cong \det(a^{\mathrm{I}}\mathscr{O}^{\oplus n}_{\mathrm{D}_{x^{\mathrm{I}}}} \mid \mathscr{O}^{\oplus n}_{\mathrm{D}_{x^{\mathrm{I}}}}) \otimes \det(b^{\mathrm{J}}\mathscr{O}^{\oplus n}_{\mathrm{D}_{x^{\mathrm{J}}}} \mid \mathscr{O}^{\oplus n}_{\mathrm{D}_{x^{\mathrm{J}}}}),$$

for S-points $(x^{\mathrm{I}}, a^{\mathrm{I}})$, $(x^{\mathrm{J}}, b^{\mathrm{J}})$ of $\mathscr{L}\mathrm{GL}_n$ with x^{I}, x^{J} disjoint.

3.3.2. Following [CH21, §4], we define the *Contou-Carrère symbol* (or *tame symbol*) to be the commutator pairing of (3.9) for $n = 1$:

$$\langle\cdot,\cdot\rangle : \mathscr{L}\mathbb{G}_m \otimes \mathscr{L}\mathbb{G}_m \to \mathbb{G}_{m,\mathrm{Ran}}, \tag{3.10}$$

Namely, $\langle\cdot,\cdot\rangle$ carries S-points $(x^{\mathrm{I}}, a^{\mathrm{I}})$, $(x^{\mathrm{I}}, b^{\mathrm{I}})$ of $\mathscr{L}\mathbb{G}_m$ to the element $(x^{\mathrm{I}}, \tilde{a}^{\mathrm{I}}\tilde{b}^{\mathrm{I}}(\tilde{a}^{\mathrm{I}})^{-1}(\tilde{b}^{\mathrm{I}})^{-1})$ of $\mathbb{G}_{m,\mathrm{Ran}}$, where $\tilde{a}^{\mathrm{I}}$ (resp. $\tilde{b}^{\mathrm{I}}$) is a lift of a^{I} (resp. b^{I}) to $\widetilde{\mathbb{G}}_m$ which exists locally on S.

The pairing (3.10) is *factorization* in the following sense: given disjoint S-points x^{I}, x^{J} of Ran and lifts a^{I}, b^{I} (resp. a^{J}, b^{J}) of x^{I} (resp. x^{J}) to $\mathscr{L}\mathbb{G}_m$, there holds:

$$\langle a^{\mathrm{I}} \sqcup a^{\mathrm{J}}, b^{\mathrm{I}} \sqcup b^{\mathrm{J}}\rangle = \langle a^{\mathrm{I}}, b^{\mathrm{I}}\rangle\langle a^{\mathrm{J}}, b^{\mathrm{J}}\rangle.$$

Furthermore, (3.10) is *perfect* in the sense that its adjoint:

$$\mathscr{L}\mathbb{G}_m \to \underline{\mathrm{Hom}}(\mathscr{L}\mathbb{G}_m, \mathbb{G}_{m,\mathrm{Ran}}) \tag{3.11}$$

is an isomorphism of factorization group presheaves [CH21, Corollary 5.4.1.1]. This pairing exhibits $\mathscr{L}^+\mathbb{G}_m$ as the Cartier dual of $\mathrm{Gr}_{\mathbb{G}_m}$ [CH21, Theorem 5.2.1].

3.3.3. More generally, let T be an X-torus with dual X-torus $\check{\mathrm{T}}$, (3.11) induces an isomorphism between $\mathscr{L}\check{\mathrm{T}}$ and $\underline{\mathrm{Hom}}(\mathscr{L}\mathrm{T}, \mathbb{G}_{m,\mathrm{Ran}})$.

In particular, for a pair of X-tori T_1, T_2 with sheaves of cocharacters Λ_1, Λ_2, any bilinear form $b : \Lambda_1 \otimes \Lambda_2 \to \mathbb{Z}$ defines a factorization pairing:

$$\langle \cdot, \cdot \rangle_b : \mathscr{L}\mathrm{T}_1 \otimes \mathscr{L}\mathrm{T}_2 \to \mathbb{G}_{m,\mathrm{Ran}}, \tag{3.12}$$

uniquely characterized by the property that its restriction along λ_1, λ_2, viewed as homomorphisms from $\mathscr{L}\mathbb{G}_m$ to $\mathscr{L}\mathrm{T}_1$ (resp. $\mathscr{L}\mathrm{T}_2$), equals $b(\lambda_1, \lambda_2)\langle \cdot, \cdot \rangle$.

Pairings $\mathscr{L}\mathrm{T}_1 \otimes \mathscr{L}\mathrm{T}_2 \to \mathbb{G}_{m,\mathrm{Ran}}$ of the form $\langle \cdot, \cdot \rangle_b$ are called *tame*. Given morphisms $\mathrm{T}_1' \to \mathrm{T}_1$, $\mathrm{T}_2' \to \mathrm{T}_2$, a tame pairing $\mathscr{L}\mathrm{T}_1 \otimes \mathscr{L}\mathrm{T}_2 \to \mathbb{G}_{m,\mathrm{Ran}}$ induces a tame pairing $\mathscr{L}\mathrm{T}_1' \otimes \mathscr{L}\mathrm{T}_2' \to \mathbb{G}_{m,\mathrm{Ran}}$. The converse also holds for surjections of tori.

Lemma 3.3.4. *Let* $\langle \cdot, \cdot \rangle : \mathscr{L}\mathrm{T}_1 \otimes \mathscr{L}\mathrm{T}_2 \to \mathbb{G}_{m,\mathrm{Ran}}$ *be a factorization pairing. Given surjections of* X*-tori* $\mathrm{T}_1' \to \mathrm{T}_1$, $\mathrm{T}_2' \to \mathrm{T}_2$, *if the induced pairing* $\mathscr{L}\mathrm{T}_1' \otimes \mathscr{L}\mathrm{T}_2' \to \mathbb{G}_{m,\mathrm{Ran}}$ *is tame, then so is* $\langle \cdot, \cdot \rangle$.

3.3.5. Before proving Lemma 3.3.4, we shall make an observation.

Suppose that we are given X-tori T_1 and T_2. *Claim*: all factorization morphisms of group presheaves over Ran below are trivial:

$$\mathscr{L}^+\mathrm{T}_1 \to \mathrm{Gr}_{\mathrm{T}_2}; \tag{3.13}$$

$$\mathrm{Gr}_{\mathrm{T}_1} \to \mathscr{L}^+\mathrm{T}_2. \tag{3.14}$$

For (3.13), this is because $\mathscr{L}^+\mathrm{T}_1 \to \mathrm{Ran}$ is pro-smooth with connected geometric fibers, whereas $\mathrm{Gr}_{\mathrm{T}_2} \to \mathrm{Ran}$ has formal geometric fibers. For (3.14), this is because $\mathrm{Gr}_{\mathrm{T}_1} \to \mathrm{Ran}$ is ind-proper, whereas $\mathscr{L}^+\mathrm{T}_2 \to \mathrm{Ran}$ is pro-affine. The combination of these two facts shows that any factorization bilinear pairing $\mathscr{L}\mathrm{T}_1 \otimes \mathscr{L}\mathrm{T}_2 \to \mathbb{G}_m$ induces, and is uniquely determined by a pairing $\mathscr{L}^+\mathrm{T}_1 \otimes \mathrm{Gr}_{\mathrm{T}_2} \to \mathbb{G}_m$.

Proof of Lemma 3.3.4. Let $\check{\mathrm{T}}_2$ (resp. $\check{\mathrm{T}}_2'$) denote the X-torus dual to T_2 (resp. T_2'). By perfectness of the Contou-Carrère symbol, $\langle \cdot, \cdot \rangle$ is equivalent to a factorization morphism of group presheaves over Ran:

$$\mathscr{L}\mathrm{T}_1 \to \mathscr{L}\check{\mathrm{T}}_2. \tag{3.15}$$

We need to prove that for any pair of cocharacters λ_1, λ_2 of T_1, T_2, the endomorphism φ of $\mathscr{L}\mathbb{G}_m$ defined by the composition:

$$\mathscr{L}\mathbb{G}_m \xrightarrow{\lambda_1} \mathscr{L}\mathrm{T}_1 \xrightarrow{(3.15)} \mathscr{L}\check{\mathrm{T}}_2 \xrightarrow{\lambda_2} \mathscr{L}\mathbb{G}_m$$

is given by $a \mapsto a^{\mathrm{N}}$ for some integer N.

The hypothesis implies that this statement holds after composing with an endomorphism of $\mathscr{L}\mathbb{G}_m$ defined by nth power map $a \mapsto a^n$ for some integer $n \geq 1$.

By the observation of §3.3.5, φ is uniquely determined by its restriction φ^+ to $\mathscr{L}^+\mathbb{G}_m$, whose image is also contained in $\mathscr{L}^+\mathbb{G}_m$. Note furthermore that for each S-point x^{I} of X^{I}, the restriction $\mathscr{L}^+_{x^{\mathrm{I}}}\mathbb{G}_m$ of $\mathscr{L}^+\mathbb{G}_m$ is the extension of a group S-scheme of multiplicative type by an iterated extension of vector group S-schemes (§3.2.3).

In particular, φ^+ induces a homomorphism of short exact sequences:

$$\begin{array}{ccccccccc} 0 & \to & \mathscr{L}^{\geq 1}_{x^{\mathrm{I}}}\mathbb{G}_m & \to & \mathscr{L}^{+}_{x^{\mathrm{I}}}\mathbb{G}_m & \to & \mathrm{R}_{\Gamma}\mathbb{G}_m & \to & 1 \\ & & \downarrow{\scriptstyle \varphi^+_a} & & \downarrow{\scriptstyle \varphi^+} & & \downarrow{\scriptstyle \varphi^+_m} & & \\ 0 & \to & \mathscr{L}^{\geq 1}_{x^{\mathrm{I}}}\mathbb{G}_m & \to & \mathscr{L}^{+}_{x^{\mathrm{I}}}\mathbb{G}_m & \to & \mathrm{R}_{\Gamma}\mathbb{G}_m & \to & 1 \end{array}$$

The fact that φ^+_m is the N'th power map after composing with the nth power map shows that $n \mid \mathrm{N}'$ and φ^+_m is the Nth power map, for $\mathrm{N} := \mathrm{N}'/n$.

Since $\mathscr{L}^{+}_{x^{\mathrm{I}}}\mathbb{G}_m \to \mathrm{S}$ is pro-smooth, it suffices to prove that φ^+ is the Nth power map on $\bar{k}$-points. In other words, given $f \in \bar{k}[[t]]^\times$ satisfying the equality:

$$\varphi^+(f)^n = (f^{\mathrm{N}})^n \text{ in } \bar{k}[[t]]^\times,$$

we need to deduce the equality $\varphi^+(f) = f^{\mathrm{N}}$. Setting $g := \varphi^+(f)/f^{\mathrm{N}}$, we may write:

$$g = 1 + \sum_{i \geq 1} a_i t^i \text{ in } \bar{k}[[t]]^\times.$$

Claim: $g^n = 1$ implies $g = 1$. For $\mathrm{char}(\bar{k}) \nmid n$, this holds because all nth roots of unity of $\bar{k}[[t]]$ are contained in $\bar{k}$. For $\mathrm{char}(\bar{k}) \mid n$, this holds because the Frobenius is injective. □

Example 3.3.6. Suppose that k has characterstic $p > 0$. Let us define a factorization central extension whose commutator is *not* tame:

$$1 \to \mathbb{G}_{m,\mathrm{Ran}} \to \mathscr{G} \to \mathscr{L}\mathbb{G}_m \to 1. \tag{3.16}$$

Given a morphism $\mathrm{Y} \to \mathrm{S}$ of k-presheaves, we write $\mathrm{Fr}_{\mathrm{Y}/\mathrm{S}} : \mathrm{Y} \to \mathrm{Y}^{(1)}_{/\mathrm{S}}$ for the pth power Frobenius of Y relative to S. Its formation is compatible with base change along S. Note that the presheaf $\mathscr{L}\mathbb{G}_m{}^{(1)}_{/\mathrm{Ran}}$ is canonically isomorphic to $\mathscr{L}\mathbb{G}_m$: an S-point of $\mathscr{L}\mathbb{G}_m{}^{(1)}_{/\mathrm{Ran}}$ is a pair (x^{I}, a) where x^{I} is an S-point of Ran and a is map $\mathring{\mathrm{D}}_{\mathrm{Fr}^*_{\mathrm{S}}(x^{\mathrm{I}})} \to \mathbb{G}_m$. However, $\mathring{\mathrm{D}}_{\mathrm{Fr}^*_{\mathrm{S}}(s^{\mathrm{I}})}$ is isomorphic to $\mathring{\mathrm{D}}_{x^{\mathrm{I}}}$ since its formation depends only on the subset $|\Gamma_{x^{\mathrm{I}}}|$ of $|\mathrm{S} \times \mathrm{X}|$. In particular, we may view $\mathrm{Fr}_{\mathscr{L}\mathbb{G}_m/\mathrm{Ran}}$ as an endomorphism of $\mathscr{L}\mathbb{G}_m$ over Ran.

The central extension (3.16) is defined to be the presheaf of sets $\mathbb{G}_{m,\mathrm{Ran}} \times_{\mathrm{Ran}} \mathscr{L}\mathbb{G}_m$ whose group structure is defined by the cocycle:

$$\mathscr{L}\mathbb{G}_m \otimes \mathscr{L}\mathbb{G}_m \to \mathbb{G}_{m,\mathrm{Ran}}, \quad (a, b) \mapsto \langle \mathrm{Fr}_{\mathscr{L}\mathbb{G}_m/\mathrm{Ran}}(a), b\rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the Contou-Carrère symbol. Since $\langle \cdot, \cdot \rangle$ is anti-symmetric, the commutator of (3.16) is the pairing:

$$\mathscr{L}\mathbb{G}_m \otimes \mathscr{L}\mathbb{G}_m \to \mathbb{G}_{m,\mathrm{Ran}}, \quad (a, b) \mapsto \langle \mathrm{Fr}_{\mathscr{L}\mathbb{G}_m/\mathrm{Ran}}(a), b\rangle\langle a, \mathrm{Fr}_{\mathscr{L}\mathbb{G}_m/\mathrm{Ran}}(b)\rangle. \tag{3.17}$$

Let us argue that this pairing is not tame over any geometric point $x : \mathrm{Spec}(\bar{k}) \to \mathrm{X}$. The choice of a uniformizer allows us to identify $\mathscr{L}_x\mathbb{G}_m$ with $\mathbb{G}_m((t))$. The morphism $\mathrm{Fr}_{\mathbb{G}_m((t))/\bar{k}}$ evaluates to the following map on R-points for any $\bar{k}$-algebra R:

$$\mathrm{R}((t))^\times \to \mathrm{R}((t))^\times, \quad \sum_n a_n t^n \mapsto \sum_n (a_n)^p t^n.$$

The commutator (3.17) is indeed $\langle \cdot, \cdot \rangle^{p+1}$ on $\bar{k}$-points. For a more general $\bar{k}$-algebra R, the Contou-Carrère pairing $\langle 1 - a_1 t, 1 - b_{-1} t^{-1}\rangle$ equals $1 - a_1 b_{-1}$ for nilpotents $a_1, b_{-1} \in \mathrm{R}$ [APR04]. Taking $\mathrm{R} := \bar{k}[\epsilon]/\epsilon^3$ with $a_1 := \epsilon x$, $b_{-1} := \epsilon y$ for $x, y \in \bar{k}^\times$ and equating the commutator $(1 - a_1^p b_{-1})(1 - a_1 b_{-1}^p)$ with $(1 - a_1 b_{-1})^{p+1}$, we find $xy = 0$, which is impossible.

3.3.7. We now show that tameness is a positive characteristic phenomenon.

The assertion below relies on [Tao21a] which uses the hypothesis $\mathrm{char}(k) = 0$.

Proposition 3.3.8. *Let* T_1, T_2 *be a pair of* X*-tori. If* $\mathrm{char}(k) = 0$*, then any factorization pairing* $\mathscr{L}\mathrm{T}_1 \otimes \mathscr{L}\mathrm{T}_2 \to \mathbb{G}_{m,\mathrm{Ran}}$ *is tame.*

Proof. Using the observations in §3.3.5, it suffices to prove that any factorization pairing $\mathscr{L}^+\mathrm{T}_1 \otimes \mathrm{Gr}_{\mathrm{T}_2} \to \mathbb{G}_{m,\mathrm{Ran}}$ is necessarily of the form $\langle\cdot,\cdot\rangle_b$ for some bilinear form $b : \Lambda_1 \otimes \Lambda_2 \to \mathbb{Z}$ (see §3.3.3).

Using the duality between $\mathrm{Gr}_{\mathrm{T}_2}$ and $\mathscr{L}^+\check{\mathrm{T}}_2$ under the Contou-Carrère symbol, we reduce the statement to the special case $\mathrm{T}_1 = \mathrm{T}_2 = \mathbb{G}_m$.

For each I-tuple $\lambda^{\mathrm{I}} = (\lambda^i)$ of integers, there is a closed immersion $\iota_{\lambda^{\mathrm{I}}} : \mathrm{X}^{\mathrm{I}} \to \mathrm{Gr}_{\mathbb{G}_m,\mathrm{X}^{\mathrm{I}}}$ sending an S-point $x^{\mathrm{I}} = (x^i)$ of X^{I} to the line bundle $\mathscr{O}(\sum_{i\in\mathrm{I}} \Gamma_{x^i})$ over $\mathrm{S}\times\mathrm{X}$ equipped with its canonical trivialization off $\Gamma_{x^{\mathrm{I}}}$. Consider the category of pairs $(\mathrm{I}, \lambda^{\mathrm{I}})$ where I is a nonempty finite set and λ^{I} is as above, where morphisms $(\mathrm{I}, \lambda^{\mathrm{I}}) \to (\mathrm{J}, \lambda^{\mathrm{J}})$ are defined by surjections $\varphi : \mathrm{I} \twoheadrightarrow \mathrm{J}$ with $\lambda^j = \sum_{i\in\varphi^{-1}(j)} \lambda^i$ for each $j \in \mathrm{J}$. The closed immersions $\iota_{\lambda^{\mathrm{I}}}$ assemble into a morphism of presheaves over Ran:

$$\mathrm{Gr}^{\mathrm{comb}}_{\mathbb{G}_m} \coloneqq \operatorname*{colim}_{(\mathrm{I},\lambda^{\mathrm{I}})} \mathrm{X}^{\mathrm{I}} \to \mathrm{Gr}_{\mathbb{G}_m},$$

which induces a bijection on field-valued points.

Claim: factorization pairings $\mathscr{L}^+\mathbb{G}_m \otimes \mathrm{Gr}^{\mathrm{comb}}_{\mathbb{G}_m} \to \mathbb{G}_{m,\mathrm{Ran}}$ are in bijection with sections of $\underline{\mathbb{Z}}$ over X. More precisely, locally on X, a generator is the colimit over $(\mathrm{I}, \lambda^{\mathrm{I}})$ of maps:

$$f_{(\mathrm{I},\lambda^{\mathrm{I}})} : \mathscr{L}^+_{\mathrm{X}^{\mathrm{I}}}\mathbb{G}_m \to \mathbb{G}_m, \quad (x^{\mathrm{I}}, a) \mapsto \prod_{i\in\mathrm{I}} (a|_{\Gamma_{x^i}})^{\lambda^i}, \tag{3.18}$$

where $a|_{\Gamma_{x^i}}$ is the restriction of a to the closed subscheme $\Gamma_{x^i} \subset \mathrm{D}_{x^{\mathrm{I}}}$.

To prove the claim, we use the presentation (3.8) of the group X^{I}-scheme $\mathscr{L}^+_{\mathrm{X}^{\mathrm{I}}}\mathbb{G}_m$ as an extension of $\mathrm{R}_\Gamma\mathbb{G}_m$ by $\mathscr{L}^{\geq 1}_{\mathrm{X}^{\mathrm{I}}}\mathbb{G}_m$. Every character $\mathscr{L}^+_{\mathrm{X}^{\mathrm{I}}}\mathbb{G}_m \to \mathbb{G}_m$ must factor through $\mathrm{R}_\Gamma\mathbb{G}_m$ and is uniquely determined by its restriction to the pairwise disjoint locus of X^{I}.

Given a factorization pairing $f' : \mathscr{L}^+\mathbb{G}_m \otimes \mathrm{Gr}^{\mathrm{comb}}_{\mathbb{G}_m} \to \mathbb{G}_{m,\mathrm{Ran}}$, we obtain a system of maps indexed by $(\mathrm{I}, \lambda^{\mathrm{I}})$:

$$f'_{(\mathrm{I},\lambda^{\mathrm{I}})} : \mathscr{L}^+_{\mathrm{X}^{\mathrm{I}}}\mathbb{G}_m \to \mathbb{G}_m. \tag{3.19}$$

By the observation above, (3.19) is uniquely determined by the case $\mathrm{I} = \{1\}$. Moreover, $f'_{(\{1\},\lambda)}$ is a character of $\mathrm{R}_\Gamma\mathbb{G}_m \cong \mathbb{G}_{m,\mathrm{X}}$, hence a section of $\underline{\mathbb{Z}}$ over X. Looking at $\mathrm{I} = \{1, 2\}$, we see that the association $\lambda \mapsto f'_{(\{1\},\lambda)}$ defines a group homomorphism $\underline{\mathbb{Z}} \to \underline{\mathbb{Z}}$. The group homomorphism corresponding to multiplication by n yields the nth power of (3.18).

Under the hypothesis $\mathrm{char}(k) = 0$, [Tao21a, Proposition 5.1.5] shows that any S-point of $\mathrm{Gr}_{\mathbb{G}_m}$ admits a factorization $\mathrm{S} \to \mathrm{S}_0 \to \mathrm{Gr}_{\mathbb{G}_m}$, where S_0 is reduced. Therefore, any pairing $\mathscr{L}^+\mathbb{G}_m \otimes \mathrm{Gr}_{\mathbb{G}_m} \to \mathbb{G}_{m,\mathrm{Ran}}$ is uniquely determined by its values on reduced test schemes, hence on field-valued points. Consequently, any such pairing is uniquely determined by its restriction to $\mathscr{L}^+\mathbb{G}_m \otimes \mathrm{Gr}^{\mathrm{comb}}_{\mathbb{G}_m}$. □

Corollary 3.3.9. *If* $\mathrm{char}(k) = 0$*, then any factorization pairing* $\mathscr{L}\mathbb{G}_m \otimes \mathscr{L}\mathbb{G}_m \to \mathbb{G}_{m,\mathrm{Ran}}$ *is an integral power of the Contou-Carrère symbol.*

Proof. This is a restatement of Proposition 3.3.8 in the special case $\mathrm{T}_1 = \mathrm{T}_2 = \mathbb{G}_m$. □

Remark 3.3.10. For any field k, we may view the proof of Proposition 3.3.8 as establishing the implication (1) ⇒ (2) between the following statements:

1. any S-point of $\mathrm{Gr}_{\mathbb{G}_m}$ admits a factorization $\mathrm{S} \to \mathrm{S}_0 \to \mathrm{Gr}_{\mathbb{G}_m}$ where S_0 is reduced;
2. any factorization pairing $\mathscr{L}\mathbb{G}_m \otimes \mathscr{L}\mathbb{G}_m \to \mathbb{G}_{m,\mathrm{Ran}}$ is an integral power of the Contou-Carrère symbol.

Since (2) fails when $\mathrm{char}(k) > 0$ (Example 3.3.6), (1) must also fail when $\mathrm{char}(k) > 0$. In other words, the hypothesis $\mathrm{char}(k) = 0$ in [Tao21a, Theorem 1.2.1] is necessary.

Remark 3.3.11. It is known that the Contou-Carrère symbol is the universal Steinberg symbol over a point of X, *cf.* [GO15]. The universal property established in Corollary 3.3.9 is of a different kind: instead of imposing the Steinberg relation, we impose compatibility with factorization.

3.4. Classification.

3.4.1. Let G be a reductive group X-scheme. Denote by $\mathrm{Rad}(\mathrm{G})$ the radical of G, *i.e.* the maximal torus of the center $\mathrm{Z_G}$ of G [ABD+66, XXII, Définition 4.3.6].

Our principal goal is to study factorization super central extensions of $\mathscr{L}\mathrm{G}$ by $\mathbb{G}_{m,\mathrm{Ran}}$ subject to the followng property: the commutator of the induced factorization super central extension of $\mathscr{L}\mathrm{Rad}(\mathrm{G})$ by $\mathbb{G}_{m,\mathrm{Ran}}$ is tame in the sense of §3.3.3. Such a factorization super central extension of $\mathscr{L}\mathrm{G}$ by $\mathbb{G}_{m,\mathrm{Ran}}$ is said to have *tame commutator*.

By Example 3.3.6 and Proposition 3.3.8, the condition of having tame commutator is vacuous when $\mathrm{char}(k) = 0$, but not so when $\mathrm{char}(k) > 0$.

3.4.2. Suppose that G has a maximal torus T with sheaf of cocharacters Λ.

Write $\mathrm{G_{sc}}$ for the simply connected form of G with induced maximal torus $\mathrm{T_{sc}}$ and sheaf of cocharacters Λ_{sc}.

Recall that any integral Weyl-invariant quadratic form $\mathrm{Q_{sc}}$ on Λ_{sc} defines a section of $\vartheta(\Lambda_{\mathrm{sc}})$ via (2.2). The corresponding central extension $\widetilde{\Lambda}_{\mathrm{sc}}$ of Λ_{sc} by $\mathbb{G}_m$ can be viewed as a monoidal morphism:

$$\nu_{\mathrm{Q_{sc}}} : \Lambda_{\mathrm{sc}} \to \mathsf{B}\mathbb{G}_m \cong \mathsf{Pic}. \tag{3.20}$$

We define a morphism of pointed X-stacks by the formula:

$$\nu_{\mathrm{Q_{sc}},+} : \Lambda_{\mathrm{sc}} \to \mathsf{Pic}, \quad \lambda \mapsto \nu_{\mathrm{Q_{sc}}}(\lambda) \otimes \omega_{\mathrm{X}}^{\mathrm{Q_{sc}}(\lambda)}. \tag{3.21}$$

3.4.3. To specify the additional structure of the map $\nu_{\mathrm{Q_{sc}},+}$ inherited from the monoidal structure of $\nu_{\mathrm{Q_{sc}}}$, we introduce a piece of terminology.

Let Λ denote, temporarily, any étale sheaf of finite free $\mathbb{Z}$-modules over X and $b : \Lambda\otimes\Lambda \to \mathbb{Z}$ be any symmetric bilinear form. A morphism of X-stacks $\nu_+ : \Lambda \to \mathsf{Pic}^{\mathrm{super}}$ is said to be *ω-monoidal with respect to b* if it is equipped with ismorphisms:

$$\mathscr{O}_{\mathrm{X}} \xrightarrow{\simeq} \nu_+(0);$$
$$\nu_+(\lambda_1) \otimes \nu_+(\lambda_2) \otimes \omega_{\mathrm{X}}^{b(\lambda_1,\lambda_2)} \xrightarrow{\simeq} \nu_+(\lambda_1 + \lambda_2),$$

for each $\lambda_1, \lambda_2 \in \Lambda$, satisfying unitality and associativity. (This notion does not refer to the commutativity constraint of the Picard groupoid $\mathsf{Pic}^{\mathrm{super}}$.)

In this terminology, the map $\nu_{\mathrm{Q_{sc}},+}$ is ω-monoidal with respect to the symmetric form b_{sc} associated to $\mathrm{Q_{sc}}$.

3.4.4. Denote by $\vartheta^{\mathrm{super}}_{\mathrm{G},+}(\Lambda)$ the Picard groupoid of triples (b, ν_+, φ), where:

1. b is a Weyl-invariant integral symmetric bilinear form on Λ, such that $b(\lambda,\lambda) \in 2\mathbb{Z}$ if $\lambda \in \Lambda_{\mathrm{sc}}$—we write $\mathrm{Q_{sc}}$ for the corresponding quadratic form on Λ_{sc};
2. ν_+ is a morphism $\Lambda \to \mathsf{Pic}^{\mathrm{super}}$ which is ω-monoidal with respect to b and commutes with the commutativity constraint up to the bilinear form $(-1)^b$;
3. φ is an isomorphism between the restriction of ν_+ to Λ_{sc} and $\nu_{\mathrm{Q_{sc}},+}$ as ω-monoidal morphisms.

The relation between $\vartheta^{\mathrm{super}}_{\mathrm{G},+}(\Lambda)$ and the Picard groupoid $\vartheta^{\mathrm{super}}_{\mathrm{G}}(\Lambda)$ defined in §2.2.1 is as follows. Given a *ϑ-characteristic* $\omega^{1/2}$, *i.e.* a line bundle over X equipped with an isomorphism $(\omega^{1/2})^{\otimes 2} \cong \omega_{\mathrm{X}}$, we obtain an isomorphism (called the *$\omega^{1/2}$-shift*):

$$\vartheta^{\mathrm{super}}_{\mathrm{G}}(\Lambda) \xrightarrow{\simeq} \vartheta^{\mathrm{super}}_{\mathrm{G},+}(\Lambda), \quad (b, \nu, \varphi) \mapsto (b, \nu_+, \varphi), \tag{3.22}$$

where ν_+ is the ω-monoidal morphism $\lambda \mapsto \nu(\lambda) \otimes (\omega^{1/2})^{b(\lambda,\lambda)}$ and ν is the monoidal morphism $\Lambda \to \mathsf{Pic}^{\mathrm{super}}$ corresponding to $\widetilde{\Lambda}$, *i.e.* (2.11).

Theorem 3.4.5. *Let* G *be a reductive group* X*-scheme. The following Picard groupoids are canonically equivalent:*

(1) factorization super central extensions of $\mathscr{L}\mathrm{G}$ *by* $\mathbb{G}_{m,\mathrm{Ran}}$ *with tame commutator;*
(2) factorization super line bundles over Gr_{G}*;*
(3) $\vartheta^{\mathrm{super}}_{\mathrm{G},+}(\Lambda)$*—if* G *is equipped with a maximal torus* T *with sheaf of cocharacters* Λ*.*
(4) rigidified sections of $\underline{\mathrm{K}}^{\mathrm{super}}_{[1,2]}$ *over the Zariski classifying stack of* G*—if* X *is equipped with a ϑ-characteristic.*

3.4.6. The equivalences of Theorem 3.4.5 are constructed in several stages. We briefly indicate the steps and explain where prior works are used.

First, we prove that factorization super central extensions of $\mathscr{L}^+\mathrm{G}$ by $\mathbb{G}_{m,\mathrm{Ran}}$ are canonically trivial (Proposition 4.1.2). By descent, we obtain the functor (1) → (2).

Next, the equivalence (2) ≅ (3) is essentially known when G is a torus or a semisimple, simply connected group scheme. The torus case is treated in [TZ21] (using a substantial theorem of [Tao21b]). The simply connected case reduces to Falting [Fal03].

One may then define the functor (2) → (3) for any reductive group scheme G, by appealing to functoriality with respect to the commutative diagram:

$$\begin{array}{ccc} \mathrm{T}_{\mathrm{sc}} & \longrightarrow & \mathrm{G}_{\mathrm{sc}} \\ \downarrow & & \downarrow \\ \mathrm{T} & \longrightarrow & \mathrm{G} \end{array}$$

We then proceed as follows. We first prove "by hand" that (1) → (2) is an equivalence for G a torus or a semisimple, simply connected group scheme. Then we prove that, for any reductive group scheme G, the composition (1) → (2) → (3) is an equivalence, whereas the second functor is fully faithful. Here, we use an idea of Finkelberg–Lysenko [FL10], an idea of Gaitsgory [Gai20], and an argument from [TZ21].

The equivalences (1) ≅ (2) ≅ (3) are completed in Proposition 4.4.13.

Finally, the equivalence (3) ≅ (4) is defined by combining Theorem 2.2.3 with the $\omega^{1/2}$-shift (3.22). The composed functor (1) →(2) → (3) → (4) thus *a priori* depends on a maximal torus T, but it shall follow from the construction that this is not the case. By étale descent, we obtain the equivalence (1) ≅ (4) without assuming the existence of a maximal torus.

Corollary 3.4.7. *Let* G *be a reductive group* X*-scheme. The following Picard groupoids are canonically equivalent:*

(1) factorization central extensions of $\mathscr{L}\mathrm{G}$ *by* $\mathbb{G}_{m,\mathrm{Ran}}$ *with tame commutator;*
(2) factorization line bundles over Gr_{G}*;*
(3) $\vartheta_{\mathrm{G},+}(\Lambda)$*, if* G *is equipped with a maximal torus* T *with sheaf of cocharacters* Λ*;*
(4) central extensions of G *by* $\underline{\mathrm{K}}_2$ *on the big Zariski site of* X*.*

Proof. Each Picard groupoid in Theorem 3.4.5 admits a canonical functor to $\mathrm{Hom}(\pi_1\mathrm{G}, \mathbb{Z}/2)$: for the Picard groupoids (1), (2), these are the functors remembering the grading; for (3), this is the second functor in (2.10); for (4), this is the lower horizontal functor in (2.28).

The equivalences of Theorem 3.4.5 commute with these functors to $\mathrm{Hom}(\pi_1\mathrm{G}, \mathbb{Z}/2)$. By restricting them to the fibers, we obtain the corollary. (Note that the restriction of (3.22) is defined without the choice of $\omega^{1/2}$.) □

4. Proofs

The goal of this section is to prove Theorem 3.4.5. We begin in §4.1 by constructing a functor from factorization super central extensions of $\mathscr{L}\mathrm{G}$ to factorization super line bundles over Gr_{G}. The subsections §4.2, §4.3, §4.4 prove the equivalences (1) ≅ (2) ≅ (3) of Theorem 3.4.5 for tori, simply connected groups, respectively all reductive groups. In §4.5, we show that the equivalence (1) ≅ (4) obtained by combining the equivalence (1) ≅ (3) and Theorem 2.2.3 is independent of the choice of a maximal torus.

We remain in the context of the previous section: we fix a ground field k and a smooth curve X.

4.1. Triviality over arc groups.

4.1.1. In this subsection, we let G denote a smooth affine group X-scheme with connected geometric fibers. Our goal is to prove the following result.

Proposition 4.1.2. *Any factorization super central extension of $\mathscr{L}^+\mathrm{G}$ by $\mathbb{G}_{m,\mathrm{Ran}}$ is canonically trivial.*

4.1.3. Consider an S-point x^{I} of Ran. Recall that the restriction $\mathscr{L}^+_{x^{\mathrm{I}}}\mathrm{G}$ is an extension of $\mathrm{R}_\Gamma\mathrm{G}$ by $\mathscr{L}^{\geq 1}_{x^{\mathrm{I}}}\mathrm{G}$ as a group S-scheme (3.8).

We begin with a Lemma which allows us to reduce the problem to $\mathrm{R}_\Gamma\mathrm{G}$.

Lemma 4.1.4. *If* S *is locally Noetherian and normal, then pullback along* (3.8) *defines an equivalence between the groupoid of central extensions of* $\mathrm{R}_\Gamma\mathrm{G}$ *by* $\mathbb{G}_{m,\mathrm{S}}$ *and that of* $\mathscr{L}^+_{x^{\mathrm{I}}}\mathrm{G}$.

Proof. Given a central extension:

$$1 \to \mathbb{G}_{m,\mathrm{S}} \to \mathscr{G} \to \mathscr{L}^+_{x^{\mathrm{I}}}\mathrm{G} \to 1, \tag{4.1}$$

we claim that (4.1) admits a unique splitting over $\mathscr{L}^{\geq 1}_{x^{\mathrm{I}}}\mathrm{G}$, and its image is normal in $\mathscr{G}$. Then the association $\mathscr{G} \mapsto \mathscr{G}/\mathscr{L}^{\geq 1}_{x^{\mathrm{I}}}\mathrm{G}$ supplies the desired inverse functor.

Since $\mathscr{L}^{\geq 1}_{x^{\mathrm{I}}}\mathrm{G}$ is an iterated extension of vector group S-schemes, to show the existence and uniqueness of the splitting, it suffices to show that any central extension of $\mathbb{G}_{a,\mathrm{S}}$ by $\mathbb{G}_{m,\mathrm{S}}$ is canonically split.

This assertion follows from the observations below:

(1) any S-morphism $\mathbb{G}_{a,\mathrm{S}} \to \mathbb{G}_{m,\mathrm{S}}$ is trivial—this uses the hypothesis that S is reduced;
(2) the pullback $\mathsf{Pic}(\mathrm{S}) \to \mathsf{Pic}(\mathbb{G}_{a,\mathrm{S}})$ is an equivalence—this uses the hypothesis that S is locally Noetherian and normal [GD67, Corollaire 21.4.13, p.361].

To prove that the resulting splitting $\mathscr{L}^{\geq 1}_{x^{\mathrm{I}}}\mathrm{G} \to \mathscr{G}$ has normal image, it suffices to show that it commutes with the conjugation action of $\mathscr{L}^+_{x^{\mathrm{I}}}\mathrm{G}$. However, this follows from the uniqueness of the splitting. □

Proof of Proposition 4.1.2. For an integer $n \geq 1$, we denote by $\mathrm{Ran}^{\leq n} \subset \mathrm{Ran}$ the subfunctor whose S-points are finite subsets of $\mathrm{Maps}(\mathrm{S}, \mathrm{X})$ of cardinality ≤ 2.

Given a factorization super central extension:

$$1 \to \mathbb{G}_{m,\mathrm{Ran}} \to \mathscr{G} \to \mathscr{L}^+\mathrm{G} \to 1, \tag{4.2}$$

we first observe that "super" is redundant since the group scheme $\mathscr{L}^+_{x^{\mathrm{I}}}\mathrm{G} \to \mathrm{S}$ for any S-point x^{I} of Ran has connected geometric fibers. Next, we claim that a trivialization of (4.2) over

$\mathrm{Ran}^{\le 2}$ compatible with the factorization morphism (3.7) for $|\mathrm{I}| = |\mathrm{J}| = 1$ uniquely extends to a trivialization of (4.2) over Ran.

For a nonempty finite set I, consider the tautological X^{I}-point x^{I} of Ran (Remark 3.1.2). Denote by $\mathrm{U} \subset \mathrm{X}^{\mathrm{I}}$ the open subset where $\Gamma_{x^{i_1}} \cap \Gamma_{x^{i_2}} \neq \varnothing$ for at most one pair of distinct elements $i_1, i_2 \in \mathrm{I}$. Let $\mathrm{Z} \subset \mathrm{X}^{\mathrm{I}}$ be its complement, a closed subset of codimension ≥ 2. (It is empty if $|\mathrm{I}| \leq 2$.) The induced U-point $x^{\mathrm{I}}|_{\mathrm{U}}$ is a disjoint union of x^i (for $i \neq i_1, i_2$) with $x^{\{i_1, i_2\}}$. The factorization morphism for $\mathscr{G}$ and its trivialization over $\mathrm{Ran}^{\le 2}$ define a trivialization of $\mathscr{G}_{x^{\mathrm{I}}}$ over $\mathrm{U} \subset \mathrm{X}^{\mathrm{I}}$.

Since X^{I} is smooth, $\mathscr{G}_{x^{\mathrm{I}}}$ is pulled back from a central extension of $\mathrm{R}_\Gamma \mathrm{G}$ by $\mathbb{G}_{m, \mathrm{X}^{\mathrm{I}}}$ (Lemma 4.1.4). Since $\mathrm{R}_\Gamma \mathrm{G}$ is also smooth, the trivialization of $\mathscr{G}_{x^{\mathrm{I}}}$ extends uniquely along $\mathrm{U} \subset \mathrm{X}^{\mathrm{I}}$ [Sta18, 031T]. Given a surjection of nonempty finite sets $\mathrm{I} \twoheadrightarrow \mathrm{J}$, we need to argue that the trivialization of $\mathscr{G}_{x^{\mathrm{I}}}$ restricts to the trivialization of $\mathscr{G}_{x^{\mathrm{J}}}$. This statement reduces to the case $|\mathrm{I}| = |\mathrm{J}| + 1$, where the diagonal $\mathrm{X}^{\mathrm{J}} \subset \mathrm{X}^{\mathrm{I}}$ intersects nontrivially with U. Since the two trivializations agree over $\mathrm{X}^{\mathrm{J}} \cap \mathrm{U}$, they must agree over X^{J}.

Finally, we construct the trivialization of (4.2) over $\mathrm{Ran}^{\le 2}$ compatible with factorization. Consider the tautological $\mathrm{X}^{\{1,2\}}$-point $x^{\{1,2\}}$ of Ran, with $\mathrm{U} \subset \mathrm{X}^{\{1,2\}}$ the complement of the diagonal. The closed immersions $\Gamma_{x^i} \subset \Gamma_{x^{\{1,2\}}}$ (for $i = 1, 2$) induce projection maps from $\mathrm{R}_\Gamma \mathrm{G}$ to G. Moreoever, the base change of $\mathrm{R}_\Gamma \mathrm{G}$ along the diagonal of $\mathrm{X}^{\{1,2\}}$ is an extension $\widetilde{\mathrm{G}}$ of G by a vector group scheme. These morphisms are summarized in the following diagram:

$$\begin{array}{ccccc} \widetilde{\mathrm{G}} & \xrightarrow{\Delta} & \mathrm{R}_\Gamma \mathrm{G} & \overset{\mathrm{pr}_1}{\underset{\mathrm{pr}_2}{\rightrightarrows}} & \mathrm{G} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{X} & \xrightarrow{\Delta} & \mathrm{X}^{\{1,2\}} & \overset{\mathrm{pr}_1}{\underset{\mathrm{pr}_2}{\rightrightarrows}} & \mathrm{X} \end{array}$$

where both compositions in the upper row are the canonical surjection $\widetilde{\mathrm{G}} \to \mathrm{G}$.

The restriction of $\mathscr{G}$ along the tautological X-point of Ran is pulled back from a central extension of G by $\mathbb{G}_{m,\mathrm{X}}$, to be denoted by $\mathscr{G}_1$. Similarly, the restriction $\mathscr{G}_{x^{\{1,2\}}}$ is pulled back from a central extension of $\mathrm{R}_\Gamma \mathrm{G}$, to be denoted by $\mathscr{G}_2$. By factorization, the restriction of $\mathscr{G}_2$ to $\mathrm{U} \subset \mathrm{X}^{\{1,2\}}$ is identified with $\mathrm{pr}_1^* \mathscr{G}_1 \otimes \mathrm{pr}_2^* \mathscr{G}_1$. This identification extends uniquely to an isomorphism between $\mathscr{G}_2$ with $\mathrm{pr}_1^* \mathscr{G}_1 \otimes \mathrm{pr}_2^* \mathscr{G}_1 \otimes \mathscr{O}(n\Delta)$ as line bundles over $\mathrm{R}_\Gamma \mathrm{G}$, for some $n \in \mathbb{Z}$. Restriction to the unit section $e : \mathrm{X}^{\{1,2\}} \to \mathrm{R}_\Gamma \mathrm{G}$ tells us that $n = 0$, so we obtain an isomorphism of central extensions of $\mathrm{R}_\Gamma \mathrm{G}$ by $\mathbb{G}_{m, \mathrm{X}^{\{1,2\}}}$:

$$\mathscr{G}_2 \xrightarrow{\simeq} \mathrm{pr}_1^* \mathscr{G}_1 \otimes \mathrm{pr}_2^* \mathscr{G}_1. \tag{4.3}$$

Restriction of (4.3) along the diagonal then yields an isomorphism $\mathscr{G}_1 \xrightarrow{\simeq} \mathscr{G}_1^{\otimes 2}$, *i.e.* a trivialization of $\mathscr{G}_1$. The trivialization of $\mathscr{G}_2$ is deduced from (4.3), so it is automatically compatible with factorization. □

4.1.5. Using Proposition 4.1.2, we construct a functor of Picard groupoids:

$$\mathrm{Hom}_{\mathrm{fact}}(\mathscr{L}\mathrm{G}, \mathsf{Pic}^{\mathrm{super}}) \to \Gamma_{\mathrm{fact}}(\mathrm{Gr}_{\mathrm{G}}, \mathsf{Pic}^{\mathrm{super}}), \tag{4.4}$$

where the target consists of factorization super line bundles over Gr_{G}.

Indeed, any factorization monoidal morphism $\mathscr{L}\mathrm{G} \to \mathsf{Pic}^{\mathrm{super}}$ is trivial over $\mathscr{L}^+\mathrm{G}$, so the monoidal structure yields its descent data to $\mathscr{L}\mathrm{G}/\mathscr{L}^+\mathrm{G} \cong \mathrm{Gr}_{\mathrm{G}}$.

One can make a stronger statement: this monoidal structure yields descent data to the local Hecke stack $\mathrm{Hec}_{\mathrm{G}} := \mathscr{L}^+\mathrm{G} \backslash \mathscr{L}\mathrm{G}/\mathscr{L}^+\mathrm{G}$, and the resulting factorization super line bundle over $\mathrm{Hec}_{\mathrm{G}}$ is compatible with the "convolution structure" of $\mathrm{Hec}_{\mathrm{G}}$. To make this precise, we need the prestack:

$$\mathrm{Hec}_{\mathrm{G}}^{[2]} := \mathscr{L}^{+}\mathrm{G}\backslash\mathscr{L}\mathrm{G} \times^{\mathscr{L}^{+}\mathrm{G}} \mathscr{L}\mathrm{G}/\mathscr{L}^{+}\mathrm{G},$$

equipped with three maps p_1, m, p_2 to $\mathrm{Hec}_{\mathrm{G}}$: projection onto the first factor, multiplication, and projection onto the second factor. There is also a unit section $e : \mathrm{B}(\mathscr{L}^{+}\mathrm{G}) \to \mathrm{Hec}_{\mathrm{G}}$, where B stands for delooping relative to Ran. A factorization super line bundle $\mathscr{L}$ over $\mathrm{Hec}_{\mathrm{G}}$ is *compatible with convolution* if there are additional isomorphisms:

$$\mathscr{O}_{\mathrm{B}(\mathscr{L}^{+}\mathrm{G})} \xrightarrow{\simeq} e^{*}\mathscr{L}; \tag{4.5}$$

$$(p_1)^{*}\mathscr{L} \otimes (p_2)^{*}\mathscr{L} \xrightarrow{\simeq} m^{*}\mathscr{L}, \tag{4.6}$$

which satisfy the conditions of an associative algebra and commute with factorization.

Let $\mathrm{Hom}_{\mathrm{fact}}(\mathrm{Hec}_{\mathrm{G}}, \mathsf{Pic}^{\mathrm{super}})$ denote the Picard groupoid of factorization super line bundles over $\mathrm{Hec}_{\mathrm{G}}$ compatible with convolution. The descent procedure then yields an equivalence of Picard groupoids:

$$\mathrm{Hom}_{\mathrm{fact}}(\mathscr{L}\mathrm{G}, \mathsf{Pic}^{\mathrm{super}}) \xrightarrow{\simeq} \mathrm{Hom}_{\mathrm{fact}}(\mathrm{Hec}_{\mathrm{G}}, \mathsf{Pic}^{\mathrm{super}}). \tag{4.7}$$

This assertion follows at once from two observations: the convolution structure on $\mathrm{Hec}_{\mathrm{G}}$ is defined by the Čech nerve of $\mathrm{B}(\mathscr{L}^{+}\mathrm{G}) \to \mathrm{B}(\mathscr{L}\mathrm{G})$ and any monoidal morphism $\mathrm{B}(\mathscr{L}^{+}\mathrm{G}) \to \mathsf{Pic}^{\mathrm{super}}$ compatible with factorization is canonically trivial (Proposition 4.1.2).

4.2. Tori.

4.2.1. Let Λ be an étale sheaf of finite free $\mathbb{Z}$-modules over X. Denote by $\vartheta_{+}^{\mathrm{super}}(\Lambda)$ the Picard groupoid of pairs (b, F_{+}), where:

1. b is an integral symmetric bilinear form on Λ;
2. $\mathrm{F}_{+} : \Lambda \to \mathsf{Pic}^{\mathrm{super}}$ is an ω-monoidal morphism with respect to b and commutes with the commutativity constraint up to the bilinear form $(-1)^{b}$.

This is the special case of $\vartheta_{\mathrm{G},+}^{\mathrm{super}}(\Lambda)$ defined in §3.4.4, for G the X-torus $\Lambda \otimes \mathbb{G}_m$.

4.2.2. Let T be an X-torus with sheaf of cocharacters Λ. We shall define a functor from the Picard groupoid of factorization super line bundles over Gr_{T} to $\vartheta_{+}^{\mathrm{super}}(\Lambda)$:

$$\Gamma_{\mathrm{fact}}(\mathrm{Gr}_{\mathrm{T}}, \mathsf{Pic}^{\mathrm{super}}) \to \vartheta_{+}^{\mathrm{super}}(\Lambda). \tag{4.8}$$

This is a variant of [BD04, §3.10.7], where objects of $\vartheta_{+}^{\mathrm{super}}(\Lambda)$ are called *ϑ-data*.

For each I-tuple $\lambda^{\mathrm{I}} = (\lambda_i)$ of elements of Λ, there is a closed immersion $\iota_{\lambda^{\mathrm{I}}} : \mathrm{X}^{\mathrm{I}} \to \mathrm{Gr}_{\mathrm{T},\mathrm{X}^{\mathrm{I}}}$ sending an S-point $x^{\mathrm{I}} = (x^{i})$ of X^{I} to the T-torsor $\mathscr{O}(\lambda_i\Gamma_{x^i})$ equipped with the canonical trivialization off $\Gamma_{x^{\mathrm{I}}}$.

To define (4.8), we take a factorization super line bundle $\mathscr{L}$ over Gr_{T} and construct a pair (b, F_{+}). Given $\lambda_1, \lambda_2 \in \Lambda$, the line bundle $(\iota_{\lambda_1,\lambda_2})^{*}\mathscr{L}$ is identified with $(\iota_{\lambda_1})^{*}\mathscr{L} \boxtimes (\iota_{\lambda_2})^{*}\mathscr{L}$ off the diagonal of X^2 by factorization—this identification extends to an isomorphism:

$$(\iota_{\lambda_1,\lambda_2})^{*}\mathscr{L} \cong (\iota_{\lambda_1})^{*}\mathscr{L} \otimes (\iota_{\lambda_2})^{*}\mathscr{L} \otimes \mathscr{O}_{\mathrm{X}^2}(b(\lambda_1,\lambda_2)\Delta), \tag{4.9}$$

for a uniquely defined integer $b(\lambda_1, \lambda_2)$. The associativity and unitality of the factorization isomorphism implies that $\lambda_1, \lambda_2 \mapsto b(\lambda_1, \lambda_2)$ is a bilinear form. Commutativity of the factorization isomorphism implies that b is symmetric and that $b(\lambda, \lambda) \bmod 2$ agrees with the grading on $(\iota_{\lambda})^{*}\mathscr{L}$. The definition of b is complete.

The second datum F_{+} is set to be $\mathrm{F}_{+}(\lambda) := (\iota_{\lambda})^{*}\mathscr{L}$, with ω-monoidal structure given by restricting (4.9) along the diagonal. The fact that F_{+} commutes with the brading up to the factor $(-1)^{b(\lambda_1,\lambda_2)}$ follows from the fact that the isomorphism $\mathscr{O}_{\mathrm{X}^2}(\Delta)|_{\Delta} \cong \omega_{\mathrm{X}}$ is equivariant against the exchange map $\mathrm{X}^2 \to \mathrm{X}^2$, $(x^1, x^2) \mapsto (x^2, x^1)$ up to the factor (-1).

Proposition 4.2.3. *The functor* (4.8) *is an equivalence of Picard groupoids.*

Proof. The proof is identical to that of [TZ21, Proposition 1.4]. □

4.2.4. Let T, T′ be X-tori whose sheaves of cocharacters are Λ, respectively Λ′. Suppose that we are given a factorization bilinear pairing:

$$\langle \cdot, \cdot \rangle : \mathscr{L}^+\mathrm{T} \otimes \mathrm{Gr}_{\mathrm{T}'} \to \mathbb{G}_m. \tag{4.10}$$

Delooping in the first variable and pulling back along the projection map $\mathrm{Gr}_\mathrm{T} \to \mathrm{B}(\mathscr{L}^+\mathrm{T})$, we obtain a morphism compatible with factorization:

$$\langle \cdot, \cdot \rangle : \mathrm{Gr}_\mathrm{T} \times_{\mathrm{Ran}} \mathrm{Gr}_{\mathrm{T}'} \to \mathrm{B}\mathbb{G}_m, \tag{4.11}$$

or equivalently a factorization line bundle over $\mathrm{Gr}_{\mathrm{T}\times\mathrm{T}'}$.

Consider the factorization pairing $\langle \cdot, \cdot \rangle_b$ defined by a bilinear form $b : \Lambda \otimes \Lambda' \to \mathbb{Z}$ as in §3.3.3. The property of Contou-Carrère symbol shows that $\langle \cdot, \cdot \rangle_b$ induces a pairing $\mathscr{L}^+\mathrm{T} \otimes \mathrm{Gr}_{\mathrm{T}'} \to \mathbb{G}_m$, hence a factorization line bundle $\mathscr{O}(b)$ over $\mathrm{Gr}_{\mathrm{T}\times\mathrm{T}'}$. We may calculate its image under the equivalence of Proposition 4.2.3.

Lemma 4.2.5. *The factorization line bundle $\mathscr{O}(b)$ corresponds under the equivalence of Proposition 4.2.3 to the pair consisting of:*

(1) the quadratic form $\Lambda \oplus \Lambda' \to \mathbb{Z}$, $(\lambda, \lambda') \mapsto b(\lambda, \lambda')$;

(2) the ω-monoidal morphism $\Lambda \oplus \Lambda' \to \mathsf{Pic}$ with $(\lambda, \lambda') \mapsto \omega_\mathrm{X}^{b(\lambda,\lambda')}$, whose ω-monoidal structure is the isomorphism:

$$(-1)^{b(\lambda_2,\lambda_1')} \cdot \mathrm{id} : \omega_\mathrm{X}^{b(\lambda_1+\lambda_2, \lambda_1'+\lambda_2')} \xrightarrow{\simeq} \omega_\mathrm{X}^{b(\lambda_1,\lambda_1')} \otimes \omega_\mathrm{X}^{b(\lambda_2,\lambda_2')} \otimes \omega_\mathrm{X}^{b(\lambda_1,\lambda_2')+b(\lambda_2,\lambda_1')},$$

for any pair of elements $(\lambda_1, \lambda_1'), (\lambda_2, \lambda_2') \in \Lambda \oplus \Lambda'$.

Proof. Let $(\lambda')^\mathrm{I} = (\lambda_i')$ be an I-tuple of elements of Λ′. The restriction of $\mathscr{O}(b)$ along:

$$(\mathrm{id}, \iota_{(\lambda')^\mathrm{I}}) : \mathrm{Gr}_{\mathrm{T},\mathrm{X}^\mathrm{I}} \to \mathrm{Gr}_{\mathrm{T},\mathrm{X}^\mathrm{I}} \times_{\mathrm{X}^\mathrm{I}} \mathrm{Gr}_{\mathrm{T}',\mathrm{X}^\mathrm{I}}$$

is the line bundle over $\mathrm{Gr}_{\mathrm{T},\mathrm{X}^\mathrm{I}}$ whose fiber at an S-point $(x^\mathrm{I}, \mathrm{P}_\mathrm{T}, \alpha)$ of $\mathrm{Gr}_{\mathrm{T},\mathrm{X}^\mathrm{I}}$ is given by $\bigotimes_{i\in\mathrm{I}} (\mathrm{P}_\mathrm{T}|_{\Gamma_{x^i}})^{b(-,\lambda_i')}$, where the superscript indicates inducing along the character $\mathrm{T} \to \mathbb{G}_m$ defined by $b(-, \lambda_i') : \Lambda \to \mathbb{Z}$.

In particular, for an I-tuple $\lambda^\mathrm{I} = (\lambda_i)$ of elements of Λ, further restricting $(\mathrm{id}, \iota_{(\lambda')^\mathrm{I}})^* \mathscr{O}(b)$ along $\iota_{\lambda^\mathrm{I}}$ yields the following line bundle over X^I:

$$\bigotimes_{i\in\mathrm{I}} (p_i^* \omega_\mathrm{X}^{b(\lambda_i,\lambda_i')} \otimes \bigotimes_{\substack{j\in\mathrm{I}\\ j\neq i}} p_{ij}^* \mathscr{O}_{\mathrm{X}^2}(b(\lambda_j, \lambda_i')\Delta)), \tag{4.12}$$

where $p_i : \mathrm{X}^\mathrm{I} \to \mathrm{X}$ (resp. $p_{ij} : \mathrm{X}^\mathrm{I} \to \mathrm{X}^2$) denotes the projection onto the factor labeled by i (resp. factors labeled by (i, j)).

Statement (1) follows by inspecting (4.12) for $\mathrm{I} = \{1, 2\}$, seeing that the quadratic form $(\lambda, \lambda') \mapsto b(\lambda, \lambda')$ has symmetric form $(\lambda_1, \lambda_1'), (\lambda_2, \lambda_2') \mapsto b(\lambda_2, \lambda_1') + b(\lambda_1, \lambda_2')$. The first part of statement (2) follows by inspecting (4.12) for $\mathrm{I} = \{1\}$, the second part for $\mathrm{I} = \{1, 2\}$, taking into account the fact that the isomorphism $\mathscr{O}_{\mathrm{X}^2}(\Delta)|_\Delta \cong \omega_\mathrm{X}$ is equivariant for the exchange map $\mathrm{X}^2 \to \mathrm{X}^2$, $(x^1, x^2) \mapsto (x^2, x^1)$ up to the factor (-1). □

4.2.6. Let T be an X-torus with sheaf of cocharacters Λ. We now complete the classification of factorization super central extensions of $\mathscr{L}\mathrm{T}$ by $\mathbb{G}_{m,\mathrm{Ran}}$ with tame commutator.

In light of the equivalence between factorization super line bundles over Gr_T and $\vartheta_+^{\mathrm{super}}(\Lambda)$ (Proposition 4.2.3), it remains to prove the following assertion.

Proposition 4.2.7. *The functor* (4.4) *induces an equivalence between:*

(1) *factorization super central extensions of* $\mathscr{L}\mathrm{T}$ *with tame commutator; and*
(2) *factorization super line bundles over* Gr_{T}.

Proof. Passing through the equivalence (4.7), we replace (4.4) by the forgetful functor:

$$\mathrm{Hom}_{\mathrm{fact}}(\mathrm{Hec}_{\mathrm{T}}, \mathsf{Pic}^{\mathrm{super}}) \to \Gamma_{\mathrm{fact}}(\mathrm{Gr}_{\mathrm{T}}, \mathsf{Pic}^{\mathrm{super}}), \tag{4.13}$$

defined via pullback along $\mathrm{Gr}_{\mathrm{T}} \to \mathrm{Hec}_{\mathrm{T}}$.

The desired equivalence amounts to showing that every factorization super line bundle $\mathscr{L}$ over Gr_{T} admits a unique collection of the following pieces of structure:

(1) an $\mathscr{L}^+\mathrm{T}$-equivariance structure;
(2) compatibility data with convolution, *i.e.* (4.5) and (4.6), on the factorization super line bundle over $\mathrm{Hec}_{\mathrm{T}}$ induced from structure (1),

subject to the *tameness condition*: the induced factorization super central extension of $\mathscr{L}\mathrm{T}$ has tame commutator.

Since the $\mathscr{L}^+\mathrm{T}$-action on Gr_{T} is trivial, we may view an $\mathscr{L}^+\mathrm{T}$-equivariance structure on $\mathscr{L}$ as a morphism:

$$\mathscr{L}^+\mathrm{T} \times_{\mathrm{Ran}} \mathrm{Gr}_{\mathrm{T}} \to \mathbb{G}_{m,\mathrm{Ran}}, \tag{4.14}$$

linear in the first variable. On the other hand, given a factorization super central extension of $\mathscr{L}\mathrm{T}$ with commutator $\langle\cdot,\cdot\rangle : \mathscr{L}\mathrm{T} \otimes \mathscr{L}\mathrm{T} \to \mathbb{G}_{m,\mathrm{Ran}}$, the $\mathscr{L}^+\mathrm{T}$-equivariance structure on the induced factorization super line bundle over Gr_{T} is precisely the map $\mathscr{L}^+\mathrm{T} \times \mathrm{Gr}_{\mathrm{T}} \to \mathbb{G}_{m,\mathrm{Ran}}$ associated to $\langle\cdot,\cdot\rangle$ by restriction (*cf.* §3.3.5). By the tameness condition, the morphism (4.14) must then be of the form $\langle\cdot,\cdot\rangle_{b_1}$ for some bilinear form:

$$b_1 : \Lambda \otimes \Lambda \to \mathbb{Z}. \tag{4.15}$$

Let us now analyze the compatibility data with convolution. The existence and uniqueness of the unital structure (4.6) follows from the canonical triviality of factorization super line bundles over Ran and the bilinearity of (4.15). The multiplicative structure (4.5) amounts to an isomorphism of factorization super line bundles over $\mathscr{L}\mathrm{T} \times^{\mathscr{L}^+\mathrm{T}} \mathrm{Gr}_{\mathrm{T}}$ equivariant against the leftmost $\mathscr{L}^+\mathrm{T}$-action. Triviality of the $\mathscr{L}^+\mathrm{T}$-action on Gr_{T} yields an isomorphism:

$$\mathscr{L}\mathrm{T} \times^{\mathscr{L}^+\mathrm{T}} \mathrm{Gr}_{\mathrm{T}} \cong \mathrm{Gr}_{\mathrm{T}} \times_{\mathrm{Ran}} \mathrm{Gr}_{\mathrm{T}},$$

Since the $\mathscr{L}^+\mathrm{T}$-equivariance is defined by b_1, the multiplicative structure (4.5) amounts to an isomorphism of factorization line bundles over $\mathrm{Gr}_{\mathrm{T}} \times \mathrm{Gr}_{\mathrm{T}}$:

$$m^*(\mathscr{L}) \cong (\mathscr{L} \boxtimes \mathscr{L}) \otimes \mathscr{O}(b_1), \tag{4.16}$$

where $\mathscr{O}(b_1)$ is the factorization line bundle associated to the form b_1 in §4.2.4.

Under the equivalence of Proposition 4.2.3, $\mathscr{L}$ corresponds to a pair (b, F_+). Applying Proposition 4.2.3 to $\mathrm{T} \times \mathrm{T}$, we see that (4.16) exists if and only if $b_1 = b$. Indeed, the quadratic form equates $\mathrm{Q}(\lambda_1 + \lambda_2)$ with $\mathrm{Q}(\lambda_1) + \mathrm{Q}(\lambda_2) + b_1(\lambda_1, \lambda_2)$ for each $\lambda_1, \lambda_2 \in \Lambda$, using Lemma 4.2.5(1). When $b_1 = b$, Lemma 4.2.5(2) yields a canonical isomorphism of ω-monoidal morphisms associated to the two sides of (4.16). The isomorphism (4.16) thus defined is the unique one satisfying the cocycle condition. □

4.2.8. Consider any factorization super central extension of $\mathscr{L}\mathrm{T}$ with tame commutator:

$$1 \to \mathbb{G}_{m,\mathrm{Ran}} \to \mathscr{T} \to \mathscr{L}\mathrm{T} \to 1. \tag{4.17}$$

The equivalences of Proposition 4.2.3, Proposition 4.2.7 show that (4.17) is classified by a pair (b, F_+) in $\vartheta_+^{\mathrm{super}}(\Lambda)$.

In the course of the proof of Proposition 4.2.7, we have also established the fact that the commutator pairing of (4.17) equals $\langle\cdot,\cdot\rangle_b$.

4.3. Simply connected groups.

4.3.1. Let G denote a semisimple and simply connected group X-scheme.

Since $\mathrm{Gr}_{\mathrm{G}} \to \mathrm{Ran}$ (resp. $\mathscr{L}\mathrm{G} \to \mathrm{Ran}$) has connected geometric fibers, every factorization super line bundle over Gr_{G} (resp. factorization super central extension of $\mathscr{L}\mathrm{G}$ by $\mathbb{G}_{m,\mathrm{Ran}}$) is pure of even grading.

Proposition 4.3.2. *If* G *is semisimple and simply connected, then the functor* (4.4) *is an equivalence between:*

1. *factorization central extensions of* $\mathscr{L}\mathrm{G}$*; and*
2. *factorization line bundles over* Gr_{G}.

4.3.3. Before proving Proposition 4.3.2, we define Schubert varieties in Gr_{G} as flat schematic morphisms to Ran. We give a detailed presentation because Lemma 4.3.4 below was also used in the proof of [TZ21, Lemma 3.6] but the justification there is inadequate.

Let us assume that G contains a Borel subgroup and a maximal torus $\mathrm{T} \subset \mathrm{B} \subset \mathrm{G}$. Denote by $\Lambda^+ \subset \Lambda$ the subsheaf of dominant cocharacters of T. For an I-tuple $\lambda^{\mathrm{I}} = (\lambda^i)$ of elements of Λ^+, we may view $\iota_{\lambda^{\mathrm{I}}}$ of §4.2.2 as a closed immersion $\mathrm{X}^{\mathrm{I}} \to \mathrm{Gr}_{\mathrm{G},\mathrm{X}^{\mathrm{I}}}$. Denote by $\mathrm{Gr}_{\mathrm{G}}^{\le\lambda^{\mathrm{I}}} \subset \mathrm{Gr}_{\mathrm{G},\mathrm{X}^{\mathrm{I}}}$ the schematic image of the map $\mathscr{L}^+_{\mathrm{X}^{\mathrm{I}}}\mathrm{G} \to \mathrm{Gr}_{\mathrm{G},\mathrm{X}^{\mathrm{I}}}$ defined by acting on $\iota_{\lambda^{\mathrm{I}}}(\mathrm{X}^{\mathrm{I}})$.

Since G is semisimple and simply connected, $\mathrm{Gr}_{\mathrm{G},\mathrm{X}^{\mathrm{I}}}$ is reduced and $\mathrm{Gr}_{\mathrm{G},\mathrm{X}^{\mathrm{I}}} \to \mathrm{X}^{\mathrm{I}}$ has connected geometric fibers. In particular, the above closed subschemes define an isomorphism of indschemes:

$$\operatorname*{colim}_{\lambda^{\mathrm{I}}} \mathrm{Gr}_{\mathrm{G}}^{\le\lambda^{\mathrm{I}}} \xrightarrow{\simeq} \mathrm{Gr}_{\mathrm{G},\mathrm{X}^{\mathrm{I}}}. \tag{4.18}$$

Lemma 4.3.4. *For each* I*-tuple* λ^{I} *of elements of* Λ^+*, the projection* $p : \mathrm{Gr}_{\mathrm{G}}^{\le\lambda^{\mathrm{I}}} \to \mathrm{X}^{\mathrm{I}}$ *is flat and the canonical map below is an isomorphism:*

$$\mathscr{O}_{\mathrm{X}^{\mathrm{I}}} \to \mathrm{R}p_*\mathscr{O}_{\mathrm{Gr}_{\mathrm{G}}^{\le\lambda^{\mathrm{I}}}}. \tag{4.19}$$

Proof. For $|\mathrm{I}| \le 2$, flatness of p is established in [Zhu09, §1.2]. The argument below which applies to general I is explained to me by João Lourenço. We call a morphism $f : \mathrm{Y}_1 \to \mathrm{Y}_2$ of schemes *derived* $\mathscr{O}$*-connected* if the induced map $\mathscr{O}_{\mathrm{Y}_2} \to \mathrm{R}f_*\mathscr{O}_{\mathrm{Y}_1}$ is an equivalence.

We begin by recalling some classical facts taking place over geometric points of X. Let x be a $\bar{k}$-point of X and $\mathrm{Gr}_{\mathrm{G},x}$ the fiber of Gr_{G}. Let W denote the Weyl group of (G,T) and $\mathrm{W}_{\mathrm{aff}} := \Lambda \rtimes \mathrm{W}$ its affinization. Write $\mathrm{I}_x \subset \mathscr{L}^+_x\mathrm{G}$ for the Iwahori group scheme associated to the Borel B. The affine flag variety $\mathrm{Fl}_{\mathrm{G},x} := \mathscr{L}_x\mathrm{G}/\mathrm{I}_x$ has I_x-orbits parametrized by $\mathrm{W}_{\mathrm{aff}}$. For each $\lambda \in \Lambda^+$, the preimage of $\mathrm{Gr}_{\mathrm{G},x}^{\le\lambda}$ along $\mathrm{Fl}_{\mathrm{G},x} \to \mathrm{Gr}_{\mathrm{G},x}$ concides with $\mathrm{Fl}_{\mathrm{G},x}^{\le w(\lambda)}$, the closure of the I_x-orbit corresponding to the longest element $w(\lambda)$ in $\mathrm{W}\lambda\mathrm{W} \subset \mathrm{W}_{\mathrm{aff}}$.

The scheme $\mathrm{Fl}_{\mathrm{G},x}^{\le w(\lambda)}$ admits a Demazure resolution $\mathrm{D}^{w(\lambda)}$ associated to any reduced expression of $w(\lambda)$:

$$\mathrm{D}^{w(\lambda)} \xrightarrow{\pi} \mathrm{Fl}_{\mathrm{G},x}^{\le w(\lambda)} \to \mathrm{Gr}_{\mathrm{G},x}^{\le\lambda}. \tag{4.20}$$

By [Fal03, Theorem 8], the morphism π is derived $\mathscr{O}$-connected. Since $\mathrm{G}/\mathrm{B} \to \mathrm{Spec}(\bar{k})$ is derived $\mathscr{O}$-connected [Kem76], the same holds for the composition (4.20). We collect two consequences of this fact:

1. $\mathrm{Gr}_{\mathrm{G},x}^{\le\lambda} \to \mathrm{Spec}(\bar{k})$ is derived $\mathscr{O}$-connected; indeed, this is because $\mathrm{D}^{w(\lambda)} \to \mathrm{Spec}(\bar{k})$ is derived $\mathscr{O}$-connected, being an iterated $\mathbb{P}^1$-bundle.

(2) the convolution map $\mathrm{Gr}^{\le\lambda_1}_{\mathrm{G},x} \widetilde{\times} \mathrm{Gr}^{\le\lambda_2}_{\mathrm{G},x} \to \mathrm{Gr}^{\le\lambda_1+\lambda_2}_{\mathrm{G},x}$ is derived $\mathscr{O}$-connected; indeed, the source and target both admit rational resolutions in the sense of [Kov22, Definition 9.1], so this claim follows from [Kov22, Theorem 9.12(i)].

To prove that $p : \mathrm{Gr}^{\le\lambda^{\mathrm{I}}}_{\mathrm{G}} \to \mathrm{X}^{\mathrm{I}}$ is flat, we consider the global convolution map $m : \widetilde{\mathrm{Gr}}^{\le\lambda^{\mathrm{I}}}_{\mathrm{G}} \to \mathrm{Gr}^{\le\lambda^{\mathrm{I}}}_{\mathrm{G}}$ over X^{I}, where the composition $\widetilde{\mathrm{Gr}}^{\le\lambda^{\mathrm{I}}}_{\mathrm{G}} \to \mathrm{X}^{\mathrm{I}}$ is evidently flat. Statement (2) implies that for each $\bar{k}$-point x^{I} of X^{I}, the base change $m_{x^{\mathrm{I}}}$ of m satisfies $\mathrm{R}^i(m_{x^{\mathrm{I}}})_*\mathscr{O} \cong 0$ for $i \ge 1$. By [Gör03, Proposition 3.13], $m_*\mathscr{O}$ is X^{I}-flat and its formation is compatible with base change along X^{I}. Since $\mathrm{Gr}^{\le\lambda^{\mathrm{I}}}_{\mathrm{G}}$ is reduced and m is surjective on $\bar{k}$-points, we see that $m_*\mathscr{O}$ coincides with the structure sheaf of $\mathrm{Gr}^{\le\lambda^{\mathrm{I}}}_{\mathrm{G}}$. This implies that p is flat and its geometric fibers are identified with the corresponding Schubert varieties. The derived $\mathscr{O}$-connectedness (4.19) then follows from its pointwise version, *i.e.* statement (1) above. □

4.3.5. Lemma 4.3.4 has the following consequence: given any S-point $x^{\mathrm{I}} : \mathrm{S} \to \mathrm{X}^{\mathrm{I}}$, a line bundle $\mathscr{L}$ over $\mathrm{Gr}_{\mathrm{G},x^{\mathrm{I}}}$ descends to S if and only if it is trivial over all geometric fibers. Indeed, if $\mathscr{L}$ is trivial over all geometric fibers, then by Lemma 4.3.4 and cohomology and base change, the derived pushforward of $\mathscr{L}$ to S yields the desired descent.

On the other hand, [Fal03, Theorem 7] proves that the Picard group of $\mathrm{Gr}_{\mathrm{G},x}$, for any $\bar{k}$-point x of X, is isomorphic to $\mathbb{Z}^r$, where r denotes the number of simple factors of G, with $(1,\cdots,1) \in \mathbb{Z}^r$ corresponding to an ample line bundle over $\mathrm{Gr}_{\mathrm{G},x}$.

Proof of Proposition 4.3.2. The problem is of étale locally nature on X, so we may assume that G contains a Borel subgroup and a maximal torus $\mathrm{T} \subset \mathrm{B} \subset \mathrm{G}$.

Let $\mathscr{L}$ be a factorization line bundle over Gr_{G}. According to §4.1.5, it suffices to prove that $\mathscr{L}$ admits a unique $\mathscr{L}^+\mathrm{G}$-equivariance structure and the induced factorization line bundle over $\mathrm{Hec}_{\mathrm{G}}$ admits unique compatibility data with respect to convolution.

Consider an S-point (x^{I}, g) of $\mathscr{L}^+_{\mathrm{X}^{\mathrm{I}}}\mathrm{G}$. The action by g defines an automorphism act_g of $\mathrm{Gr}_{\mathrm{G},x^{\mathrm{I}}}$. There is a unique isomorphism:

$$(\mathrm{act}_g)^*\mathscr{L} \xrightarrow{\simeq} \mathscr{L} \tag{4.21}$$

extending the identity over the unit section $e : \mathrm{S} \to \mathrm{Gr}_{\mathrm{G},x^{\mathrm{I}}}$. Indeed, this is because the difference $(\mathrm{act}_g)^*\mathscr{L} \otimes \mathscr{L}^{-1}$ is trivial along geometric fibers, so we may apply the observation of §4.3.5. The uniqueness of (4.21) implies that it satisfies the cocycle condition.

The compatibility data with respect to convolution consist of isomorphisms (4.5) and (4.6). The second isomorphism is clear. The first one amounts to an isomorphism of line bundles over:

$$\mathrm{Gr}_{\mathrm{G}} \widetilde{\times} \mathrm{Gr}_{\mathrm{G}} := \mathscr{L}\mathrm{G} \times^{\mathscr{L}^+\mathrm{G}} \mathrm{Gr}_{\mathrm{G}},$$

compatible with the left $\mathscr{L}^+\mathrm{G}$-equivariance. This isomorphism is constructed in the same way as (4.21), by reducing to geometric fibers over Ran. □

4.3.6. Suppose that G contains a maximal torus T with sheaf of cocharacters Λ. The Weyl group W acts naturally on Λ. Restricting along $\mathrm{T} \subset \mathrm{G}$ and applying Proposition 4.2.3, each factorization line bundle over Gr_{G} defines a pair (b, F_+) with $b(\lambda,\lambda) \in 2\mathbb{Z}$, hence quadratic form $\mathrm{Q} : \lambda \mapsto b(\lambda,\lambda)/2$ on Λ.

By [TZ21, Proposition 2.5], the quadratic form Q is Weyl-invariant and this procedure defines an equivalence of Picard groupoids between factorization line bundles over Gr_{G} and Weyl-invariant quadratic forms $\mathrm{Quad}(\Lambda,\mathbb{Z})^{\mathrm{W}}$ on Λ. (Evaluation on short coroots belonging to each simple factor of G defines an isomorphism $\mathrm{Quad}(\Lambda,\mathbb{Z})^{\mathrm{W}} \cong \mathbb{Z}^{\oplus r}$.)

In summary, all of the Picard groupoids below are canonically equivalent when G is semisimple and simply connected:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{fact}}(\mathscr{L}\mathrm{G}, \mathsf{Pic}) & \xrightarrow{\simeq} & \mathrm{Hom}_{\mathrm{fact}}(\mathscr{L}\mathrm{G}, \mathsf{Pic}^{\mathrm{super}}) \\ \downarrow\cong & & \downarrow\cong \\ \Gamma_{\mathrm{fact}}(\mathrm{Gr}_{\mathrm{G}}, \mathsf{Pic}) & \xrightarrow{\simeq} & \Gamma_{\mathrm{fact}}(\mathrm{Gr}_{\mathrm{G}}, \mathsf{Pic}^{\mathrm{super}}) \\ \downarrow\cong & & \\ \mathrm{Quad}(\Lambda, \mathbb{Z})^{\mathrm{W}} & & \end{array} \tag{4.22}$$

In particular, composing these equivalences with the restriction along $\mathrm{T} \subset \mathrm{G}$ and the functor (4.8), we obtain a functor:

$$\mathrm{Quad}(\Lambda, \mathbb{Z})^{\mathrm{W}} \to \vartheta_+^{\mathrm{super}}(\Lambda). \tag{4.23}$$

In [TZ21, §2.4.7], we have verified that the image of Q is the pair (b, F_+), where F_+ is the ω-twist of the monoidal morphism F_{Q} defined in §3.4.2.

Remark 4.3.7. The equivalences in (4.22) remain valid when factorization central extensions of $\mathscr{L}\mathrm{G}$ (resp. factorization line bundles over Gr_{G}) are replaced by central extensions of $\mathscr{L}_x\mathrm{G}$ (resp. line bundles over $\mathrm{Gr}_{\mathrm{G},x}$) for any geometric point x of X.

4.3.8. Let us relax the hypothesis and let G be any reductive group X-scheme. We shall now use our knowledge about the simply connected case to perform a commutator calculation.

Denote by G_{sc} the simply connected form of G and G_{ad} the adjoint form of G. The G_{ad}-action on G by conjugation extends to a G_{ad}-action on G_{sc}, which we still refer to as the *conjugation* action.

Consider any factorization central extension:

$$1 \to \mathbb{G}_{m,\mathrm{Ran}} \to \mathscr{G}_{\mathrm{sc}} \to \mathscr{L}\mathrm{G}_{\mathrm{sc}} \to 1. \tag{4.24}$$

Claim: the conjugation $\mathscr{L}\mathrm{G}_{\mathrm{ad}}$-action on $\mathscr{L}\mathrm{G}_{\mathrm{sc}}$ extends uniquely to $\mathscr{G}_{\mathrm{sc}}$.

Indeed, let (x^{I}, g) be an S-point of $\mathscr{L}\mathrm{G}_{\mathrm{ad}}$. Action by g defines an automorphism act_g of $\mathscr{L}_{x^{\mathrm{I}}}\mathrm{G}_{\mathrm{sc}}$. Viewing $\mathscr{G}_{\mathrm{sc}}$ as a multiplicative line bundle over $\mathscr{L}\mathrm{G}_{\mathrm{sc}}$, we shall argue that there is a unique isomorphism:

$$(\mathrm{act}_g)^*\mathscr{G}_{\mathrm{sc},x^{\mathrm{I}}} \xrightarrow{\simeq} \mathscr{G}_{\mathrm{sc},x^{\mathrm{I}}}, \tag{4.25}$$

compatible with the multiplicative structure of $\mathscr{G}_{\mathrm{sc},x^{\mathrm{I}}}$. According to §4.3.5, it suffices to show that the two sides of (4.25) are isomorphic on geometric fibers over S. This statement holds by Remark 4.3.7, seeing that $(\mathrm{act}_g)^*$ induces the identity map on $\mathrm{Quad}(\Lambda, \mathbb{Z})^{\mathrm{W}}$.

4.3.9. Suppose that G contains a maximal torus T with sheaf of cocharacters Λ. Write T_{sc}, T_{ad} for the induced maximal tori in G_{sc}, G_{ad}, with sheaves of cocharacters Λ_{sc}, Λ_{ad}.

The $\mathscr{L}\mathrm{G}_{\mathrm{ad}}$-action on $\mathscr{G}_{\mathrm{sc}}$ constructed in §4.3.8 restricts to an $\mathscr{L}\mathrm{T}_{\mathrm{ad}}$-action, and we obtain a factorization pairing:

$$\mathscr{L}\mathrm{T}_{\mathrm{ad}} \otimes \mathscr{L}\mathrm{T}_{\mathrm{sc}} \to \mathbb{G}_{m,\mathrm{Ran}}, \quad (t_{\mathrm{ad}}, t_{\mathrm{sc}}) \mapsto (t_{\mathrm{ad}}\tilde{t}_{\mathrm{sc}}t_{\mathrm{ad}}^{-1}) \cdot \tilde{t}_{\mathrm{sc}}^{-1}, \tag{4.26}$$

where $\tilde{t}_{\mathrm{sc}}$ is an arbitrary lift of t_{sc} to $\mathscr{G}_{\mathrm{sc}}$, which exists locally.

To compute this pairing, we recall that $\mathscr{G}_{\mathrm{sc}}$ is classified by a Weyl-invariant quadratic form Q_{sc} on Λ_{sc}. Since Λ_{ad} is canonically dual to the root lattice, the formula:

$$(\lambda, \alpha) \mapsto \mathrm{Q}_{\mathrm{sc}}(\alpha)\langle\lambda, \check{\alpha}\rangle$$

for each $\lambda \in \Lambda_{\mathrm{ad}}$ and coroot $\alpha \in \Lambda_{\mathrm{sc}}$ yields a pairing $b_1 : \Lambda_{\mathrm{ad}} \otimes \Lambda_{\mathrm{sc}} \to \mathbb{Z}$.

Lemma 4.3.10. *The factorization pairing* (4.26) *equals* $\langle\cdot,\cdot\rangle_{b_1}$ *(in the notation of §3.3.3).*

Proof. The problem is of étale local nature on X, so we may assume that G is split and T is a split maximal torus. Each coroot α induces a morphism $f_\alpha : \mathrm{SL}_2 \to \mathrm{G}_{\mathrm{sc}}$, sending the upper-triangular unipotent matrices to the root subgroup $\mathrm{U}_\alpha \subset \mathrm{G}_{\mathrm{sc}}$ and restricts to α on the diagonal torus $\mathbb{G}_m \subset \mathrm{SL}_2$.

The central extension (4.26) restricts along f_α to the $\mathrm{Q}_{\mathrm{sc}}(\alpha)$-multiple of the factorization central extension:

$$1 \to \mathbb{G}_{m,\mathrm{Ran}} \to \widetilde{\mathrm{SL}}_2 \to \mathscr{L}\mathrm{SL}_2 \to 1, \tag{4.27}$$

defined by restricting the Tate central extension $\widetilde{\mathrm{GL}}_2$ along $\mathscr{L}\mathrm{SL}_2 \subset \mathscr{L}\mathrm{GL}_2$. Indeed, factorization central extensions of $\mathscr{L}\mathrm{SL}_2$ are uniquely determined by their quadratic forms (§4.3.6), hence by the commutator of their restrictions to $\mathscr{L}\mathbb{G}_m$, so we conclude by §4.2.8 and our definition of the Contou-Carrère symbol.

By functoriality of the construction, it suffices to show that (4.26) equals the Contou-Carrère pairing for $\mathrm{G} = \mathrm{SL}_2$ and $\mathscr{G}_{\mathrm{sc}}$ being the central extension (4.27). Note that the T_{ad} $(= \mathbb{G}_m)$-action on G $(= \mathrm{SL}_2)$ extends to the inner action of $\mathbb{G}_m$ on GL_2 as the subgroup:

$$\mathbb{G}_m \subset \mathrm{GL}_2, \quad a \mapsto \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.28}$$

Using the group structure on $\widetilde{\mathrm{GL}}_2$, we extend the induced inner $\mathscr{L}\mathbb{G}_m$-action on $\mathscr{L}\mathrm{GL}_2$ to an action on $\widetilde{\mathrm{GL}}_2$. This action must restrict to the T_{ad}-action on the subgroup $\widetilde{\mathrm{SL}}_2 \subset \widetilde{\mathrm{GL}}_2$, by the uniqueness of the latter (§4.3.8). Therefore, it remains to prove that the commutator pairing in $\widetilde{\mathrm{GL}}_2$ between the subtorus (4.28) and the subtorus:

$$\mathbb{G}_m \subset \mathrm{GL}_2, \quad a \mapsto \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$$

is the Contou-Carrère symbol. This assertion holds because $\widetilde{\mathrm{GL}}_2$ restricts to the Tate central extension $\widetilde{\mathbb{G}}_m$ of $\mathscr{L}\mathbb{G}_m$ along (4.28). □

4.4. Descent.

4.4.1. Let $\mathscr{K}$ and $\mathscr{H}$ be factorization group presheaves whose base changes along any S-point of Ran are fppf sheaves.

An action of $\mathscr{H}$ on $\mathscr{K}$ as group presheaves is *compatible with factorization* if for any disjoint S-points $x^{\mathrm{I}}, x^{\mathrm{J}}$ of Ran, the $\mathscr{H}_{x^{\mathrm{I}}\sqcup x^{\mathrm{J}}}$-action on $\mathscr{K}_{x^{\mathrm{I}}\sqcup x^{\mathrm{J}}}$ coincides with the $\mathscr{H}_{x^{\mathrm{I}}} \times \mathscr{H}_{x^{\mathrm{J}}}$-action on $\mathscr{K}_{x^{\mathrm{I}}} \times \mathscr{K}_{x^{\mathrm{J}}}$ under the factorization isomorphisms of $\mathscr{H}$ and $\mathscr{K}$. When this happens, the group presheaf $\mathscr{K} \rtimes \mathscr{H}$ over Ran inherits a factorization structure.

Consider a factorization super central extension:

$$1 \to \mathbb{G}_m \to \widetilde{\mathscr{K}} \to \mathscr{K} \to 1.$$

Suppose that $\widetilde{\mathscr{K}}$ is equipped with an $\mathscr{H}$-action which is trivial on A. Then we say that the $\mathscr{H}$-action on $\widetilde{\mathscr{K}}$ is *compatible with factorization* if for disjoint S-points x^{I}, x^{J} of Ran, the $\mathscr{H}_{x^{\mathrm{I}}\sqcup x^{\mathrm{J}}}$-action on $\widetilde{\mathscr{K}}_{x^{\mathrm{I}}\sqcup x^{\mathrm{J}}}$ coincides with the induced $\mathscr{H}_{x^{\mathrm{I}}} \times \mathscr{H}_{x^{\mathrm{J}}}$-action on the quotient:

$$\widetilde{\mathscr{K}}_{x^{\mathrm{I}}} \times \widetilde{\mathscr{K}}_{x^{\mathrm{J}}} \twoheadrightarrow \widetilde{\mathscr{K}}_{x^{\mathrm{I}}\sqcup x^{\mathrm{J}}}.$$

The following lemma is a variant of [BD01, Construction 1.7].

Lemma 4.4.2. *Let $\mathscr{K}$, $\mathscr{H}$ be as in §4.4.1 with $\mathscr{H}$ acting on $\mathscr{K}$ compatibly with factorization. The following categories are equivalent:*

(1) *factorization super central extensions of $\mathscr{K} \rtimes \mathscr{H}$ by $\mathbb{G}_{m,\mathrm{Ran}}$;*

(2) *triples* $(\widetilde{\mathscr{K}}, \widetilde{\mathscr{H}}, \alpha)$*, where* $\widetilde{\mathscr{K}}$ *(resp.* $\widetilde{\mathscr{H}}$*) is a factorization super central extension of* $\mathscr{K}$ *(resp.* $\mathscr{H}$*) by* $\mathbb{G}_{m,\mathrm{Ran}}$*, and* α *is an* $\mathscr{H}$*-action on* $\widetilde{\mathscr{K}}$ *which is trivial on* $\mathbb{G}_{m,\mathrm{Ran}}$*, compatible with factorization, and induces the given* $\mathscr{H}$*-action on* $\mathscr{K}$*.*

Proof. The functor (1) $\Rightarrow$ (2) is given by restricting a factorization central extension of $\mathscr{K} \rtimes \mathscr{H}$ along the group sub-presheaves $\mathscr{K} \subset \mathscr{K} \rtimes \mathscr{H}$, $\mathscr{H} \subset \mathscr{K} \rtimes \mathscr{H}$ to obtain $\widetilde{\mathscr{K}}$, $\widetilde{\mathscr{H}}$, and observing that the $\widetilde{\mathscr{H}}$-action on $\widetilde{\mathscr{K}}$ factors through $\mathscr{H}$.

The functor (2) $\Rightarrow$ (1) is given by forming the central extension $\widetilde{\mathscr{K}} \rtimes \widetilde{\mathscr{H}}$ of $\mathscr{K} \rtimes \mathscr{H}$ by $\mathbb{G}_{m,\mathrm{Ran}} \times \mathbb{G}_{m,\mathrm{Ran}}$ using the action α, and pushing out along the product map on $\mathbb{G}_{m,\mathrm{Ran}}$. □

Proposition 4.4.3. *Let* G *be a reductive group* X*-scheme with a maximal torus* T*. Denote by* G_{sc} *the simply connected form of* G*, equipped with the conjugation* T*-action. The following categories are canonically equivalent:*

(1) *factorization super central extensions of* $\mathscr{L}\mathrm{G}_{\mathrm{sc}} \rtimes \mathscr{L}\mathrm{T}$ *by* $\mathbb{G}_{m,\mathrm{Ran}}$*;*
(2) *pairs* $(\mathscr{G}_{\mathrm{sc}}, \mathscr{T})$*, where* $\mathscr{G}_{\mathrm{sc}}$ *(resp.* $\mathscr{T}$*) is a factorization super central extension of* $\mathscr{L}\mathrm{G}_{\mathrm{sc}}$ *(resp.* $\mathscr{L}\mathrm{T}$*) by* $\mathbb{G}_{m,\mathrm{Ran}}$*.*

Proof. We appeal to the equivalence of Lemma 4.4.2. It suffices to prove that given any pair $(\mathscr{G}_{\mathrm{sc}}, \mathscr{T})$ in (2), there is a unique $\mathscr{L}\mathrm{T}$-action on $\mathscr{G}_{\mathrm{sc}}$ which is trivial on $\mathbb{G}_{m,\mathrm{Ran}}$, compatible with factorization, and induces the conjugation action on $\mathscr{L}\mathrm{G}_{\mathrm{sc}}$. This is established in §4.3.8 when T is replaced by G_{ad}, but the argument for uniqueness carries over. □

4.4.4. Under the equivalence of Proposition 4.4.3, we shall write the factorization super central extension of $\mathscr{L}\mathrm{G}_{\mathrm{sc}} \rtimes \mathscr{L}\mathrm{T}$ induced from $(\mathscr{G}_{\mathrm{sc}}, \mathscr{T})$ as follows:

$$1 \to \mathbb{G}_{m,\mathrm{Ran}} \to \mathscr{G}_{\mathrm{sc}} \widetilde{\rtimes} \mathscr{T} \to \mathscr{L}\mathrm{G}_{\mathrm{sc}} \rtimes \mathscr{L}\mathrm{T} \to 1.$$

It is by construction the pushout of $\mathscr{G}_{\mathrm{sc}} \rtimes \mathscr{T}$ along the product map on $\mathbb{G}_{m,\mathrm{Ran}}$.

Lemma 4.4.5. *Let* $\widetilde{\mathrm{G}} \to \mathrm{G}$ *be a surjection of reductive group* X*-schemes whose kernel is a torus. The induced morphism* $\mathscr{L}\widetilde{\mathrm{G}} \to \mathscr{L}\mathrm{G}$ *is surjective in the topology generated by fpqc and proper covers.*

Proof. We shall deduce this from two statements:

(1) $\mathscr{L}^+\widetilde{\mathrm{G}} \to \mathscr{L}^+\mathrm{G}$ is surjective in the fpqc topology;
(2) $\mathrm{Gr}_{\widetilde{\mathrm{G}}} \to \mathrm{Gr}_{\mathrm{G}}$ is surjective in the topology generated by proper covers.

Let us prove the lemma assuming both statements. Indeed, the morphism $\mathscr{L}\widetilde{\mathrm{G}} \to \mathscr{L}\mathrm{G}$ factors as:

$$\mathscr{L}\widetilde{\mathrm{G}} \to \mathscr{L}\mathrm{G} \times_{\mathrm{Gr}_{\mathrm{G}}} \mathrm{Gr}_{\widetilde{\mathrm{G}}} \to \mathscr{L}\mathrm{G}.$$

By statement (2), the second morphism is surjective in the proper topology. We claim that the first morphism is surjective in the fpqc topology. Indeed, consider an S-point (g, x) of $\mathscr{L}\mathrm{G} \times_{\mathrm{Gr}_{\mathrm{G}}} \mathrm{Gr}_{\widetilde{\mathrm{G}}}$. Since $\mathscr{L}\widetilde{\mathrm{G}} \to \mathrm{Gr}_{\widetilde{\mathrm{G}}}$ is surjective in the étale topology, we may lift x to an S_1-point $\tilde{g}$ of $\mathscr{L}\widetilde{\mathrm{G}}$ over some étale cover $\mathrm{S}_1 \to \mathrm{S}$. The image of $\tilde{g}$ in $\mathscr{L}\mathrm{G}$ differs from g by an S_1-point h of $\mathscr{L}^+\mathrm{G}$. Statement (1) allows us to lift h to an S_2-point $\tilde{h}$ in $\mathscr{L}^+\widetilde{\mathrm{G}}$ over some fpqc cover $\mathrm{S}_2 \to \mathrm{S}_1$, which we may then use to modify $\tilde{g}$ to obtain a lift of (g, x).

To prove statement (1), we consider an S-point x^{I} of X^{I} with graph $\Gamma_{x^{\mathrm{I}}} \subset \mathrm{S} \times \mathrm{X}$. The base change $\mathscr{L}^+_{x^{\mathrm{I}}}\widetilde{\mathrm{G}} \to \mathscr{L}^+_{x^{\mathrm{I}}}\mathrm{G}$ is the inverse limit of morphisms:

$$\mathrm{R}_{\Gamma^{(n)}}\widetilde{\mathrm{G}} \to \mathrm{R}_{\Gamma^{(n)}}\mathrm{G}, \tag{4.29}$$

which are the Weil restrictions of $\widetilde{\mathrm{G}} \to \mathrm{G}$ along the finite locally free morphisms $\Gamma^{(n)}_{x^{\mathrm{I}}} \to \mathrm{S}$, see §3.2.3. Since $\widetilde{\mathrm{G}} \to \mathrm{G}$ is affine, smooth, and surjective, the same holds for (4.29). Hence $\mathscr{L}^+_{x^{\mathrm{I}}}\widetilde{\mathrm{G}} \to \mathscr{L}^+_{x^{\mathrm{I}}}\mathrm{G}$ is affine, *flat*, and surjective.

We now turn to statement (2). Since the formation of the affine Grassmannian is compatible with étale base change, we may assume that the kernel of $\widetilde{\mathrm{G}} \to \mathrm{G}$ is a *split* torus T. This implies that $\mathscr{L}\widetilde{\mathrm{G}} \to \mathscr{L}\mathrm{G}$ is surjective on field-valued points. Indeed, given any k-field F, the graph of an F-point of X^{I} is the disjoint union of schemes isomorphic to $\mathrm{Spec}(\mathrm{F}((t)))$, but the map $\widetilde{\mathrm{G}}(\mathrm{F}((t))) \to \mathrm{G}(\mathrm{F}((t)))$ is surjective because $\mathrm{H}^1(\mathrm{F}((t)), \mathrm{T}) = 0$. In particular, any F-point of Gr_{G} lifts to $\mathrm{Gr}_{\widetilde{\mathrm{G}}}$ after a finite extension $\mathrm{F} \subset \mathrm{F}_1$.

On the other hand, the morphism $\mathrm{Gr}_{\widetilde{\mathrm{G}}} \to \mathrm{Gr}_{\mathrm{G}}$ is ind-proper because $\widetilde{\mathrm{G}}$ is reductive. Since Gr_{G} is of finite presentation, taking schematic points of Gr_{G} puts us in the following situation: an affine k-scheme S of finite type, an ind-proper S-indscheme Y, such that $\mathrm{Y} \to \mathrm{S}$ is surjective on field-valued points up to finite extension. We claim that *some closed subscheme* $\mathrm{Y}_i \subset \mathrm{Y}$ *surjects onto* S. Then $\mathrm{Y}_i \to \mathrm{S}$ is a proper cover which lifts to Y.

Let us prove the claim. Since S is of finite type over k, we reduce to the case where S is irreducible. Then its generic point lifts to some Y_i after a finite extension. Since $\mathrm{Y}_i \to \mathrm{S}$ is proper, its image contains the closure of the generic point which is all of S. □

4.4.6. Let G denote a reductive group X-scheme with a maximal torus T. Denote by G_{sc} the simply connected form of G and $\mathrm{T}_{\mathrm{sc}} \subset \mathrm{G}_{\mathrm{sc}}$ the preimage of T. The T-action on G by conjugation extends to G_{sc}. There is a short exact sequence:

$$1 \to \mathrm{T}_{\mathrm{sc}} \to \mathrm{G}_{\mathrm{sc}} \rtimes \mathrm{T} \to \mathrm{G} \to 1, \tag{4.30}$$

where the first map is the anti-diagonal embedding $t \mapsto (t, t^{-1})$. Furthermore, its image is central in $\mathrm{G}_{\mathrm{sc}} \rtimes \mathrm{T}$.

The exact sequence (4.30) induces an exact sequence of factorization group presheaves:

$$1 \to \mathscr{L}\mathrm{T}_{\mathrm{sc}} \to \mathscr{L}\mathrm{G}_{\mathrm{sc}} \rtimes \mathscr{L}\mathrm{T} \to \mathscr{L}\mathrm{G}, \tag{4.31}$$

where the last map is surjective in the topology generated by fpqc and proper covers. Since perfect complexes satisfy derived proper descent [Cho22, Theorem 1.8] and loop groups are classical [GR14, Theorem 9.3.5], central extensions of $\mathscr{L}\mathrm{G}$ by $\mathbb{G}_{m,\mathrm{Ran}}$ are equivalent to those of $\mathscr{L}\mathrm{G}_{\mathrm{sc}} \rtimes \mathscr{L}\mathrm{T}$ by $\mathbb{G}_{m,\mathrm{Ran}}$ equipped with a splitting over $\mathscr{L}\mathrm{T}_{\mathrm{sc}}$ whose image is normal. (This idea is due to Gaitsgory, *cf.* [Gai20, Corollary 5.2.7].)

4.4.7. Let (G, T) be as above. Appealing to Proposition 4.4.3, we obtain an equivalence of Picard groupoids between:

(1) factorization super central extensions of $\mathscr{L}\mathrm{G}$ by $\mathbb{G}_{m,\mathrm{Ran}}$; and
(2) triples $(\mathscr{T}, \mathscr{G}_{\mathrm{sc}}, \varphi)$, where $\mathscr{T}$ and $\mathscr{G}_{\mathrm{sc}}$ are factorization super central extensions:

$$1 \to \mathbb{G}_{m,\mathrm{Ran}} \to \mathscr{T} \to \mathscr{L}\mathrm{T} \to 1, \tag{4.32}$$

$$1 \to \mathbb{G}_{m,\mathrm{Ran}} \to \mathscr{G}_{\mathrm{sc}} \to \mathscr{L}\mathrm{G}_{\mathrm{sc}} \to 1, \tag{4.33}$$

and φ is an isomorphism of their pullbacks to $\mathscr{L}\mathrm{T}_{\mathrm{sc}}$, subject to the *normality condition* that the section $\mathscr{L}\mathrm{T}_{\mathrm{sc}} \to \mathscr{G}_{\mathrm{sc}} \widetilde{\rtimes} \mathscr{T}$ induced from φ has normal image.

Lemma 4.4.8. *A factorization super central extension of* $\mathscr{L}\mathrm{G}$ *by* $\mathbb{G}_{m,\mathrm{Ran}}$ *has tame commutator if and only if its restriction to* $\mathscr{L}\mathrm{T}$ *does.*

Proof. Note that any factorization (super) central extension of $\mathscr{L}\mathrm{G}_{\mathrm{sc}}$ by $\mathbb{G}_{m,\mathrm{Ran}}$ has tame commutator (Lemma 4.3.10). The claim now follows from Lemma 3.3.5, seeing that $\mathrm{Rad}(\mathrm{G}) \times \mathrm{T}_{\mathrm{sc}} \to \mathrm{T}$ is an isogeny of X-tori. □

4.4.9. Consider a triple $(\mathscr{T}, \mathscr{G}_{\mathrm{sc}}, \varphi)$ as in §4.4.7 and assume that $\mathscr{T}$ has tame commutator.

In particular, this implies that its commutator is $\langle \cdot, \cdot \rangle_b$ where b is the symmetric bilinear form appearing in the classifying data of $\mathscr{T}$, see §4.2.8.

Under this assumption, we shall make the normality condition of §4.4.7(2) explicit.

Lemma 4.4.10. *If $\mathscr{T}$ has commutator $\langle\cdot,\cdot\rangle_b$, then the normality condition holds if and only if b is Weyl-invariant.*

Proof. Let Λ (resp. Λ_{ad}, Λ_{sc}) denote the sheaf of cocharacters of T (resp. T_{ad}, T_{sc}). Note that $\mathscr{T}$ has commutator $\langle\cdot,\cdot\rangle_b$ while $\mathscr{G}_{\mathrm{sc}}$ defines the pairing $\langle\cdot,\cdot\rangle_{b_1}$ via its $\mathscr{L}\mathrm{T}_{\mathrm{ad}}$-action (see Lemma 4.3.10). The existence of φ implies that b and b_1 agree on $\Lambda_{\mathrm{sc}}\otimes\Lambda_{\mathrm{sc}}$. In particular, the restriction of b to Λ_{sc} comes from a Weyl-invariant quadratic form Q.

We shall prove that the normality condition holds if and only if b and b_1 coincide over $\Lambda\otimes\Lambda_{\mathrm{sc}}$. This latter condition means that for each $\lambda\in\Lambda$ and root $\alpha\in\Lambda_{\mathrm{sc}}$, there holds:

$$b(\lambda,\alpha)=\mathrm{Q}(\alpha)\langle\check{\alpha},\lambda\rangle. \tag{4.34}$$

The equality (4.34) is equivalent to the Weyl-invariance of b.

Let us now analyze the normality condition. The section induced from φ has the following description on S-points:

$$\mathscr{L}\mathrm{T}_{\mathrm{sc}}\to\mathscr{G}_{\mathrm{sc}}\widetilde{\rtimes}\mathscr{T},\quad t_{\mathrm{sc}}\mapsto(\tilde{t}_{\mathrm{sc}},\varphi(\tilde{t}_{\mathrm{sc}})^{-1}), \tag{4.35}$$

where $\tilde{t}_{\mathrm{sc}}$ is any lift of t_{sc} to $\mathscr{G}_{\mathrm{sc}}$ and $\varphi(\tilde{t}_{\mathrm{sc}})$ is its image in $\mathscr{T}$ under φ. Since $\mathscr{L}\mathrm{T}_{\mathrm{sc}}$ is a central subgroup of $\mathscr{L}\mathrm{G}_{\mathrm{sc}}\rtimes\mathscr{L}\mathrm{T}$, the image of (4.35) is normal if and only if it is central. This condition translates to the following equality in $\mathscr{G}_{\mathrm{sc}}\widetilde{\rtimes}\mathscr{T}$:

$$(g_{\mathrm{sc}},t)\cdot(\tilde{t}_{\mathrm{sc}},\varphi(\tilde{t}_{\mathrm{sc}})^{-1})\cdot(g_{\mathrm{sc}},t)^{-1}=(\tilde{t}_{\mathrm{sc}},\varphi(\tilde{t}_{\mathrm{sc}})^{-1}), \tag{4.36}$$

for all S-points (g_{sc},t) of $\mathscr{L}\mathrm{G}_{\mathrm{sc}}\rtimes\mathscr{L}\mathrm{T}$ and t_{sc} of $\mathscr{L}\mathrm{T}_{\mathrm{sc}}$.

The left-hand-side of (4.36) computes to $(\langle(t,\tilde{t}_{\mathrm{sc}}\rangle_{b_1}\tilde{t}_{\mathrm{sc}},\langle t,\varphi(\tilde{t}_{\mathrm{sc}})^{-1}\rangle_b\varphi(\tilde{t}_{\mathrm{sc}})^{-1})$. Its equality with the right-hand-side amounts to the equality:

$$\langle t,\tilde{t}_{\mathrm{sc}}\rangle_{b_1}=\langle t,\varphi(\tilde{t}_{\mathrm{sc}})\rangle_b,$$

for all S-points t of $\mathscr{L}\mathrm{T}$ and t_{sc} of $\mathscr{L}\mathrm{T}_{\mathrm{sc}}$, *i.e.* the agreement of b and b_1 over $\Lambda\otimes\Lambda_{\mathrm{sc}}$. □

4.4.11. Define the Picard groupoid $\widetilde{\vartheta}^{\mathrm{super}}_{\mathrm{G},+}(\Lambda)$ by the Cartesian diagram:

$$\begin{array}{ccc}\widetilde{\vartheta}^{\mathrm{super}}_{\mathrm{G},+}(\Lambda) & \to & \mathrm{Quad}(\Lambda_{\mathrm{sc}},\mathbb{Z})^{\mathrm{W}}\\ \downarrow & & \downarrow{\scriptstyle (4.23)}\\ \theta^{\mathrm{super}}_{+}(\Lambda) & \longrightarrow & \theta^{\mathrm{super}}_{+}(\Lambda_{\mathrm{sc}})\end{array}$$

Then $\vartheta^{\mathrm{super}}_{\mathrm{G},+}(\Lambda)$ can be viewed as the full subgroupoid of $\widetilde{\vartheta}^{\mathrm{super}}_{\mathrm{G},+}(\Lambda)$ consisting of objects whose images in $\theta^{\mathrm{super}}_{+}(\Lambda)$ have a *Weyl-invariant* form b.

Pulling back along $\mathrm{T}\subset\mathrm{G}$, $\mathrm{G}_{\mathrm{sc}}\subset\mathrm{G}$, and using the compatibility over T_{sc}, we obtain a functor:

$$\Gamma_{\mathrm{fact}}(\mathrm{Gr}_{\mathrm{G}},\mathsf{Pic}^{\mathrm{super}})\to\widetilde{\vartheta}^{\mathrm{super}}_{\mathrm{G},+}(\Lambda). \tag{4.37}$$

4.4.12. We are now ready to establish the equivalence (1) ≅ (2) ≅ (3) in Theorem 3.4.5. We shall do so using the equivalence of §4.4.7, together with an argument from [TZ21] showing that factorization super line bundles over Gr_{G} embed fully faithfully into $\widetilde{\vartheta}^{\mathrm{super}}_{\mathrm{G},+}(\Lambda)$.

Denote by:

$$\mathrm{Hom}^{\mathrm{tame}}_{\mathrm{fact}}(\mathscr{L}\mathrm{G},\mathsf{Pic}^{\mathrm{super}})\subset\mathrm{Hom}_{\mathrm{fact}}(\mathscr{L}\mathrm{G},\mathsf{Pic}^{\mathrm{super}})$$

the full subgroupoid of factorization super central extensions of $\mathscr{L}\mathrm{G}$ by $\mathbb{G}_{m,\mathrm{Ran}}$, characterized by the property of having tame commutator.

Proposition 4.4.13. *Let* G *be a reductive group* X*-scheme equipped with a maximal torus* T *with sheaf of cocharacters* Λ*. The functors* (4.4) *and* (4.37) *induce equivalences among the following Picard groupoids:*

$$\mathrm{Hom}^{\mathrm{tame}}_{\mathrm{fact}}(\mathscr{L}\mathrm{G}, \mathsf{Pic}^{\mathrm{super}}) \xrightarrow{\simeq} \Gamma_{\mathrm{fact}}(\mathrm{Gr}_{\mathrm{G}}, \mathsf{Pic}^{\mathrm{super}}) \xrightarrow{\simeq} \vartheta^{\mathrm{super}}_{\mathrm{G},+}(\Lambda). \tag{4.38}$$

Proof. The functors (4.4) and (4.37) *a priori* define:

$$\mathrm{Hom}^{\mathrm{tame}}_{\mathrm{fact}}(\mathscr{L}\mathrm{G}, \mathsf{Pic}^{\mathrm{super}}) \to \Gamma_{\mathrm{fact}}(\mathrm{Gr}_{\mathrm{G}}, \mathsf{Pic}^{\mathrm{super}}) \to \widetilde{\vartheta}^{\mathrm{super}}_{\mathrm{G},+}(\Lambda),$$

The composition is an equivalence onto $\vartheta^{\mathrm{super}}_{\mathrm{G},+}(\Lambda)$—this is the equivalence of §4.4.7 restricted to the subgroupoid characterized by the tameness condition, as we see from Lemma 4.4.8 and Lemma 4.4.10. Hence it suffices to prove that (4.37) is fully faithful and its essential image is contained in $\vartheta^{\mathrm{super}}_{\mathrm{G},+}(\Lambda)$.

The fully faithfulness is the content of [TZ21, §3.2]. The assertion that its image lies in $\vartheta^{\mathrm{super}}_{\mathrm{G},+}(\Lambda)$ amounts to establishing the Weyl-invariance of the bilinear form b associated to any factorization super line bundle over Gr_{G}. Since $\mathrm{Rad}(\mathrm{G}) \times \mathrm{T}_{\mathrm{sc}} \to \mathrm{T}$ is a Weyl-equivariant isogeny of X-tori, the statement can be proved when G is replaced by $\mathrm{Rad}(\mathrm{G}) \times \mathrm{G}_{\mathrm{sc}}$. In this case, we claim that the external product:

$$\boxtimes : \Gamma_{\mathrm{fact}}(\mathrm{Gr}_{\mathrm{Rad}(\mathrm{G})}, \mathsf{Pic}^{\mathrm{super}}) \times \Gamma_{\mathrm{fact}}(\mathrm{Gr}_{\mathrm{G}_{\mathrm{sc}}}, \mathsf{Pic}) \to \Gamma_{\mathrm{fact}}(\mathrm{Gr}_{\mathrm{Rad}(\mathrm{G}) \times \mathrm{G}_{\mathrm{sc}}}, \mathsf{Pic}^{\mathrm{super}})$$

is an equivalence of Picard groupoids. Indeed, this follows from the fiberwise characterization of line bundles over $\mathrm{Gr}_{\mathrm{Rad}(\mathrm{G}) \times \mathrm{G}_{\mathrm{sc}}}$ which descend to $\mathrm{Gr}_{\mathrm{Rad}(\mathrm{G})}$ (§4.3.5). □

4.5. Poor man's transgression.

4.5.1. Let G be a reductive group X-scheme. Denote by $\mathsf{B}\mathrm{G}$ the X-stack classifying *Zariski* locally trivial G-torsors.

The following result completes the equivalence (1) ≅ (4) in Theorem 3.4.5.

Proposition 4.5.2. *Fix a* ϑ*-characteristic* $\omega^{1/2}$ *over* X*. There is a canonical equivalence of Picard groupoids:*

$$\int_{(\mathring{\mathrm{D}}, \omega^{1/2})} : \Gamma_e(\mathsf{B}\mathrm{G}, \underline{\mathrm{K}}^{\mathrm{super}}_{[1,2]}) \xrightarrow{\simeq} \mathrm{Hom}^{\mathrm{tame}}_{\mathrm{fact}}(\mathscr{L}\mathrm{G}, \mathsf{Pic}^{\mathrm{super}}). \tag{4.39}$$

Furthermore, if G *is equipped with a maximal torus* T *with sheaf of cocharacters* Λ*. Then the following diagram is canonically commutative:*

$$\begin{array}{ccc} \Gamma_e(\mathsf{B}\mathrm{G}, \underline{\mathrm{K}}^{\mathrm{super}}_{[1,2]}) & \xrightarrow{\int_{(\mathring{\mathrm{D}}, \omega^{1/2})}} & \mathrm{Hom}^{\mathrm{tame}}_{\mathrm{fact}}(\mathscr{L}\mathrm{G}, \mathsf{Pic}^{\mathrm{super}}) \\ \downarrow{\scriptstyle (2.12)} & & \downarrow{\scriptstyle (4.38)} \\ \vartheta^{\mathrm{super}}_{\mathrm{G}}(\Lambda) & \xrightarrow{(3.22)} & \vartheta^{\mathrm{super}}_{\mathrm{G},+}(\Lambda) \end{array} \tag{4.40}$$

4.5.3. In an ideal world, we would define (4.39) by a "transgression" on K-theory and verify the commutativity of (4.40), but we do not know how to do so.

In what follows, we offer a poor man's substitute: we first fix a maximal torus contained in Borel subgroup $\mathrm{T} \subset \mathrm{B} \subset \mathrm{G}$ (which exists étale locally on X) and *define* (4.39) as the composition of the three equivalences in (4.40). We denote this functor by $\Phi_{(\mathrm{T},\mathrm{B})}$ to emphasize its *a priori* dependence on (T, B).

Then we prove that $\Phi_{(\mathrm{T},\mathrm{B})}$ is canonically independent of (T, B), *i.e.* given two pairs $\mathrm{T}_i \subset \mathrm{B}_i \subset \mathrm{G}$ (for $i = 1, 2$) of maximal tori contained in Borel subgroups, there is a canonical isomorphism of functors:

$$\alpha_{(\mathrm{T}_1,\mathrm{B}_1),(\mathrm{T}_2,\mathrm{B}_2)} : \Phi_{(\mathrm{T}_1,\mathrm{B}_1)} \xrightarrow{\simeq} \Phi_{(\mathrm{T}_2,\mathrm{B}_2)}. \tag{4.41}$$

satisfiying the cocycle condition for three such pairs $(\mathrm{T}_i, \mathrm{B}_i)$ $(i = 1, 2, 3)$.

These canonical isomorphisms allow us to glue the functors $\Phi_{(\mathrm{T},\mathrm{B})}$ over an étale cover of X, which yields the equivalence (4.39).

Remark 4.5.4. The definition of $\Phi_{(\mathrm{T},\mathrm{B})}$ uses only T. However, the isomorphism (4.41) depends on B_1 and B_2, so we prefer keep the Borel subgroups in the notation.

Proof of Proposition 4.5.2. It suffices to construct (4.41) satisfying the cocycle condition.

Suppose that t is an X-point of T. The inner automorphism $\mathrm{int}_t : \mathrm{G} \to \mathrm{G}$, $g \mapsto tgt^{-1}$ preserves $\mathrm{T} \subset \mathrm{G}$ and induces a commutative diagram:

$$\begin{array}{ccc} \Gamma_e(\mathsf{BG}, \underline{\mathrm{K}}^{\mathrm{super}}_{[1,2]}) & \xrightarrow{(2.12)} & \vartheta^{\mathrm{super}}_{\mathrm{G}}(\Lambda) \\ \downarrow{\scriptstyle \mathrm{int}_t^*} & & \downarrow{\scriptstyle \mathrm{int}_t^*} \\ \Gamma_e(\mathsf{BG}, \underline{\mathrm{K}}^{\mathrm{super}}_{[1,2]}) & \xrightarrow{(2.12)} & \vartheta^{\mathrm{super}}_{\mathrm{G}}(\Lambda) \end{array} \tag{4.42}$$

Rigidified sections of $\underline{\mathrm{K}}^{\mathrm{super}}_{[1,2]}$ over BG are equivalent to monoidal functors $\mathrm{G} \to \Omega(\underline{\mathrm{K}}^{\mathrm{super}}_{[1,2]})$. Since the target has a *symmetric* monoidal structure, int_t^* acts as the identity on the groupoid of such monoidal functors.

On the other hand, the right vertical functor int_t^* in (4.42) carries a triple $(b, \widetilde{\Lambda}, \varphi)$ to the triple $(b, \widetilde{\Lambda}, \varphi \cdot t^b)$, where $t^b : \Lambda \to \mathbb{G}_m$ denotes the character sending $\lambda \in \Lambda$ to the character $b(\lambda, -) : \mathrm{T} \to \mathbb{G}_m$ evaluated at t, and $\varphi \cdot t^b$ is the sum of φ with the restriction of t^b to Λ_{sc}. The 2-isomorphism rendering (4.42) commutative evalutes to the isomorphism:

$$(b, \widetilde{\Lambda}, \varphi) \xrightarrow{\simeq} \mathrm{int}_t^*(b, \widetilde{\Lambda}, \varphi) \xrightarrow{\simeq} (b, \widetilde{\Lambda}, \varphi \cdot t^b),$$

induced by the automorphism of the central extension $\widetilde{\Lambda}$ defined by t^b. These calculations are performed using the description of the commutator in [BD01, Proposition 3.13].

The situation is parallel for the functor (4.38): an X-point t of T induces a commutative diagram:

$$\begin{array}{ccc} \mathrm{Hom}^{\mathrm{tame}}_{\mathrm{fact}}(\mathscr{L}\mathrm{G}, \mathsf{Pic}^{\mathrm{super}}) & \xrightarrow{(4.38)} & \vartheta^{\mathrm{super}}_{\mathrm{G},+}(\Lambda) \\ \downarrow{\scriptstyle \mathrm{int}_t^*} & & \downarrow{\scriptstyle \mathrm{int}_t^*} \\ \mathrm{Hom}^{\mathrm{tame}}_{\mathrm{fact}}(\mathscr{L}\mathrm{G}, \mathsf{Pic}^{\mathrm{super}}) & \xrightarrow{(4.38)} & \vartheta^{\mathrm{super}}_{\mathrm{G},+}(\Lambda) \end{array} \tag{4.43}$$

where the left vertical functor is isomorphic to the identity, the right vertical functor sends $(b, \widetilde{\Lambda}_+, \varphi)$ to $(b, \widetilde{\Lambda}_+, \varphi \cdot t^b)$, and the 2-isomorphism rendering (4.43) commutative is given by the automorphism t^b of $\widetilde{\Lambda}_+$. These calculations are performed using the description of the commutator in §4.2.8.

Suppose now that $\mathrm{T}_i \subset \mathrm{B}_i \subset \mathrm{G}$ (for $i = 1, 2$) are a pair of maximal tori contained in Borel subgroups and g is an X-point of G with $\mathrm{int}_g(\mathrm{T}_2) = \mathrm{T}_1$, $\mathrm{int}_g(\mathrm{B}_2) = \mathrm{B}_1$. Denote by Λ_i the sheaf of cocharacters of T_i.

The inner automorphism int_g gives rise to a commutative diagram:

$$\begin{array}{ccccccc} \Gamma_e(\mathrm{BG}, \underline{\mathrm{K}}^{\mathrm{super}}_{[1,2]}) & \xrightarrow{(2.12)} & \vartheta^{\mathrm{super}}_{\mathrm{G}}(\Lambda_1) & \simeq & \vartheta^{\mathrm{super}}_{\mathrm{G},+}(\Lambda_1) & \xleftarrow{(4.38)} & \mathrm{Hom}^{\mathrm{tame}}_{\mathrm{fact}}(\mathscr{L}\mathrm{G}, \mathsf{Pic}^{\mathrm{super}}) \\ \downarrow{\scriptstyle \mathrm{int}_g^*} & & \downarrow{\scriptstyle \mathrm{int}_g^*} & & \downarrow{\scriptstyle \mathrm{int}_g^*} & & \downarrow{\scriptstyle \mathrm{int}_g^*} \\ \Gamma_e(\mathrm{BG}, \underline{\mathrm{K}}^{\mathrm{super}}_{[1,2]}) & \xrightarrow{(2.12)} & \vartheta^{\mathrm{super}}_{\mathrm{G}}(\Lambda_2) & \simeq & \vartheta^{\mathrm{super}}_{\mathrm{G},+}(\Lambda_1) & \xleftarrow{(4.38)} & \mathrm{Hom}^{\mathrm{tame}}_{\mathrm{fact}}(\mathscr{L}\mathrm{G}, \mathsf{Pic}^{\mathrm{super}}) \end{array} \tag{4.44}$$

where the middle equivalences are defined by $\omega^{1/2}$-shift (3.22). Since the outer vertical functors are equivalent to the identity, the 2-isomorphism rendering (4.44) commutative defines an isomorphism of functors:

$$\alpha_g : \Phi_{(\mathrm{T}_1,\mathrm{B}_1)} \xrightarrow{\simeq} \Phi_{(\mathrm{T}_2,\mathrm{B}_2)}. \tag{4.45}$$

Claim: α_g depends only on the pairs $(\mathrm{T}_1, \mathrm{B}_1)$ and $(\mathrm{T}_2, \mathrm{B}_2)$ (as opposed to g). Indeed, any other choice of an X-point of G conjugating $(\mathrm{T}_2, \mathrm{B}_2)$ into $(\mathrm{T}_1, \mathrm{B}_1)$ differs from g by an X-point of T_1, so the claim is equivalent to the following assertion: for $(\mathrm{T}_1, \mathrm{B}_1) = (\mathrm{T}_2, \mathrm{B}_2) = (\mathrm{T}, \mathrm{B})$, the isomorphism α_t is the identity on $\Phi_{(\mathrm{T},\mathrm{B})}$. However, this follows from the description of the 2-isomorphisms in (4.42) and (4.43).

Finally, we set (4.41) to be the isomorphism (4.45) for any X-point g conjugating $(\mathrm{T}_2, \mathrm{B}_2)$ into $(\mathrm{T}_1, \mathrm{B}_1)$, which exists étale locally over X—these choices glue thanks to the independence of α_g on g. The cocycle condition follows from the equality $\alpha_{g_1 g_2} = \alpha_{g_1} \cdot \alpha_{g_2}$. □

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