WHAT ARE THE EXTENDED PURE INNER FORMS OF A COVER?

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Abstract. Kottwitz suggested to study all extended pure inner forms together in the local Langlands correspondence for linear reductive groups. We extend this philosophy to a large class of covers, including those defined by Brylinski and Deligne, and explain its relation with Weissman's observation that L-packets for covers are sometimes empty.

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Introduction

The goal of this article is to define a notion of "extended pure inner forms" of a covering group and argue that it is relevant for the local Langlands program for covers.

Let us begin by describing the puzzle that motivated our consideration.

0.1. The "missing" L-packets. In the usual local Langlands program, one takes as input a local field F and a reductive group F-scheme G. To these data, one attaches the set $\Pi(\mathrm{G}(\mathrm{F}))$ of isomorphism classes of irreducible smooth $\mathrm{G}(\mathrm{F})$-representations and the set $\Phi({}^{\mathrm{L}}\mathrm{G})$ of L-parameters, and posits the existence of a natural map

$$\mathrm{LLC} : \Pi(\mathrm{G}(\mathrm{F})) \to \Phi({}^{\mathrm{L}}\mathrm{G}). \tag{0.1}$$

The local Langlands program for covers requires a cohomological input μ, in addition to F and G. Traditionally, μ is defined in terms of algebraic K-theory (*cf.* [BD01, Wei18]). For now, let us ignore the precise meaning of μ and accept that it gives rise to a topological central extension

$$1 \to \mathrm{A} \to \widetilde{\mathrm{G}} \to \mathrm{G}(\mathrm{F}) \to 1$$

for a finite subgroup A of $\mathbf{C}^\times$, as well as an "L-group" $\widetilde{\mathrm{H}}$. Imitating the linear situation, one posits the existence of a natural map

$$\mathrm{LLC} : \Pi(\widetilde{\mathrm{G}}) \to \Phi(\widetilde{\mathrm{H}}), \tag{0.2}$$

where $\Pi(\widetilde{\mathrm{G}})$ is the set of isomorphism classes of irreducible genuine smooth $\widetilde{\mathrm{G}}$-representations and $\Phi(\widetilde{\mathrm{H}})$ is the set of L-parameters defined in terms of $\widetilde{\mathrm{H}}$. (The adjective "genuine" means that A acts through its inclusion in $\mathbf{C}^\times$.) We refer the reader to [GGW18, Wei18, GG18] where foundations of this program are laid out.

The map (0.2) has been constructed by Weissman when G is a split torus. He observed an intriguing phenomenon: It may *not* be surjective. This stands in contrast with (0.1), which is expected to be surjective when G is quasi-split.

The goal of our article is to relate this phenomenon to Kottwitz's philosophy of treating all extended pure inner forms of G together in the formulation of (0.1) (*cf.* [Kot85, Kot97, Kal14]). Namely, for each basic G-isocrystal β, we shall construct a cover $\widetilde{\mathrm{G}}_\beta$ of the extended pure inner form $\mathrm{G}_\beta(\mathrm{F})$ associated to β. We expect (0.2) to fit into a family of maps

$$\mathrm{LLC}_\beta : \Pi(\widetilde{\mathrm{G}}_\beta) \to \Phi(\widetilde{\mathrm{H}}), \tag{0.3}$$

parametrized by β. Weissman's observation above, in fact, leads to an obstruction $\Omega_\beta(\sigma)$ for the nonemptiness of $\mathrm{LLC}_\beta^{-1}(\sigma)$. Our main result determines the set of β for which $\Omega_\beta(\sigma)$ vanishes. The key point is that this set of β is always *nonempty*, though it may not contain the trivial element. When G is a torus, we prove that $\mathrm{LLC}_\beta^{-1}(\sigma)$ is indeed nonempty and finite whenever $\Omega_\beta(\sigma)$ vanishes.

Informally, our results indicate that the "missing" L-packets observed by Weissman may appear on an "extended pure inner form" $\widetilde{\mathrm{G}}_\beta$ of the cover $\widetilde{\mathrm{G}}$.

0.2. Results. Let us give a more precise account of the content of this article.

In the remainder of this introduction, we fix a local field F, a reductive group F-scheme G, and a finite subgroup A of $\mathbf{C}^\times$ whose order is invertible in F.

Our first task is to define the cover $\widetilde{\mathrm{G}}_\beta$ for an arbitrary G-isocrystal β, given the cohomological input μ. It turns out that the K-theoretic formalism of Brylinski–Deligne (*cf.* [BD01]) is too restrictive for this purpose. Instead, we take μ to be an *étale metaplectic cover*, *i.e.* a morphism of pointed (higher) étale stacks (*cf.* [Del96, GL18, Zha22])

$$\mathsf{B}\mathrm{G} \to \mathsf{B}^4\mathrm{A}(1).$$

The reason is as follows: An étale metaplectic cover μ of G induces an étale metaplectic cover μ_β of each G_β, hence a cover $\widetilde{G}_\beta$ of $G_\beta(F)$. However, even if μ comes from algebraic K-theory, μ_β may not (*cf.* Remark 4.5.4). In other words, étale metaplectic covers are necessary even if one is only interested in Brylinski–Deligne covers.

Next, we turn our attention to Langlands duality. The L-group of an étale metaplectic cover is defined in [Zha22]. Let us sketch (a minor variant of) this construction, as it is important for the formulation of our main results.

This construction consists of three steps.

In the first step, we replace the pair (G, μ) by another one $(G^\sharp, \mu_{G^\sharp})$. Here, $G^\sharp$ is a reductive group F-scheme, endowed with an étale metaplectic cover $\mu_{G^\sharp}$ which is "as commutative as possible".

To explain the last phrase, we recall that every reductive group F-scheme G maps to the stack quotient $G_{ab} := G/G_{sc}$, where G_{sc} is the simply connected form of G. In fact, G_{ab} is a commutative group stack and the gentlest kind of étale metaplectic covers are pulled back from "**Z**-linear" morphisms[1]

$$\mathsf{B}G_{ab} \to \mathsf{B}^4 A(1)$$

Such morphisms are parametrized by maps of complexes $\pi_1 G \to A[2]$. Kaletha (*cf.* [Kal22]) has studied the covers defined by them, at least for quasi-split G, and reduced the Langlands correspondence for them to that for linear reductive groups.[2]

The next, and slightly less gentle, kind of étale metaplectic covers is pulled back from symmetric monoidal morphisms[3] from $\mathsf{B}G_{ab}$ to $\mathsf{B}^4 A(1)$. The étale metaplectic cover $\mu_{G^\sharp}$ is of this kind. The passage from (G, μ) to $(G^\sharp, \mu_{G^\sharp})$ is analogous to the "sharp cover" construction of Weissman (*cf.* [Wei18]).

The second step has to do with the subtle, but important difference between symmetric monoidal and **Z**-linear morphisms from $\mathsf{B}G^\sharp_{ab}$ to $\mathsf{B}^4 A(1)$. Namely, there is a canonical decomposition

$$\mu_{G^\sharp} \cong \mu^{(1)}_{G^\sharp} + \mu^{(2)}_{G^\sharp} \tag{0.4}$$

where $\mu^{(1)}_{G^\sharp}$ is 2-torsion and $\mu^{(2)}_{G^\sharp}$ comes from a **Z**-linear morphism. The decomposition (0.4) appeared first in Gaitsgory and Lysenko's work (*cf.* [GL18]), who used it to explain a sign occurring in the twisted geometric Satake equivalence.

The third step is the passage to the Galois side: We take H to be the Langlands dual group of $G^\sharp$ and define a sum of $Z_H(\mathbf{C})$-gerbes over the étale site of Spec F

$$\widetilde{Z}_H := \widetilde{Z}^{(1)}_H + \widetilde{Z}^{(2)}_H, \tag{0.5}$$

where Z_H is the center of H. The summands in (0.5) are constructed from the respective summands in (0.4). The L-group $\widetilde{H}$ is obtained formally from $\widetilde{Z}_H$, by rewriting an étale gerbe as a Galois cocycle. In the K-theoretic context, Weissman defined the L-group as a Baer sum similar to the above (*cf.* [Wei18]). However, the decomposition (0.4) has no K-theoretic counterpart, so our formalism renders the situation more symmetric.

Slightly extending Kaletha's work, one can relate the Langlands duality for the "sharp cover" $(G^\sharp, \mu_{G^\sharp})$ to that for linear reductive groups (*cf.* §2.5). The passage from (G, μ) to $(G^\sharp, \mu_{G^\sharp})$ is more mysterious and is responsible for the "missing" L-packets.

However, before we can go any further, we must first construct the Langlands duality for sharp covers of tori. The following result is proved in §2.

[1]In homotopical terms, this means morphisms of sheaves of H**Z**-module spectra.

[2]However, Kaletha's construction of the covers is different from ours. His uses the Langlands duality for tori, and ours does not. The equivalence of these two constructions is a consequence of our results in §2.

[3]This means morphisms of sheaves of (grouplike) $\mathbb{E}_\infty$-monoids, or equivalently of spectra.

Theorem A. *Let* T *be an* F*-torus equipped with a symmetric monoidal morphism* $\mu : \mathsf{B}\mathrm{T} \to \mathsf{B}^4\mathrm{A}(1)$*, defining a cover* $\widetilde{\mathrm{T}}$ *and an L-group* $\widetilde{\mathrm{H}}$*. There is a canonical bijection*

$$\Pi(\widetilde{\mathrm{T}}) \xrightarrow{\simeq} \Phi(\widetilde{\mathrm{H}}),$$

where $\Pi(\widetilde{\mathrm{T}})$ *is the set of genuine smooth characters of* $\widetilde{\mathrm{T}}$ *and* $\Phi(\widetilde{\mathrm{H}})$ *is the set of L-parameters defined in terms of* $\widetilde{\mathrm{H}}$*.*

The special case of Theorem A where μ comes from algebraic K-theory and T is split is established by Weissman (*cf.* [Wei18, Part 3]). This serves as justification for his definition of the L-group and is not at all a trivial consequence of class field theory.

Our proof of Theorem A is independent of *op.cit.*. It establishes that, in a precise sense, the decompositions (0.4) and (0.5) match under Langlands duality. The main novelty in our proof is the treatment of a subtle 2-torsion phenomenon, which explains how Gaitsgory and Lysenko's "sign gerbe" (*cf.* [GL18, §4.8]) and Weissman's meta-Galois group (*cf.* [Wei18, §4]) are interchanged under Langlands duality (*cf.* Corollary 2.2.14).

We shall use Theorem A to formulate the compatibility of the conjectural local Langlands correspondence for (G, μ) with "central core characters". According to Weissman's vision, this is a substitute for the compatibility with central characters for the usual local Langlands correspondence.

Let us be more precise. There is a natural map from the center $\mathrm{Z}^\sharp$ of $\mathrm{G}^\sharp$ to the center Z of G. (There is, however, no natural maps between $\mathrm{G}^\sharp$ and G.) This map is compatible with their étale metaplectic covers, so it induces a map on the covers of their F-points

$$\widetilde{\mathrm{Z}}^\sharp \to \widetilde{\mathrm{Z}}. \tag{0.6}$$

Given an irreducible genuine smooth representation V of $\widetilde{\mathrm{G}}$, the $\widetilde{\mathrm{Z}}^\sharp$-action on V through (0.6) is given by a genuine smooth character: This is the *central core character* of V. Compatibility of the local Langlands correspondence for (G, μ) with central core characters asserts that the following diagram commutes:

$$\begin{array}{ccc} \Pi(\widetilde{\mathrm{G}}) & \xrightarrow{\text{LLC}} & \Phi(\widetilde{\mathrm{H}}) \\ \downarrow & & \downarrow \\ \Pi(\widetilde{\mathrm{Z}}^\sharp) & \xrightarrow{\simeq} & \Phi(\widetilde{\mathrm{H}}_{\mathrm{ab}}) \end{array} \tag{0.7}$$

Here, the left vertical arrow extracts the central core character, the right vertical arrow is the "abelianization" of an L-parameter σ, and the lower horizontal equivalence is defined by Theorem A, or rather, its mild generalization to disconnected groups.

Assuming the commutativity of (0.7), we can now explain the failure of surjectivity of LLC. Let K be the kernel of (0.6). Given an L-parameter $\sigma \in \Phi(\widetilde{\mathrm{H}})$, whose abelianization corresponds to a genuine smooth character $\chi_\sigma : \widetilde{\mathrm{Z}}^\sharp \to \mathbf{C}^\times$, if the restriction

$$\Omega(\sigma) := \chi_\sigma|_{\mathrm{K}} \tag{0.8}$$

is nonzero, then the fiber of LLC at σ is empty. This is because the central core character of any $\mathrm{V} \in \Pi(\widetilde{\mathrm{G}})$ must annihilate K.

We call (0.8) *Weissman's obstruction*, as he first discovered it for tori (*cf.* [Wei09, Wei16]). The main goal of this article is to generalize it to $\widetilde{\mathrm{G}}_\beta$ for every G-isocrystal β and to characterize those β for which this obstruction vanishes.

The first task is easy, given our definition of $\widetilde{\mathrm{G}}_\beta$. Indeed, there is a natural map $\widetilde{\mathrm{Z}}^\sharp \to \widetilde{\mathrm{G}}_\beta$ for every G-isocrystal β with central image. This allows us to formulate the compatibility of the conjectural local Langlands correspondence (0.3) with central core characters, in analogy with (0.7). For each L-parameter $\sigma \in \Phi(\widetilde{\mathrm{H}})$, we may then define a character

$$\Omega_\beta(\sigma) : \mathrm{K} \to \mathbf{C}^\times.$$

If $\Omega_\beta(\sigma) \neq 1$, then the fiber of LLC_β at σ is empty.

Let us now state our main theorem. It will be established in §4.

Theorem B. *Let* G *be a reductive group* F*-scheme endowed with an étale metaplectic cover* μ*. Let* $\mathrm{G}^\sharp$ *and* K *be defined as above.*

(1) There is a canonical exact sequence of abelian groups

$$(\pi_1 \mathrm{G}^\sharp)_{\mathrm{Gal}_\mathrm{F}} \to (\pi_1 \mathrm{G})_{\mathrm{Gal}_\mathrm{F}} \xrightarrow{\gamma} \mathrm{Hom}(\mathrm{K}, \mathbf{C}^\times) \to 1.$$

(2) For each G*-isocrystal* β *and L-parameter* σ*, the character* $\Omega_\beta(\sigma)$ *vanishes if and only if the Kottwitz invariant of* β *maps to* $\Omega(\sigma)^{-1}$ *under* γ*.*

In particular, for any L-parameter σ, there always exists a basic G-isocrystal β for which $\Omega_\beta(\sigma)$ vanishes and the set of isomorphism classes of such β is a torsor under the image of $(\pi_1 \mathrm{G}^\sharp)_{\mathrm{Gal}_\mathrm{F}}$. When G is a torus, we prove that the vanishing of $\Omega_\beta(\sigma)$ is necessary and sufficient for $\mathrm{LLC}_\beta^{-1}(\sigma)$ to be nonempty (*cf.* Proposition 4.4.4).

Let us say one word about the proof of Theorem B.

The key ingredient is the "canonical quadratic structure" of an étale metaplectic cover μ with respect to the BZ-action on BG (*cf.* Proposition 3.1.3). This appears to be a fundamental piece of structure of étale characteristic classes, valid over an arbitrary base scheme. Informally, it expresses

$$\mu(\mathscr{E} \otimes \mathscr{Z}) - \mu(\mathscr{E}) - \mu(\mathscr{Z}_\mathrm{G}),$$

for any G-torsor $\mathscr{E}$ and Z-torsor $\mathscr{Z}$ (with induced G-torsor $\mathscr{Z}_\mathrm{G}$), in terms of an explicit bilinear expression in $\mathscr{Z}$ and the G_{ab}-torsor induced from $\mathscr{E}$. One classical manifestation of this phenomenon is the formula of Chern classes

$$c_2(\mathscr{E} \otimes \mathscr{L}) - c_2(\mathscr{E}) - c_2(\mathscr{L}^{\oplus n}) = (n-1) \cdot c_1(\det \mathscr{E}) \cup c_1(\mathscr{L})$$

for any rank-n vector bundle $\mathscr{E}$ and any line bundle $\mathscr{L}$.

0.3. Conventions. This paper uses homotopical algebra as developed by Lurie (*cf.* [Lur09, Lur17]). While certain aspects of the theory of covers can be handled using traditional methods of homological algebra[4], manipulations of higher symmetric monoidal structures in this article are infeasible without Lurie's theory.

Following Lurie's convention, we refer to ∞-groupoids as *spaces*. We invoke the equivalence between connective spectra and grouplike $\mathbb{E}_\infty$-monoids (*cf.* [Lur17, Remark 5.2.6.26]) and view them as spaces with additional structure. We abbreviate "connective H**Z**-module spectra" as **Z***-linear spaces*. In particular, there are forgetful functors from **Z**-linear spaces to grouplike $\mathbb{E}_\infty$-monoids, and from grouplike $\mathbb{E}_\infty$-monoids to pointed spaces.[5] These are essentially the only higher algebraic structures we will need.

Given a scheme S, we use B to denote the deloop functor for fppf sheaves over S. In particular, for a group S-scheme G, BG is the usual classifying stack of G. If G is smooth and affine, then its deloops in the fppf and étale topologies coincide.

[4] For example, by taking a simplicial resolution of BG, one can encode an étale metaplectic cover as an étale hypercocycle, which is how this notion was originally conceived of by Deligne (*cf.* [Del96]). The **Z**-linear part of $\mu_{\mathrm{G}^\sharp}$ can also be encoded by a map of complexes over Spec F, hence by a Galois hypercocycle. This is how Kaletha describes them (*cf.* [Kal22]).

[5] For a usual groupoid, *i.e.* a 1-truncated space, a **Z**-linear structure is the structure of a strictly commutative Picard groupoid, whereas a grouplike $\mathbb{E}_\infty$-monoid structure is the structure of a Picard groupoid.

Given an fppf sheaf of abelian groups $\mathscr{A}$ over S and an integer $n \geq 1$, we view the n-fold deloop $\mathsf{B}^n\mathscr{A}$ as an fppf sheaf of $\mathbf{Z}$-linear spaces over S. If $\mathscr{A}$ is pulled back from the small étale site of S (*e.g.* $\mathscr{A} \cong \mathrm{A}(1)$ where A is a finite abelian group of invertible order), then its deloops in the fppf and étale topologies coincide (*cf.* [Sta18, 0DDT]). We invoke this equivalence when applying the formalism of [Zha22].

We shall use "Kummer theory" extensively in the following form. For any integer $n \geq 1$, we have the coboundary $\mathbb{G}_m \to \mathsf{B}\mathbf{Z}/n(1)$ of the Kummer exact sequence. This yields a morphism in the pro-category of fppf sheaves of $\mathbf{Z}$-linear spaces

$$\Psi : \mathbb{G}_m \to \mathsf{B}\hat{\mathbf{Z}}(1) := \lim_n \mathsf{B}\mathbf{Z}/n(1),$$

where the formal inverse limit is taken over the divisibility poset of positive integers. We shall also frequently use the deloop $\mathsf{B}\Psi : \mathsf{B}\mathbb{G}_m \to \mathsf{B}^2\hat{\mathbf{Z}}(1)$ of Ψ.

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1. Covers of $\mathrm{G}_\beta(\mathrm{F})$

Let F be a local field with residue field f. We fix an algebraic closure $\bar{\mathsf{f}}$ of f.

In this section, we first recall the notion of G-isocrystals following Kottwitz (*cf.* [Kot85, Kot97]). Then we define the cover $\widetilde{\mathrm{G}}_\beta$ for a reductive group F-scheme G together with an étale metaplectic cover μ and a G-isocrystal β. Next, we recall some combinatorial data associated to μ in order to define the set of L-parameters and extend Weissman's conjectural local Langlands correspondence to $\widetilde{\mathrm{G}}_\beta$ (*cf.* Conjecture 1.4.16).

1.1. G-isocrystals.

1.1.1. Denote by $\breve{\mathrm{F}}$ the completed maximal unramified extension of F determined by $\bar{\mathsf{f}}$. Denote by q the cardinality of f. The qth power Frobenius of $\bar{\mathsf{f}}$ extends by functoriality to an automorphism of $\breve{\mathrm{F}}$, which we denote by σ.

Denote by X the prestack quotient $\mathrm{Spec}(\breve{\mathrm{F}})/\sigma^{\mathbf{Z}}$. The inclusion $\mathrm{F} \subset \breve{\mathrm{F}}$ induces a morphism of prestacks

$$\mathrm{X} \to \mathrm{Spec}\,\mathrm{F}. \tag{1.1}$$

In what follows, we treat Spec F as the base scheme, so fiber products taken over Spec F will be written without Spec F.

1.1.2. For any affine group F-scheme G of finite type, we write Isoc_G for the groupoid of G-torsors over X, which we refer to as G-*isocrystals*.

Equivalently, a G-isocrystal β consists of a G-torsor $\mathscr{E}$ over $\mathrm{Spec}\,\breve{\mathrm{F}}$ and an isomorphism of G-torsors $\varphi : \sigma^*\mathscr{E} \xrightarrow{\simeq} \mathscr{E}$.

Note that any element g of $\mathrm{G}(\breve{\mathrm{F}})$ defines a G-isocrystal $(\mathscr{E}, \varphi)$, where $\mathscr{E}$ is the trivial G-torsor over $\mathrm{Spec}\,\breve{\mathrm{F}}$ and φ is multiplication by g. Conversely, if a G-isocrystal is endowed with a trivialization over $\mathrm{Spec}\,\breve{\mathrm{F}}$, then φ is given by multiplication by an element of $\mathrm{G}(\breve{\mathrm{F}})$.

Remark 1.1.3. If F is of characteristic zero and G is connected, then any G-torsor over $\mathrm{Spec}\,\breve{\mathrm{F}}$ is trivial (*cf.* [Ste65, Theorem 1.9]). If F is of characteristic $p \neq 0$ and G is reductive[6], then any G-torsor over $\mathrm{Spec}\,\breve{\mathrm{F}}$ is trivial (*cf.* [BS68, §8.6]).

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[6] Our convention is that "reductive" implies "connected".

For our purposes, however, we will sometimes need the case for disconnected G such as the center of a reductive group F-scheme.

1.1.4. Given a G-isocrystal β, we write G_β for the group F-sheaf of automorphisms of β. Namely, for any F-algebra R, an R-point of G_β is an automorphism of the pullback of β to the prestack $X \times \operatorname{Spec} R$.

By [Kot97, §3.3], the group F-sheaf G_β is represented by an affine group F-scheme.

1.1.5. For a reductive group scheme G over any base scheme S, we employ the following notation: G_{sc} (respectively G_{ad}) stands for the simply connected (respectively adjoint) form of G. Write T (respectively T_{sc}, T_{ad}) for the abstract Cartan of G (respectively G_{sc}, G_{ad}). Denote by Λ (respectively $\check{\Lambda}$) the fppf sheaf of cocharacters (respectively characters) of T. We use the same notation Λ_{sc}, Λ_{ad}, *etc.* for T_{sc} and T_{ad}.

Denote by $\Delta \subset \Lambda$ (respectively $\check{\Delta} \subset \check{\Lambda}$) the subsheaf of simple coroots (respectively simple roots). Thus Δ generates Λ_{sc} and $\check{\Delta}$ generates the subsheaf dual to Λ_{ad}.

Denote by Z the center of G, so we have a canonical isomorphism

$$Z \cong \operatorname{Fib}(\Lambda \to \Lambda_{ad}) \otimes \mathbb{G}_m,$$

where Fib stands for the fiber of complexes of fppf sheaves of abelian groups, and the tensor product is understood in the derived sense.

Denote by $\pi_1 G$ the quotient of fppf sheaves Λ/Λ_{sc}. We view

$$G_{ab} := \pi_1 G \otimes \mathbb{G}_m$$

as an fppf sheaf of Picard groupoids and refer to it as the *cocenter* of G. (It coincides with the abelianization of G when $\pi_1 G$ is torsion-free.) The identification $Z/Z_{sc} \cong G/G_{sc}$ induces a monoidal morphism

$$G \to G_{ab}. \tag{1.2}$$

1.1.6. For a reductive group F-scheme G, we have the Kottwitz invariant

$$\mathrm{Kott} : \mathsf{Isoc}_G \to (\pi_1 G)_{\mathrm{Gal}_F}, \tag{1.3}$$

where $(\pi_1 G)_{\mathrm{Gal}_F}$ denotes the group of Galois coinvariants of $\pi_1 G$.[7]

Since (1.3) plays a major role in this text, let us recall its definition.

Denote by $\mathsf{Isoc}_{G_{ab}}$ the space of G_{ab}-torsors over X, *i.e.* maps from X to $\mathsf{B}G_{ab}$. Since T_{sc} is reductive, every T_{sc}-torsor over $\operatorname{Spec} \breve{F}$ is trivial (*cf.* Remark 1.1.3). Combining this with the fact that $\mathbf{Z}$ has cohomological dimension 1, we see that $H^2(X, T_{sc}) \cong 0$, so the quotient map yields an equivalence

$$\mathsf{Isoc}_T/\mathsf{Isoc}_{T_{sc}} \xrightarrow{\cong} \mathsf{Isoc}_{G_{ab}}. \tag{1.4}$$

The functorial isomorphism $\pi_0(\mathsf{Isoc}_T) \cong (X_* T)_{\mathrm{Gal}_F}$ for tori (*cf.* [Kot85, §2.4]) induces an isomorphism $\pi_0(\mathsf{Isoc}_{G_{ab}}) \cong (\pi_1 G)_{\mathrm{Gal}_F}$ via (1.4). We set (1.3) to be the composition

$$\begin{aligned} &\mathsf{Isoc}_G \to \mathsf{Isoc}_{G_{ab}} \\ &\qquad \to \pi_0(\mathsf{Isoc}_{G_{ab}}) \xrightarrow{\cong} (\pi_1 G)_{\mathrm{Gal}_F}, \end{aligned}$$

where the first map is defined by functoriality with respect to (1.2).

1.1.7. We keep the assumption that G is reductive.

Recall that a G-isocrystal β is *basic* if its induced G_{ad}-isocrystal is the pullback of a G_{ad}-torsor over $\operatorname{Spec} F$ (*cf.* [Kot85, §4.5, §5.1]). Thus, if β is basic, then G_β is an inner form

[7]The formation of Galois coinvariants does not require the choice of an algebraic closure of F.

of G. Inner forms arising in this manner are called *extended pure inner forms* of G. Denote by $\mathsf{Basic}_{\mathrm{G}}$ the full subgroupoid of $\mathsf{Isoc}_{\mathrm{G}}$ consisting of basic G-isocrystals.

According to [Kot85, Proposition 5.6], (1.3) induces a bijection

$$\pi_0(\mathsf{Basic}_{\mathrm{G}}) \xrightarrow{\simeq} (\pi_1 \mathrm{G})_{\mathrm{Gal}_{\mathrm{F}}}. \tag{1.5}$$

1.2. Construction of covers.

1.2.1. Let G be an affine group F-scheme of finite type and A be a finite abelian group whose order is invertible in F.

For each $n \in \mathbf{Z}$, write $\mathrm{A}(n)$ for the corresponding Tate twist of A, viewed as an étale sheaf of finite abelian groups over Spec F.

1.2.2. Denote by $\mathrm{Maps}_e(\mathsf{BG}, \mathsf{B}^4\mathrm{A}(1))$ the space of rigidified morphisms $\mathsf{BG} \to \mathsf{B}^4\mathrm{A}(1)$, *i.e.* morphisms of pointed F-stacks. It admits a $\mathbf{Z}$-linear structure induced from the abelian group structure on $\mathrm{A}(1)$.

For a topological group K, we refer to a topological central extension

$$1 \to \mathrm{A} \to \widetilde{\mathrm{K}} \to \mathrm{K} \to 1,$$

where $\widetilde{\mathrm{K}} \to \mathrm{K}$ is a local homeomorphim, as a *cover* of K. The collection of covers of K form a $\mathbf{Z}$-linear groupoid under Baer sum, which we denote by $\mathsf{Cov}(\mathrm{K}, \mathrm{A})$.

Let us equip G(F) with the topology inherited from F. The construction of [Zha22, §2.1] yields a $\mathbf{Z}$-linear functor

$$\int_{\mathrm{F}} : \mathrm{Maps}_e(\mathsf{BG}, \mathsf{B}^4\mathrm{A}(1)) \to \mathsf{Cov}(\mathrm{G}(\mathrm{F}), \mathrm{A}). \tag{1.6}$$

1.2.3. For any G-isocrystal β, we shall construct a functor

$$\mathrm{T}_\beta : \mathrm{Maps}_e(\mathsf{BG}, \mathsf{B}^4\mathrm{A}(1)) \to \mathrm{Maps}_e(\mathsf{BG}_\beta, \mathsf{B}^4\mathrm{A}(1)), \tag{1.7}$$

to be conceived of as "translation by β".

If G_β is of finite type, then by composing (1.7) with the functor (1.6) applied to G_β, we obtain a functor

$$\int_{\mathrm{F},\beta} : \mathrm{Maps}_e(\mathsf{BG}, \mathsf{B}^4\mathrm{A}(1)) \to \mathsf{Cov}(\mathrm{G}_\beta(\mathrm{F}), \mathrm{A}). \tag{1.8}$$

Under (1.8), every rigidified morphism $\mathsf{BG} \to \mathsf{B}^4\mathrm{A}(1)$ defines a cover $\widetilde{\mathrm{G}}_\beta$ of $\mathrm{G}_\beta(\mathrm{F})$.

1.2.4. *Construction of* T_β. Let us view β as a morphism $\mathrm{X} \to \mathsf{BG}$. Since G_β is the group F-sheaf of its automorphisms, β extends to a morphism of fppf stacks

$$\mathrm{X} \times \mathsf{BG}_\beta \to \mathsf{BG}. \tag{1.9}$$

More precisely, there is a natural morphism of group X-sheaves $\mathrm{X} \times \mathrm{G}_\beta \to \mathrm{X} \times_{\mathsf{BG}} \mathrm{X}$, where the target is the fiber product of β with itself, and (1.9) is obtained as its deloop.

Given a rigidified morphism $\mu : \mathsf{BG} \to \mathsf{B}^4\mathrm{A}(1)$, the pullback of μ along (1.9) is a morphism $\mathrm{X} \times \mathsf{BG}_\beta \to \mathsf{B}^4\mathrm{A}(1)$ whose restriction along the neutral section $e : \mathrm{X} \to \mathrm{X} \times \mathsf{BG}_\beta$ is isomorphic to $\beta^*\mu$. Thus, sending μ to the difference $\mu - p^*\beta^*\mu$, where $p : \mathrm{X} \times \mathsf{BG}_\beta \to \mathrm{X}$ is the projection, defines a functor

$$\mathrm{Maps}_e(\mathsf{BG}, \mathsf{B}^4\mathrm{A}(1)) \to \mathrm{Maps}_e(\mathrm{X} \times \mathsf{BG}_\beta, \mathsf{B}^4\mathrm{A}(1)), \tag{1.10}$$

where the target is the space of maps $\mathrm{X} \times \mathsf{BG}_\beta \to \mathsf{B}^4\mathrm{A}(1)$ rigidified along e.

Thanks to Lemma 1.2.5 below, pullback along the projection $\mathrm{X} \times \mathsf{BG}_\beta \to \mathsf{BG}_\beta$ induces an isomorphism

$$\mathrm{Maps}_e(\mathsf{BG}_\beta, \mathsf{B}^4\mathrm{A}(1)) \xrightarrow{\simeq} \mathrm{Maps}_e(\mathrm{X} \times \mathsf{BG}_\beta, \mathsf{B}^4\mathrm{A}(1)). \tag{1.11}$$

The desired functor (1.7) is the composition of (1.10) with the inverse of (1.11).

Lemma 1.2.5. *For any* F*-scheme* S*, pullback along the projection* $S \times X \to S$ *induces an isomorphism of étale cochains*

$$\Gamma(S, A(1)) \xrightarrow{\simeq} \Gamma(S \times X, A(1)). \tag{1.12}$$

Proof. Denote by $\nu : \operatorname{Spec} \breve{F} \to \operatorname{Spec} F$ the natural map. The complex $\nu_* A(1)$ is endowed with an automorphism σ^* defined by pullback along $\sigma : \operatorname{Spec} \breve{F} \to \operatorname{Spec} \breve{F}$. We claim that it is sufficient to identify the fiber of

$$\sigma^* - \mathrm{id} : \nu_* A(1) \to \nu_* A(1) \tag{1.13}$$

with $A(1)$, along the unit map $A(1) \to \nu_* A(1)$.

Indeed, the complex $\Gamma(S \times X, A(1))$ is the (derived) $\mathbf{Z}$-invariants of the complex $\Gamma(S \times \operatorname{Spec} \breve{F}, A(1))$, with $1 \in \mathbf{Z}$ acting by σ^*. On the other hand, base change (*cf.* [Sta18, 0F1I]) yields an isomorphism

$$\Gamma(S \times \operatorname{Spec} \breve{F}, A(1)) \xrightarrow{\simeq} \Gamma(S, \nu_* A(1)).$$

Hence, $\Gamma(S \times X, A(1))$ is identified with the image under $\Gamma(S, -)$ of the fiber of (1.13). If the latter is identified with $A(1)$ along the unit map, then $\Gamma(S \times X, A(1))$ is identified with $\Gamma(S, A(1))$ along the pullback map.

We now identify the fiber of (1.13). Let F^{unr} denote the maximal unramified extension of F determined by $\bar{\mathsf{f}}$, so $\breve{F}$ is the completion of F^{unr}. Pulling back along $\operatorname{Spec} \breve{F} \to \operatorname{Spec} F^{\mathrm{unr}}$ induces an equivalence of étale sites, so we may replace $\operatorname{Spec} \breve{F}$ by $\operatorname{Spec} F^{\mathrm{unr}}$.

The map $\operatorname{Spec} F^{\mathrm{unr}} \to \operatorname{Spec} F$ is a $\hat{\mathbf{Z}}$-torsor and the desired isomorphism can be verified at a geometric point of $\operatorname{Spec} F$. We thus reduce to the following assertion: For any torsion $\hat{\mathbf{Z}}$-module M, its $\hat{\mathbf{Z}}$-invariants coincide with its $\mathbf{Z}$-invariants along the natural map $\mathbf{Z} \to \hat{\mathbf{Z}}$. This follows from the computation of group cohomology of $\hat{\mathbf{Z}}$ (*cf.* [Ser79, XIII, §1]). □

Remark 1.2.6. The proof of Lemma 1.2.5 applies when $A(1)$ is replaced by any torsion étale sheaf over $\operatorname{Spec} F$ of order invertible in F. It expresses the fact that the morphism (1.1) induces, universally, an isomorphism on étale cohomology with such coefficients.

1.3. Combinatorics of covers.

1.3.1. Let G be a reductive group scheme over a base scheme S. Let A be a finite abelian group whose order is invertible on S. We employ the notation of §1.1.5.

Denote by $\operatorname{Maps}_e(\mathsf{BG}, \mathsf{B}^4 A(1))$ the space of rigidified morphisms $\mathsf{BG} \to \mathsf{B}^4 A(1)$. We shall recall certain combinatorial data associated to it.

1.3.2. Given a quadratic form $Q : \Lambda \to A(-1)$, we write

$$b : \Lambda \otimes \Lambda \to A(-1)$$

for the associated symmetric form, sending $\lambda_1, \lambda_2 \in \Lambda$ to

$$b(\lambda_1, \lambda_2) := Q(\lambda_1 + \lambda_2) - Q(\lambda_1) - Q(\lambda_2).$$

We say that Q is *strictly Weyl-invariant* if the equality

$$b(\alpha, \lambda) = Q(\alpha)\langle \check{\alpha}, \lambda \rangle \tag{1.14}$$

holds for any $\lambda \in \Lambda$ and any simple coroot $\alpha \in \Delta$.

The right-hand-side of (1.14) makes sense for any $\lambda \in \Lambda_{\mathrm{ad}}$, if we understand $\langle \cdot, \cdot \rangle$ as the canonical pairing between the root lattice and Λ_{ad}. Thus it extends to a bilinear form

$$b_1 : \Lambda_{\mathrm{sc}} \otimes \Lambda_{\mathrm{ad}} \to A(-1).$$

1.3.3. *The pairing b_2.* The coincidence between b and b_1 over $\Lambda_{\mathrm{sc}} \otimes \Lambda$ implies that their adjoints make the following diagram commute:

$$\begin{array}{ccc} \Lambda & \longrightarrow & \mathscr{H}om(\Lambda, \mathrm{A}(-1)) \\ \downarrow & & \downarrow \\ \Lambda_{\mathrm{ad}} & \to & \mathscr{H}om(\Lambda_{\mathrm{sc}}, \mathrm{A}(-1)) \end{array} \tag{1.15}$$

Taking fibers of the vertical maps, we obtain a map

$$\mathrm{Fib}(\Lambda \to \Lambda_{\mathrm{ad}}) \to \mathscr{H}om(\pi_1 \mathrm{G}, \mathrm{A}(-1)). \tag{1.16}$$

Denote by b_2 the bilinear pairing obtained from (1.16) by passing to the adjoint

$$b_2 : \pi_1 \mathrm{G} \otimes \mathrm{Fib}(\Lambda \to \Lambda_{\mathrm{ad}}) \to \mathrm{A}(-1). \tag{1.17}$$

1.3.4. Denote by $\Lambda^\sharp \subset \Lambda$ the kernel of b. Denote by $\Lambda^\sharp_{\mathrm{sc}} \subset \Lambda_{\mathrm{sc}}$, $\Lambda^\sharp_{\mathrm{ad}} \subset \Lambda_{\mathrm{ad}}$ the kernels of b_1. Write $\check{\Lambda}^\sharp$ for the dual of $\Lambda^\sharp$.

For each simple coroot $\alpha \in \Delta$, we shall also write

$$\alpha^\sharp := \mathrm{ord}(\mathrm{Q}(\alpha)) \cdot \alpha, \quad \check{\alpha}^\sharp := \mathrm{ord}(\mathrm{Q}(\alpha))^{-1} \cdot \check{\alpha},$$

where $\mathrm{ord}(\mathrm{Q}(\alpha))$ denotes the order of $\mathrm{Q}(\alpha) \in \mathrm{A}(-1)$. The set $\Delta^\sharp$ of $\alpha^\sharp$ (respectively $\check{\Delta}^\sharp$ of $\check{\alpha}^\sharp$) forms a subsheaf of $\Lambda^\sharp$ (respectively $\check{\Lambda}^\sharp$).

Observe that $\Lambda^\sharp_{\mathrm{sc}}$ is the span of $\Delta^\sharp$: An element $\sum_{\alpha \in \Delta} d_\alpha \cdot \alpha$ of Λ_{sc} belongs to $\Lambda^\sharp_{\mathrm{sc}}$ if and only if it pairs to zero under b_1 against each fundamental coweight ω_α, and this occurs if and only if $d_\alpha \cdot \mathrm{Q}(\alpha) = 0$ for each $\alpha \in \Delta$. Likewise, $\Lambda^\sharp_{\mathrm{ad}}$ is dual to the span of $\check{\Delta}^\sharp \subset \check{\Lambda}^\sharp$.

Moreover, the quadruple

$$(\Delta^\sharp \subset \Lambda^\sharp, \check{\Delta}^\sharp \subset \check{\Lambda}^\sharp) \tag{1.18}$$

is a locally constant étale sheaf of based root data over S. In particular, (1.18) is the root data of a reductive group F-scheme $\mathrm{G}^\sharp$ with sheaf of cocharacters $\Lambda^\sharp$. We decorate with $(\cdot)^\sharp$ all the objects associated to $\mathrm{G}^\sharp$ in §1.1.5.

1.3.5. Write $\mathscr{M}aps_e(\mathsf{BG}, \mathsf{B}^4\mathrm{A}(1))$ for the étale sheaf over S whose sections over an S-scheme S_1 are rigidified morphisms $\mathsf{BG} \times_{\mathrm{S}} \mathrm{S}_1 \to \mathsf{B}^4\mathrm{A}(1)$.

By [Zha22, Proposition 5.1.11], there is a canonical fiber sequence

$$\mathscr{H}om_{\mathbf{Z}}(\pi_1 \mathrm{G}, \mathsf{B}^2\mathrm{A}) \to \mathscr{M}aps_e(\mathsf{BG}, \mathsf{B}^4\mathrm{A}(1)) \to \mathscr{Q}uad(\Lambda, \mathrm{A}(-1))_{\mathrm{st}}, \tag{1.19}$$

where $\mathscr{H}om_{\mathbf{Z}}$ denotes the étale sheaf of $\mathbf{Z}$-linear morphisms and $\mathscr{Q}uad(\Lambda, \mathrm{A}(-1))_{\mathrm{st}}$ denotes the étale sheaf of strictly Weyl-invariant quadratic forms on Λ. The first map in (1.19) is defined by tensoring with $\mathsf{B}\Psi$ and pulling back along $\mathsf{BG} \to \mathsf{BG}_{\mathrm{ab}}$.

In particular, to each rigidified morphism $\mu : \mathsf{BG} \to \mathsf{B}^4\mathrm{A}(1)$, we may associate a strictly Weyl-invariant quadratic form Q and pairings b, b_1, b_2 as well as the étale sheaf of based root data (1.18).

Proposition 1.3.6. *Let μ be a rigidified morphism $\mathsf{BG} \to \mathsf{B}^4\mathrm{A}(1)$. The restriction of μ to $\mathsf{BZ}^\sharp$ canonically lifts to an $\mathbb{E}_\infty$-monoidal morphism*

$$\mu_{\mathrm{Z}^\sharp} : \mathsf{BZ}^\sharp \to \mathsf{B}^4\mathrm{A}(1), \tag{1.20}$$

equipped with a trivialization over $\mathsf{BZ}^\sharp_{\mathrm{sc}}$.

Proof. In [Zha22, §6.1], we construct from μ a canonical $\mathbb{E}_\infty$-monoidal morphism $\mu_{\mathrm{T}^\sharp} : \mathsf{BT}^\sharp \to \mathsf{B}^4\mathrm{A}(1)$ endowed with a trivialization over $\mathsf{BT}^\sharp_{\mathrm{sc}}$. It is enough to identify the restriction of $\mu_{\mathrm{T}^\sharp}$ to $\mathsf{BZ}^\sharp$ with the restriction of μ.

To do this, we recall that $\mu_{T^\sharp}$ is constructed, étale locally over $\operatorname{Spec} F$, by choosing a Borel subgroup $B \subset G$ and restricting μ to $\mathsf{B}B$. The latter descends to $\mathsf{B}T$ and $\mu_{T^\sharp}$ is its pullback to $\mathsf{B}T^\sharp$. This provides an identification between the restrictions of $\mu_{T^\sharp}$ and μ to $\mathsf{B}Z^\sharp$, which *a priori* depends on B. The independence is proved as in [Zha22, §5.2.6]. □

1.3.7. Note that $Z^\sharp/Z^\sharp_{sc}$ is canonically identified with $G^\sharp_{ab}$, so the $\mathbb{E}_\infty$-monoidal morphism (1.20) together with its trivialization over $\mathsf{B}Z^\sharp_{sc}$ defines an $\mathbb{E}_\infty$-monoidal morphism

$$\mu_{G^\sharp_{ab}} : \mathsf{B}G^\sharp_{ab} \to \mathsf{B}^4 A(1). \tag{1.21}$$

The sheaf of $\mathbb{E}_\infty$-monoidal morphisms from $\mathsf{B}G^\sharp_{ab}$ to $\mathsf{B}^4A(1)$ fits into a canonical fiber sequence

$$\mathscr{H}om_{\mathbf{Z}}(\pi_1 G^\sharp, \mathsf{B}^2 A) \to \mathscr{M}aps_{\mathbb{E}_\infty}(\mathsf{B}G^\sharp_{ab}, \mathsf{B}^4 A(1)) \to \mathscr{H}om(\pi_1 G^\sharp, A(-1)_{\text{2-tors}}), \tag{1.22}$$

where $A(-1)_{\text{2-tors}}$ denotes the subsheaf of 2-torsion elements of $A(-1)$. Indeed, this follows from expressing $\mathsf{B}G^\sharp_{ab}$ as the cofiber of $\mathsf{B}T^\sharp_{sc} \to \mathsf{B}T^\sharp$ and reducing to the analogous statement for tori (*cf.* [Zha22, Proposition 4.6.2]).

In particular, it follows from *op.cit.* that the image of $\mu_{G^\sharp_{ab}}$ along the second map of (1.22) is the restriction of Q to $\Lambda^\sharp$, which is valued in $A(-1)_{\text{2-tors}}$ and annihilates $\Lambda^\sharp_{sc}$.

1.4. The local Langlands correspondence.

1.4.1. We specialize to the case where G is a reductive group F-scheme. Fix a finite abelian group A whose order is invertible in F, equipped with an injective character

$$\zeta : A \to \mathbf{C}^\times.$$

Note that ζ identifies A with the subgroup $\mu_N(\mathbf{C})$ for $N := |A|$.

Let μ be a rigidified morphism $\mathsf{B}G \to \mathsf{B}^4 A(1)$. We shall recall Weissman's conjectural local Langlands correspondence for the cover of $G(F)$ defined by μ and explain its extension to extended pure inner forms of G.

1.4.2. For each $\beta \in \mathsf{Isoc}_G$, we apply the construction functor (1.8) to μ to obtain a cover

$$\widetilde{G}_\beta := \int_{F,\beta} \mu.$$

Denote by $\Pi(\widetilde{G}_\beta)$ the set of isomorphism classes of irreducible ζ-genuine smooth representations of $\widetilde{G}_\beta$. Being "ζ-genuine" means that A acts through the character ζ.

As above, we omit the subscript β when it is the trivial G-isocrystal.

1.4.3. On the other hand, the rigidified morphism μ defines the reductive group F-scheme $G^\sharp$ (*cf.* §1.3.4) and the $\mathbb{E}_\infty$-monoidal morphism $\mu_{G^\sharp_{ab}}$ (*cf.* §1.3.7). The Galois side of the local Langlands correspondence depends only on $(G^\sharp, \mu_{G^\sharp_{ab}})$, as opposed to (G, μ).

Denote by H the Langlands dual of $G^\sharp$, viewed as a locally constant étale sheaf of pinned split reductive group $\mathbf{Z}$-schemes. In particular, H is equipped with a Killing pair $T_H \subset B_H \subset H$, where T_H has sheaf of characters $\Lambda^\sharp$.

1.4.4. We shall construct a canonical splitting of the fiber sequence (1.22). The idea of this construction is originally due to Gaitsgory and Lysenko (*cf.* [GL18, §4.8]).

If A has odd degree, then $A(-1)_{\text{2-tors}}$ vanishes and (1.22) trivially splits.

If A has even degree[8], then ζ identifies $A(-1)_{\text{2-tors}}$ with $\mathbf{Z}/2$. To split (1.22), we associate to each character $\epsilon : \pi_1 G^\sharp \to \mathbf{Z}/2$ the $\mathbb{E}_\infty$-monoidal morphism

$$\mathsf{B}G^\sharp_{ab} \xrightarrow{\epsilon \otimes \mathsf{B}\Psi} \mathsf{B}^2\{\pm 1\} \xrightarrow{\text{sgn}} \mathsf{B}^4\{\pm 1\}^{\otimes 2} \to \mathsf{B}^4 A(1), \tag{1.23}$$

[8]By the assumption that $|A|$ is invertible in F, this implies that F has characteristic $\neq 2$.

where sgn is the $\mathbb{E}_\infty$-monoidal morphism constructed below.

1.4.5. *Construction of* sgn. We work over the base scheme $\mathrm{S} := \operatorname{Spec} \mathbf{Z}[\frac{1}{2}]$. The étale sheaf $\mathscr{M}aps_{\mathbb{E}_\infty}(\mathsf{B}^2\{\pm 1\}, \mathsf{B}^4\{\pm 1\}^{\otimes 2})$ is the fiber of the map

$$\mathscr{M}aps_{\mathbb{E}_\infty}(\mathsf{B}\mathbb{G}_m, \mathsf{B}^4\{\pm 1\}^{\otimes 2}) \to \mathscr{M}aps_{\mathbb{E}_\infty}(\mathsf{B}\mathbb{G}_m, \mathsf{B}^4\{\pm 1\}^{\otimes 2}) \tag{1.24}$$

given by pullback along $(\cdot)^2 : \mathsf{B}\mathbb{G}_m \to \mathsf{B}\mathbb{G}_m$.

On the other hand, the functor of taking loop spaces and applying $\mathscr{M}aps_e(\mathbb{G}_m, \cdot)$ yields an equivalence (*cf.* [Zha22, Proposition 4.6.6])

$$\mathscr{M}aps_{\mathbb{E}_\infty}(\mathsf{B}\mathbb{G}_m, \mathsf{B}^4\{\pm 1\}^{\otimes 2}) \xrightarrow{\simeq} \mathscr{M}aps_{\mathbb{E}_\infty}(\mathbf{Z}, \mathsf{B}^2\{\pm 1\}). \tag{1.25}$$

This induces an identification of the fiber of (1.24)

$$\mathscr{M}aps_{\mathbb{E}_\infty}(\mathsf{B}^2\{\pm 1\}, \mathsf{B}^4\{\pm 1\}^{\otimes 2}) \xrightarrow{\simeq} \mathscr{M}aps_{\mathbb{E}_\infty}(\mathbf{Z}/2, \mathsf{B}^2\{\pm 1\}). \tag{1.26}$$

Note that an $\mathbb{E}_\infty$-monoidal morphism $\mathbf{Z}/2 \to \mathsf{B}^2\{\pm 1\}$ is equivalent to a symmetric monoidal extension of $\mathbf{Z}/2$ by $\mathsf{B}\{\pm 1\}$. We define $\mathrm{sgn} : \mathsf{B}^2\{\pm 1\} \to \mathsf{B}^4\{\pm 1\}^{\otimes 2}$ to be the $\mathbb{E}_\infty$-monoidal morphism whose image under (1.26) is the trivial monoidal extension of $\mathbf{Z}/2$ by $\mathsf{B}\{\pm 1\}$, with commutativity constraint specified by the pairing

$$\mathbf{Z}/2 \otimes \mathbf{Z}/2 \to \{\pm 1\}, \quad a, b \mapsto (-1)^{ab}.$$

1.4.6. Under the splitting of (1.22), we may write $\mu_{\mathrm{G}^\sharp_{\mathrm{ab}}}$ as a sum

$$\mu_{\mathrm{G}^\sharp_{\mathrm{ab}}} \xrightarrow{\simeq} \mu^{(1)}_{\mathrm{G}^\sharp_{\mathrm{ab}}} + \mu^{(2)}_{\mathrm{G}^\sharp_{\mathrm{ab}}}, \tag{1.27}$$

where $\mu^{(1)}_{\mathrm{G}^\sharp_{\mathrm{ab}}}$ is the composition (1.23) applied to the character ϵ defined by the restriction of Q to $\Lambda^\sharp$ (*cf.* §1.3.7) and $\mu^{(2)}_{\mathrm{G}^\sharp_{\mathrm{ab}}}$ is defined by a $\mathbf{Z}$-linear morphism

$$\pi_1 \mathrm{G}^\sharp \to \mathsf{B}^2 \mathrm{A}. \tag{1.28}$$

1.4.7. We shall now convert the data $\epsilon : \pi_1 \mathrm{G}^\sharp \to \mathbf{Z}/2$ and (1.28) to the Galois side. For this, it helps to introduce a bit of formalism.

Given a pro-space $\mathrm{X} = \lim_{i\in \mathrm{I}} \mathrm{X}_i$ and a sheaf of abelian groups $\mathscr{A}$ over some X_i, we write

$$\Gamma(\mathrm{X}, \mathscr{A}[n]) := \operatorname*{colim}_{j \in \mathrm{I}_{/i}} \Gamma(\mathrm{X}_j, \mathscr{A}[n]),$$

where the transition maps are given by pullbacks. We refer to objects of the $\mathbf{Z}$-linear space underlying $\Gamma(\mathrm{X}, \mathscr{A}[2])$ as $\mathscr{A}$*-gerbes* over X. Thus, the total space of an $\mathscr{A}$-gerbe over X is a pro-space over X.

Givan pro-group $\Sigma = \lim_{i\in\mathrm{I}} \Sigma_i$, we may apply the above formalism to the pro-space $*/\Sigma := \lim_{i\in \mathrm{I}} */\Sigma_i$. Any Σ_i-module $\mathscr{A}$ may be regarded as a sheaf of abelian groups over $*/\Sigma_i$, and we use $\mathsf{Z}^n(\Sigma, \mathscr{A})$ to denote the $\mathbf{Z}$-linear space underlying $\Gamma(*/\Sigma, \mathscr{A}[n])$. In particular, we have an isomorphism whenever $0 \le m \le n$:

$$\pi_m \mathsf{Z}^n(\Sigma, \mathscr{A}) \xrightarrow{\simeq} \mathrm{H}^{n-m}(\Sigma, \mathscr{A}),$$

the right-hand-side being the continuous group cohomology of Σ with coefficients in $\mathscr{A}$.

1.4.8. Fix an algebraic closure $\bar{\mathrm{F}}$ of F lifting $\bar{\mathsf{f}}$.

Denote by W_F the Weil group of F, which we view as a pro-group $\mathrm{W}_\mathrm{F} := \lim \Sigma$, where the formal limit is taken over discrete quotients $\mathrm{W}_\mathrm{F} \twoheadrightarrow \Sigma$.

By taking fibers at the geometric point $\operatorname{Spec} \bar{\mathrm{F}}$, we may view the étale sheaf $\mathrm{H}(\mathbf{C})$ as a group with a W_F-action through a finite quotient Σ. In particular, $\mathrm{H}(\mathbf{C})$ may be viewed as a sheaf of groups over $*/\Sigma$.

Analogously, $\mathrm{Z_H}(\mathbf{C})$ may be viewed as a sheaf of abelian groups over $*/\Sigma$. The formalism of §1.4.7 allows us to make sense of $\mathrm{Z_H}(\mathbf{C})$-gerbes over $*/\mathrm{W_F}$.

1.4.9. *The meta-Weil group.* Consider the central extension

$$1 \to \{\pm 1\} \to \widetilde{\mathrm{F}}^{\times}_{\mathrm{Hilb}} \to \mathrm{F}^{\times} \to 1 \tag{1.29}$$

defined by the quadratic Hilbert symbol $\{\cdot,\cdot\}$ as cocycle, *i.e.* we have $\widetilde{\mathrm{F}}^{\times}_{\mathrm{Hilb}} := \mathrm{F}^{\times} \times \{\pm 1\}$ with the group structure $(a,1)\cdot(b,1) := (ab, \{a,b\})$.

The *meta-Weil group* is defined to be the pullback of (1.29) along the Artin reciprocity map $\mathrm{W_F} \to \mathrm{F}^{\times}$ (*cf.* [Wei18, §4])

$$1 \to \{\pm 1\} \to \widetilde{\mathrm{W}}_{\mathrm{F}} \to \mathrm{W_F} \to 1. \tag{1.30}$$

Taking classifying spaces, (1.30) yields a $\{\pm 1\}$-gerbe over $*/\mathrm{W_F}$.

1.4.10. Denote by $\widetilde{\mathrm{Z}}^{(1)}_{\mathrm{H}}$ the $\mathrm{Z_H}(\mathbf{C})$-gerbe over $*/\mathrm{W_F}$ induced from (1.30) along the dual $\epsilon^{\vee} : \{\pm 1\} \to \mathrm{Z_H}(\mathbf{C})$ of the character ϵ.

Denote by $\widetilde{\mathrm{Z}}^{(2)}_{\mathrm{H}}$ the $\mathrm{Z_H}(\mathbf{C})$-gerbe over $*/\mathrm{W_F}$ defined by composing (1.28) with ζ. Here, we invoked the passage from étale $\mathrm{Z_H}(\mathbf{C})$-gerbes over $\operatorname{Spec}\mathrm{F}$ to $\mathrm{Z_H}(\mathbf{C})$-gerbes over $*/\mathrm{W_F}$.

Consider the sum of $\mathrm{Z_H}(\mathbf{C})$-gerbes

$$\widetilde{\mathrm{Z}}_{\mathrm{H}} := \widetilde{\mathrm{Z}}^{(1)}_{\mathrm{H}} + \widetilde{\mathrm{Z}}^{(2)}_{\mathrm{H}}. \tag{1.31}$$

1.4.11. Inducing (the total space of) $\widetilde{\mathrm{Z}}_{\mathrm{H}}$ along the morphism $\mathrm{Z_H}(\mathbf{C}) \to \mathrm{H}(\mathbf{C})$ of sheaves of groups over $*/\mathrm{W_F}$, we obtain a pro-space $\widetilde{\mathrm{H}}$ over $*/\mathrm{W_F}$.

By an *L-parameter*, we shall mean a section of the projection $\widetilde{\mathrm{H}} \to */\mathrm{W_F}$.

For any standard parabolic subgroup $\mathrm{P_H} \subset \mathrm{H}$ with standard Levi subgroup $\mathrm{M_H} \subset \mathrm{P_H}$, one may induce $\widetilde{\mathrm{Z}}_{\mathrm{H}}$ along $\mathrm{Z_H}(\mathbf{C}) \to \mathrm{M_H}(\mathbf{C}) \to \mathrm{P_H}(\mathbf{C})$ to obtain pro-spaces $\widetilde{\mathrm{M}}_{\mathrm{H}}$ and $\widetilde{\mathrm{P}}_{\mathrm{H}}$ over $*/\mathrm{W_F}$. An L-parameter $*/\mathrm{W_F} \to \widetilde{\mathrm{H}}$ is called *semisimple* if, whenever it factors through $\widetilde{\mathrm{P}}_{\mathrm{H}}$ for some standard parabolic subgroup $\mathrm{P_H}$, it factors through $\widetilde{\mathrm{M}}_{\mathrm{H}}$.

Denote by $\Phi(\widetilde{\mathrm{H}})$ the set of isomorphism classes of semisimple L-parameters. We shall often refer to elements of $\Phi(\widetilde{\mathrm{H}})$ simply as "L-parameters".

Remark 1.4.12. Let us remark on why we define L-parameters in terms of $\widetilde{\mathrm{H}}$ rather than the more concrete definition in terms of an "L-group". The discrepancy has to do with the choice of base points.

Indeed, choosing a base point of $\widetilde{\mathrm{Z}}_{\mathrm{H}}$ and taking loop spaces, we obtain an extension of pro-groups

$$1 \to \mathrm{Z_H}(\mathbf{C}) \to \Omega(\widetilde{\mathrm{Z}}_{\mathrm{H}}) \to \mathrm{W_F} \to 1.$$

Likewise, the induced base point of $\widetilde{\mathrm{H}}$ gives rise to an extension $\Omega(\widetilde{\mathrm{H}})$ of $\mathrm{W_F}$ by $\mathrm{H}(\mathbf{C})$, which may be considered as the "L-group".

An L-parameter $*/\mathrm{W_F} \to \widetilde{\mathrm{H}}$ is (non-canonically) isomorphic to one which preserves the base point, which is equivalent to a section $\mathrm{W_F} \to \Omega(\widetilde{\mathrm{H}})$. Note that $\Omega(\widetilde{\mathrm{H}})$ is the pullback of some extension $\Omega(\widetilde{\mathrm{H}})_f$ of a discrete quotient of $\mathrm{W_F}$ by $\mathrm{H}(\mathbf{C})$. The composite

$$\mathrm{W_F} \to \Omega(\widetilde{\mathrm{H}}) \twoheadrightarrow \Omega(\widetilde{\mathrm{H}})_f$$

is a morphism of pro-groups, so it factors through a discrete quotient of $\mathrm{W_F}$. Thus, we recover the classical notion of an L-parameter (or more precisely, a Weil parameter).[9]

[9] Weissman provides a different recipe for restoring the independence of base points, by explicitly lifting the "category of L-groups" to a 2-category (*cf.* [Wei18, §5.1]).

1.4.13. The following is Weissman's version of the local Langlands correspondence (*cf.* [Wei18, Conjecture 0.1]).

Conjecture 1.4.14 (Weissman)**.** *There is a natural finite-to-one map*

$$\mathrm{LLC} : \Pi(\widetilde{\mathrm{G}}) \to \Phi(\widetilde{\mathrm{H}}). \tag{1.32}$$

1.4.15. Let us include the covers $\widetilde{\mathrm{G}}_\beta$ in the formulation of Conjecture 1.4.14.

Conjecture 1.4.16. *For each* $\beta \in \mathsf{Basic}_{\mathrm{G}}$*, there is a natural finite-to-one map*

$$\mathrm{LLC}_\beta : \Pi(\widetilde{\mathrm{G}}_\beta) \to \Phi(\widetilde{\mathrm{H}}). \tag{1.33}$$

2. Sharp covers

Let F be a local field with a fixed algebraic closure $\bar{\mathrm{F}}$.

The goal of this subsection is to construct the local Langlands correspondence for sharp tori (*cf.* (2.7)). This is a consequence of Theorem 2.1.6 whose proof occupies §2.2 and §2.3.

In §2.4, we will use this result to establish the Langlands correspondence for the "sharp center". This will be needed for the formulation of compatibility with central core character (*cf.* §4). In §2.5, we explain another consequence of Theorem 2.1.6 which will not be needed later. Its purpose is to justify why the local Langlands correspondence for sharp covers is not far from the local Langlands correspondence for linear reductive groups.

2.1. Duality for tori.

2.1.1. Given topological abelian groups A_1, A_2, we write $\mathsf{E}^1(\mathrm{A}_1, \mathrm{A}_2)$ for the groupoid of commutative extensions of A_1 by A_2. We endow $\mathbf{C}^\times$ with the discrete topology.

For any F-torus T, we shall construct a $\mathbf{Z}$-linear functor (natural in T)

$$\mathsf{L}_{\mathrm{T}} : \mathsf{Z}^2(\mathrm{W}_{\mathrm{F}}, \check{\mathrm{T}}(\mathbf{C})) \to \mathsf{E}^1(\mathrm{T}(\mathrm{F}), \mathbf{C}^\times), \tag{2.1}$$

where $\check{\mathrm{T}}$ stands for the Langlands dual of T and the left-hand-side is defined in §1.4.7.

2.1.2. *Construction of* (2.1)*.* Since the group $\mathrm{H}^2(\mathrm{W}_{\mathrm{F}}, \check{\mathrm{T}}(\mathbf{C}))$ vanishes (*cf.* [Kar13, Theorem 3.2.2]), the space $\mathsf{Z}^2(\mathrm{W}_{\mathrm{F}}, \check{\mathrm{T}}(\mathbf{C}))$ is connected and thus identified with the classifying space of $\mathsf{Z}^1(\mathrm{W}_{\mathrm{F}}, \check{\mathrm{T}}(\mathbf{C}))$.

The automorphism group of the zero object in $\mathsf{E}^1(\mathrm{T}(\mathrm{F}), \mathbf{C}^\times)$ is the group $\mathrm{Hom}(\mathrm{T}(\mathrm{F}), \mathbf{C}^\times)$ of continuous characters. To define (2.1), it suffices to define a $\mathbf{Z}$-linear functor

$$\mathsf{Z}^1(\mathrm{W}_{\mathrm{F}}, \check{\mathrm{T}}(\mathbf{C})) \to \mathrm{Hom}(\mathrm{T}(\mathrm{F}), \mathbf{C}^\times). \tag{2.2}$$

The functor (2.2) is set to be the projection $\mathsf{Z}^1(\mathrm{W}_{\mathrm{F}}, \check{\mathrm{T}}(\mathbf{C})) \to \mathrm{H}^1(\mathrm{W}_{\mathrm{F}}, \check{\mathrm{T}}(\mathbf{C}))$, followed by Langlands duality for tori (*cf.* [Yu09, §7.5])

$$\mathrm{H}^1(\mathrm{W}_{\mathrm{F}}, \check{\mathrm{T}}(\mathbf{C})) \xrightarrow{\simeq} \mathrm{Hom}(\mathrm{T}(\mathrm{F}), \mathbf{C}^\times). \tag{2.3}$$

Remark 2.1.3. By construction, $\pi_1 \mathsf{L}_{\mathrm{T}}$ is the Langlands duality (2.3) for T.

Remark 2.1.4. There is some asymmetry in the way we defined L_{T}: The left-hand-side is a 2-groupoid, while the right-hand-side is a 1-groupoid. This is due to similar asymmetry in the usual formulation of Langlands duality for tori (2.3). A better formulation would be an equivalence of groupoids

$$\mathsf{Z}^1(\mathrm{W}_{\mathrm{F}}, \check{\mathrm{T}}(\mathbf{C})) \xrightarrow{\simeq} \mathrm{Hom}(\mathsf{Isoc}_{\mathrm{T}}, */\mathbf{C}^\times), \tag{2.4}$$

where $\mathsf{Isoc}_{\mathrm{T}}$ is understood as a pro-Picard groupoid. The equivalence (2.4) ought to recover (2.3) on π_0 and the Pontryagin duality between $\check{\mathrm{T}}(\mathbf{C})^{\mathrm{Gal}_{\mathrm{F}}}$ and $(\pi_1 \mathrm{T})_{\mathrm{Gal}_{\mathrm{F}}}$ on π_1.

We will not adopt this point of view in the present article, since the benefits it brings are not visible at the level of our results.

2.1.5. We now let A be a finite abelian group of order invertible in F, equipped with an injective character $\zeta : \mathrm{A} \to \mathbf{C}^\times$.

Let T be an F-torus endowed with an $\mathbb{E}_\infty$-monoidal morphism $\mu : \mathsf{B}\mathrm{T} \to \mathsf{B}^4\mathrm{A}(1)$. Applying the construction functor (1.6) to μ, we obtain a cover $\widetilde{\mathrm{T}}$ of T(F). It is commutative since μ is $\mathbb{E}_\infty$-monoidal. Inducing along ζ, we obtain a commutative extension

$$1 \to \mathbf{C}^\times \to \widetilde{\mathrm{T}}_\zeta \to \mathrm{T}(\mathrm{F}) \to 1. \tag{2.5}$$

We shall view $\widetilde{\mathrm{T}}_\zeta$ as an object of $\mathsf{E}^1(\mathrm{T}(\mathrm{F}), \mathbf{C}^\times)$.

We shall now apply the construction of the dual datum (1.31) to (T, μ). In the present context, we have $\mathrm{Z}_\mathrm{H} \cong \mathrm{H} \cong \check{\mathrm{T}}$. Thus (1.31) is a $\check{\mathrm{T}}(\mathbf{C})$-gerbe $\widetilde{\check{\mathrm{T}}}$ over $*/\mathrm{W}_\mathrm{F}$, which we view as an object of $\mathsf{Z}^2(\mathrm{W}_\mathrm{F}, \check{\mathrm{T}}(\mathbf{C}))$.

Theorem 2.1.6. *There is a canonical isomorphism in* $\mathsf{E}^1(\mathrm{T}(\mathrm{F}), \mathbf{C}^\times)$ *functorial in* (T, μ)*:*

$$\mathsf{L}_\mathrm{T}(\widetilde{\check{\mathrm{T}}}) \xrightarrow{\cong} \widetilde{\mathrm{T}}_\zeta. \tag{2.6}$$

2.1.7. The construction of (2.6) requires some effort and will be completed in §2.3.

The idea is as follows: The decomposition (1.27) exhibits μ as the sum of a "sign component" $\mu^{(1)}$ and a $\mathbf{Z}$-linear component $\mu^{(2)}$. The resulting cover of $\widetilde{\mathrm{T}}_\zeta$ is thus a Baer sum of two covers. Correspondingly, $\widetilde{\check{\mathrm{T}}}$ is also the sum of two $\check{\mathrm{T}}(\mathbf{C})$-gerbes. We will construct the isomorphism (2.6) for these two summands separately and obtain the general case by adding them up, using the $\mathbf{Z}$-linearity of L_T.

2.1.8. Theorem 2.1.6 yields the local Langlands correspondence (1.32) for (T, μ).

Indeed, the functor L_T carries trivializations of $\widetilde{\check{\mathrm{T}}}$ to trivializations of $\widetilde{\mathrm{T}}_\zeta$. The latter are in bijection with the set $\Pi(\widetilde{\mathrm{T}})$ of ζ-genuine characters of $\widetilde{\mathrm{T}}$.

Furthermore, this map intertwines the $\mathsf{Z}^1(\mathrm{W}_\mathrm{F}, \check{\mathrm{T}}(\mathbf{C}))$-action on trivializations of $\widetilde{\check{\mathrm{T}}}$ with the $\mathrm{Hom}(\mathrm{T}(\mathrm{F}), \mathbf{C}^\times)$-action on trivializations of $\widetilde{\mathrm{T}}_\zeta$, via the map (2.2). Since (2.2) induces an isomorphism on π_0, L_T induces an isomorphism

$$\Phi(\widetilde{\check{\mathrm{T}}}) \xrightarrow{\cong} \Pi(\widetilde{\mathrm{T}}). \tag{2.7}$$

The local Langlands correspondence for (T, μ) is define to be the inverse of (2.7).

2.2. The sign component.

2.2.1. In this subsection, we assume $\mathrm{char}\,\mathrm{F} \neq 2$. Our goal is to construct (2.6) when $\mu = \mu^{(1)}$, *i.e.* when it arises from the $\mathbb{E}_\infty$-monoidal morphism $\mathrm{sgn} : \mathsf{B}^2\{\pm 1\} \to \mathsf{B}^4\{\pm 1\}^{\otimes 2}$ (*cf.* §1.4.5) by pre-composing with $\epsilon \otimes \Psi$ and post-composing with the inclusion of $\{\pm 1\}$ in A.

Let us begin by treating the "universal" case, where ϵ is the identity on $\mathbf{Z}/2$.

2.2.2. Viewing sgn as a section of the fiber of (1.24) and applying the construction functor (1.6) for $\mathrm{G} := \mathbb{G}_m$, we obtain a cover of $\mathrm{F}^\times$ whose pullback along $(\cdot)^2 : \mathrm{F}^\times \to \mathrm{F}^\times$ is endowed with a splitting.

These data can be packaged by a diagram of topological groups

$$\begin{array}{ccccccccc} & & & & & & \mathrm{F}^\times & & \\ & & & & & \overset{\tau}{\swarrow} & \downarrow{\scriptstyle (\cdot)^2} & & \\ 1 & \longrightarrow & \{\pm 1\} & \longrightarrow & \widetilde{\mathrm{F}}^\times_{\mathrm{sgn}} & \longrightarrow & \mathrm{F}^\times & \longrightarrow & 1 \end{array} \tag{2.8}$$

where the lower row is a double cover of $\mathrm{F}^\times$.

Our main result of this subsection is the explicit identification of (2.8). The answer involves the cover (1.29) defined by the quadratic Hilbert symbol.

Proposition 2.2.3. *There is a canonical isomorphism of covers*

$$\widetilde{\mathrm{F}}^\times_{\mathrm{sgn}} \xrightarrow{\simeq} \widetilde{\mathrm{F}}^\times_{\mathrm{Hilb}} \tag{2.9}$$

such that τ *corresponds, under the natural bijection* $\widetilde{\mathrm{F}}^\times_{\mathrm{Hilb}} \cong \mathrm{F}^\times \times \{\pm 1\}$*, to the map*

$$\mathrm{F}^\times \to \mathrm{F}^\times \times \{\pm 1\}, \quad a \mapsto (a^2, \{a, a\}).$$

2.2.4. In order to construct the isomorphism (2.9), we first need to describe the rigidified morphism $\mathsf{B}\mathbb{G}_m \to \mathsf{B}^4\{\pm 1\}^{\otimes 2}$ defining the cover $\widetilde{\mathrm{F}}^\times_{\mathrm{Hilb}}$.

Recall that the fiber sequence (1.19) for $\mathrm{G} := \mathbb{G}_m$ and $\mathrm{A} := \{\pm 1\}$ admits a canonical splitting (*cf.* [Zha22, Remark 4.2.8])

$$\mathscr{M}aps_e(\mathsf{B}\mathbb{G}_m, \mathsf{B}^4\{\pm 1\}^{\otimes 2}) \xrightarrow{\simeq} \mathsf{B}^2\{\pm 1\} \oplus \mathbf{Z}/2. \tag{2.10}$$

The inclusion of $\mathbf{Z}/2$ is defined by sending 1 to cup product $\mathsf{B}\Psi \cup \mathsf{B}\Psi$, where $\mathsf{B}\Psi : \mathsf{B}\mathbb{G}_m \to \mathsf{B}^2\{\pm 1\}$ is the deloop of the Kummer map.[10]

Claim: The image of $\mathsf{B}\Psi \cup \mathsf{B}\Psi$ under the construction functor (1.6) for $\mathrm{G} := \mathbb{G}_m$ is canonically identified with $\widetilde{\mathrm{F}}^\times_{\mathrm{Hilb}}$.

2.2.5. *Proof of Claim.* Let us make the functor (1.6) more explicit. Given a rigidified morphism $\mu : \mathsf{B}\mathbb{G}_m \to \mathsf{B}^4\{\pm 1\}^{\otimes 2}$, we obtain a $\mathbb{E}_1$-monoidal morphism $\mathbb{G}_m \to \mathsf{B}^3\{\pm 1\}^{\otimes 2}$ by taking loop spaces. The fiber $\mathbb{G}_m^\dagger$ of the latter fits into a fiber sequence of $\mathbb{E}_1$-monoidal stacks

$$\mathsf{B}^2\{\pm 1\}^{\otimes 2} \to \mathbb{G}_m^\dagger \to \mathbb{G}_m. \tag{2.11}$$

Evaluating (2.11) at $\mathrm{Spec}\,\mathrm{F}$ and using the vanishing of $\mathrm{H}^3(\mathrm{Spec}\,\mathrm{F}, \{\pm 1\}^{\otimes 2})$, we obtain a short exact sequence of groups

$$1 \to \mathrm{H}^2(\mathrm{Spec}\,\mathrm{F}, \{\pm 1\}^{\otimes 2}) \to \widetilde{\mathbb{G}}_m \to \mathrm{F}^\times \to 1. \tag{2.12}$$

The image of μ under (1.6) is given by (2.12) under Tate duality $\mathrm{H}^2(\mathrm{Spec}\,\mathrm{F}, \{\pm 1\}^{\otimes 2}) \cong \{\pm 1\}$, endowed with the topology defined by distinguished sections (*cf.* [Zha22, §2.1.4]).

In the special case $\mu := \mathsf{B}\Psi \cup \mathsf{B}\Psi$, the fiber sequence (2.11) canonically splits as a fiber sequence of *pointed* stacks (*cf.* [Zha22, Proposition 4.4.5]). Its monoidal product can thus be described by a cocycle $\mathbb{G}_m \times \mathbb{G}_m \to \mathsf{B}^2\{\pm 1\}^{\otimes 2}$, which one identifies with the external cup product of Ψ with itself. This implies that the induced short exact sequence (2.12) has a canonical set-theoretic splitting, with cocycle given by the Galois symbol

$$\mathrm{F}^\times \otimes \mathrm{F}^\times \to \mathrm{H}^2(\mathrm{Spec}\,\mathrm{F}, \{\pm 1\}^{\otimes 2}), \quad a \otimes b \mapsto [\Psi(a)] \cup [\Psi(b)], \tag{2.13}$$

where $[\Psi(a)]$ is the Kummer class of $a \in \mathrm{F}^\times$. However, (2.13) becomes the quadratic Hilbert symbol after identifying $\mathrm{H}^2(\mathrm{Spec}\,\mathrm{F}, \{\pm 1\}^{\otimes 2})$ with $\{\pm 1\}$ under Tate duality. □

2.2.6. Note that every rigidified morphism $\mathsf{B}\mathbb{G}_m \to \mathsf{B}^4\{\pm 1\}^{\otimes 2}$ is canonically $\mathbb{E}_\infty$-monoidal because its associated symmetric form vanishes (*cf.* [Zha22, Proposition 4.6.2]), so we may view $\mathsf{B}\Psi \cup \mathsf{B}\Psi$ as an $\mathbb{E}_\infty$-monoidal morphism $\mathsf{B}\mathbb{G}_m \to \mathsf{B}^4\{\pm 1\}^{\otimes 2}$. Let us identify its image in $\mathscr{M}aps_{\mathbb{E}_\infty}(\mathbf{Z}, \mathsf{B}^2\{\pm 1\})$ under (1.25), viewed as a symmetric monoidal extension:

$$\mathsf{B}\{\pm 1\} \to \widetilde{\mathbf{Z}}_{\mathrm{Hilb}} \to \mathbf{Z}. \tag{2.14}$$

[10] We temporarily depart from our convention where Ψ has coefficients in $\hat{\mathbf{Z}}(1)$.

By construction, (2.14) is related to (2.11) (for $\mu := \mathsf{B}\Psi \cup \mathsf{B}\Psi$) as follows: We apply the functor $\mathscr{M}aps_e(\mathbb{G}_m, -)$ to (2.11) and form the pullback and pushout along the maps

$$\mathbf{Z} \to \mathscr{M}aps_e(\mathbb{G}_m, \mathbb{G}_m), \quad \mathscr{M}aps_e(\mathbb{G}_m, \mathsf{B}^2\{\pm 1\}^{\otimes 2}) \xrightarrow{\simeq} \mathsf{B}\{\pm 1\},$$

where the first map sends $a \in \mathbf{Z}$ to the character $x \mapsto x^a$ and the second map is defined by the étale cohomology of $\mathbb{G}_m$, *i.e.* the inverse to tensoring with Ψ.

2.2.7. *Description of* $\widetilde{\mathbf{Z}}_{\mathrm{Hilb}}$. Since (2.11) (for $\mu := \mathsf{B}\Psi \cup \mathsf{B}\Psi$) admits a canonical splitting as a sequence of pointed stacks, so does (2.14). Let us record this splitting as an isomorphism of pointed stacks

$$\widetilde{\mathbf{Z}}_{\mathrm{Hilb}} \xrightarrow{\simeq} \mathbf{Z} \times \mathsf{B}\{\pm 1\}. \tag{2.15}$$

Using (2.15), we may write the monoidal product on $\widetilde{\mathbf{Z}}_{\mathrm{Hilb}}$ as a cocycle

$$\mathbf{Z} \times \mathbf{Z} \to \mathsf{B}\{\pm 1\}. \tag{2.16}$$

Let us write $\Psi(-1)$ for the $\{\pm 1\}$-torsor of square roots of -1. There is a natural isomorphism $\Psi \cup \Psi \cong \Psi \otimes p^*\Psi(-1)$ in $\mathscr{M}aps_e(\mathbb{G}_m, \mathsf{B}^2\{\pm 1\}^{\otimes 2})$, where $p : \mathbb{G}_m \to \operatorname{Spec} \mathrm{F}$ is the projection (*cf.* [Zha22, Theorem 3.1.5]). Thus the cocycle (2.16) sends $(a, b) \in \mathbf{Z} \times \mathbf{Z}$ to the ab-multiple of $\Psi(-1)$. The associator of the monoidal product is given by the bilinearity of (2.16).

It remains to describe the commutativity constraint on $\widetilde{\mathbf{Z}}_{\mathrm{Hilb}}$. By the above description of the monoidal product, this is specified by an isomorphism $ab \cdot \Psi(-1) \cong ba \cdot \Psi(-1)$ for each $a, b \in \mathbf{Z}$, *i.e.* by a bilinear pairing

$$\mathbf{Z} \otimes \mathbf{Z} \to \{\pm 1\}. \tag{2.17}$$

(The bilinearity is a consequence of the hexagon axiom.) By [Zha22, Proposition 4.6.6], the value of this pairing at $1 \otimes 1$ is -1. Thus (2.17) is given by $a \otimes b \mapsto (-1)^{ab}$.

Remark 2.2.8. It is also possible to arrive at the above description of the monoidal structure on $\widetilde{\mathbf{Z}}_{\mathrm{Hilb}}$ by comparing with Brylinski and Deligne's classification of central extensions of $\mathbb{G}_m$ by K_2 (*cf.* [BD01, §3]).

Indeed, $\mathsf{B}\Psi \cup \mathsf{B}\Psi$ is the image, under étale realization (*cf.* [Zha22, §2.3.2]), of the central extension E of $\mathbb{G}_m$ by K_2 defined using the canonical pairing $\mathbb{G}_m \otimes \mathbb{G}_m \to \mathsf{K}_2$ as cocycle. The étale realization is compatible with second Brylinski–Deligne invariants, in the sense that we have a commutative square of $\mathbb{E}_1$-monoidal stacks

$$\begin{array}{ccc} \mathbf{Z} & \longrightarrow & \mathsf{B}\mathbb{G}_m \\ \downarrow \simeq & & \downarrow \Psi \\ \mathbf{Z} & \longrightarrow & \mathsf{B}^2\{\pm 1\} \end{array} \tag{2.18}$$

where the top horizontal arrow is the second Brylinski–Deligne invariant of E and the bottom horizontal arrow is the $\mathbb{E}_1$-monoidal morphism corresponding to $\widetilde{\mathbf{Z}}_{\mathrm{Hilb}}$.[11] Now, the second Brylinski–Deligne invariant of E is the central extension of $\mathbf{Z}$ by $\mathbb{G}_m$, defined using $a, b \mapsto (-1)^{ab}$ as cocycle. This implies the above description of $\widetilde{\mathbf{Z}}_{\mathrm{Hilb}}$ as a monoidal stack.

2.2.9. *Monoidal splitting of* $\widetilde{\mathbf{Z}}_{\mathrm{Hilb}}$. Let us construct a splitting of (2.14) as a fiber sequence of *monoidal* stacks. Under the identification (2.15), this splitting is given by

$$\mathbf{Z} \to \widetilde{\mathbf{Z}}_{\mathrm{Hilb}}, \quad a \mapsto (a, \binom{a}{2} \cdot \Psi(-1)). \tag{2.19}$$

[11] We make an important cautionary remark. Since the cocycle $a, b \mapsto (-1)^{ab}$ is commutative, the top horizontal arrow in (2.18) is symmetric monoidal. The bottom horizontal arrow is also $\mathbb{E}_\infty$-monoidal because $\mathsf{B}\Psi \cup \mathsf{B}\Psi$ is. However, (2.18) is *not* a commutative diagram of $\mathbb{E}_\infty$-monoidal stacks: The top circuit is $\mathbf{Z}$-linear while the bottom circuit is not.

Because the cocycle of $\widetilde{\mathbf{Z}}_{\mathrm{Hilb}}$ is given by $a, b \mapsto ab \cdot \Psi(-1)$, the fact that this is a monoidal splitting follows from the equality of integers

$$\binom{a+b}{2} - \binom{a}{2} - \binom{b}{2} = ab.$$

Denote by $\widetilde{\mathbf{Z}}_{\mathrm{sgn}}$ the trivial monoidal extension of $\mathbf{Z}$ by $\mathsf{B}\{\pm 1\}$ with commutativity constraint specified by $\mathbf{Z} \otimes \mathbf{Z} \to \{\pm 1\}$, $a, b \mapsto (-1)^{ab}$. The monoidal splitting (2.19) exhibits an isomorphism of symmetric monoidal extensions of $\mathbf{Z}$ by $\mathsf{B}\{\pm 1\}$:

$$\widetilde{\mathbf{Z}}_{\mathrm{sgn}} \xrightarrow{\simeq} \widetilde{\mathbf{Z}}_{\mathrm{Hilb}}. \tag{2.20}$$

2.2.10. Finally, the construction of (2.20) renders it *incompatible* with the natural splittings of the two sides over $2 : \mathbf{Z} \to \mathbf{Z}$. Let us be more precise.

The extension $\widetilde{\mathbf{Z}}_{\mathrm{sgn}}$ is monoidally equivalent to $\mathbf{Z} \times \mathsf{B}\{\pm 1\}$ by construction, so it admits a splitting over $2 : \mathbf{Z} \to \mathbf{Z}$ sending $a \in \mathbf{Z}$ to $(2a, 1)$. In other words, this is the splitting induced from sgn (as a symmetric monoidal extension of $\mathbf{Z}/2$ by $\mathsf{B}\{\pm 1\}$), by pulling back along $\mathbf{Z} \to \mathbf{Z}/2$.

The composition of this splitting with (2.20) is the map

$$\mathbf{Z} \to \widetilde{\mathbf{Z}}_{\mathrm{Hilb}}, \quad a \mapsto (2a, a \cdot \Psi(-1)), \tag{2.21}$$

because of the identity

$$\binom{2a}{2} = a \mod 2.$$

2.2.11. We are now ready to construct the isomorphism (2.9).

Proof of Proposition 2.2.3. We shall construct an isomorphism of $\mathbb{E}_\infty$-monoidal morphism $\mathsf{B}\mathbb{G}_m \to \mathsf{B}^4\{\pm 1\}^{\otimes 2}$ which produces (2.9) under the construction functor (1.6). Using the equivalence (1.25), it suffices to construct an isomorphism of $\mathbb{E}_\infty$-monoidal morphisms $\mathbf{Z} \to \mathsf{B}\{\pm 1\}$ classifying the "sign", respectively "Hilbert" covers. The desired isomorphism is supplied by (2.20).

It remains to identify the section τ. By §2.2.10, this section is defined by the section of symmetric monoidal stacks

$$\begin{array}{ccccc} & & & & \mathbf{Z} \\ & & & \overset{(2.21)}{\swarrow} & \downarrow{\scriptstyle 2} \\ \mathsf{B}\{\pm 1\} & \longrightarrow & \widetilde{\mathbf{Z}}_{\mathrm{Hilb}} & \longrightarrow & \mathbf{Z} \end{array}$$

Recall the extension $\mathbb{G}_m^\dagger$ associated to $\mathsf{B}\Psi \cup \mathsf{B}\Psi$ (*cf.* (2.11)) endowed with its natural splitting $\mathbb{G}_m^\dagger \cong \mathbb{G}_m \times \mathsf{B}^2\{\pm 1\}^{\otimes 2}$ as a pointed stack. We want to identify the section

$$\mathbb{G}_m \to \mathbb{G}_m^\dagger \tag{2.22}$$

which produces (2.21) under the construction of §2.2.6. (Recall that the construction of *loc.cit.* is a reformulation of the equivalence (1.25).) The section (2.22) will, upon evaluating at $\operatorname{Spec} \mathrm{F}$ and applying Tate duality, give rise to the section τ:

$$\begin{aligned} \tau : \mathrm{F}^\times &\to \Gamma(\operatorname{Spec} \mathrm{F}, \mathbb{G}_m^\dagger) \\ &\xrightarrow{\simeq} \mathrm{F}^\times \times \Gamma(\operatorname{Spec} \mathrm{F}, \mathsf{B}^2\{\pm 1\}^{\otimes 2}) \to \mathrm{F}^\times \times \mathrm{H}^2(\operatorname{Spec} \mathrm{F}, \{\pm 1\}^{\otimes 2}) \cong \mathrm{F}^\times \times \{\pm 1\}. \end{aligned}$$

By construction, the projection of (2.22) onto $\mathsf{B}^2\{\pm 1\}^{\otimes 2}$ is $\Psi \otimes p^*\Psi(-1)$, where $p : \mathbb{G}_m \to \operatorname{Spec} \mathrm{F}$ is the projection. By [Zha22, Theorem 3.1.5], the latter is isomorphic to the self cup-product of Ψ. Hence, the second component of τ sends $a \in \mathrm{F}^\times$ to the image of $[\Psi(a)] \cup [\Psi(a)] \in \mathrm{H}^2(\operatorname{Spec} \mathrm{F}, \{\pm 1\}^{\otimes 2})$ under Tate duality, which is $\{a, a\}$. □

Remark 2.2.12. Proposition 2.2.3 shows that (2.8) is generally *not* induced from a cover of the cokernel of $(\cdot)^2 : \mathrm{F}^\times \to \mathrm{F}^\times$, the obstruction being given by $\{-1,-1\} \in \mathrm{F}^\times$. This element is nontrivial if and only if F is an odd degree extension of $\mathbf{Q}_2$.

2.2.13. Let us now apply Proposition 2.2.3 to the Langlands duality for tori.

Denote by $\mathsf{E}^1(\mathrm{F}^\times_{/2}, \mathbf{C}^\times)$ the fiber of the endomorphism of $\mathsf{E}^1(\mathrm{F}^\times, \mathbf{C}^\times)$ defined by precomposition with $(\cdot)^2 : \mathrm{F}^\times \to \mathrm{F}^\times$. The commutative diagram (2.8) together with the tautological inclusion $\{\pm 1\} \subset \mathbf{C}^\times$, defines an object

$$\widetilde{\mathrm{F}}^\times_{\mathrm{sgn},/2} \in \mathsf{E}^1(\mathrm{F}^\times_{/2}, \mathbf{C}^\times).$$

Applying the functor (2.1) for $\mathrm{T} \coloneqq \mathbb{G}_m$ and using its naturality with respect to $(\cdot)^2 : \mathbb{G}_m \to \mathbb{G}_m$, we obtain a functor

$$\mathsf{L}_{\mathsf{B}\{\pm 1\}} : \mathsf{Z}^2(\mathrm{W}_\mathrm{F}, \{\pm 1\}) \to \mathsf{E}^1(\mathrm{F}^\times_{/2}, \mathbf{C}^\times).$$

Let us view the meta-Weil group $\widetilde{\mathrm{W}}_\mathrm{F}$ (*cf.* §1.30) as an object of $\mathsf{Z}^2(\mathrm{W}_\mathrm{F}, \{\pm 1\})$.

Corollary 2.2.14. *There is a canonical isomorphism in* $\mathsf{E}^1(\mathrm{F}^\times_{/2}, \mathbf{C}^\times)$*:*

$$\mathsf{L}_{\mathsf{B}\{\pm 1\}}(\widetilde{\mathrm{W}}_\mathrm{F}) \xrightarrow{\simeq} \widetilde{\mathrm{F}}^\times_{\mathrm{sgn},/2}. \tag{2.23}$$

Proof. For an abelian group M, denote by $\mathsf{Z}^2_e(\mathrm{W}_\mathrm{F}, \mathrm{M})$ the fiber of the map $e^* : \mathsf{Z}^2(\mathrm{W}_\mathrm{F}, \mathrm{M}) \to \mathsf{Z}^2(*, \mathrm{M})$ given by pullback along the neutral point $e : * \to */\mathrm{W}_\mathrm{F}$. Thus $\mathsf{Z}^2_e(\mathrm{W}_\mathrm{F}, \mathrm{M})$ is canonically equivalent to the groupoid of central extensions of W_F by M.

The restriction of $\mathsf{L}_{\mathbb{G}_m}$ to $\mathsf{Z}^2_e(\mathrm{W}_\mathrm{F}, \mathbf{C}^\times)$ admits the following explicit description: Pulling back a commutative extension of $\mathrm{F}^\times$ by $\mathbf{C}^\times$ along the Artin reciprocity map $\mathrm{W}_\mathrm{F} \to \mathrm{F}^\times$ yields an equivalence of groupoids

$$\mathsf{E}^1(\mathrm{F}^\times, \mathbf{C}^\times) \xrightarrow{\simeq} \mathsf{Z}^2_e(\mathrm{W}_\mathrm{F}, \mathbf{C}^\times), \tag{2.24}$$

whose inverse coincides with the restriction of $\mathsf{L}_{\mathbb{G}_m}$.

In what follows, we view $\widetilde{\mathrm{F}}^\times_{\mathrm{sgn}}$ as an object of $\mathsf{E}^1(\mathrm{F}^\times, \mathbf{C}^\times)$. It suffices to identify its image under (2.24) with the extension of W_F by $\mathbf{C}^\times$ induced from $\widetilde{\mathrm{W}}_\mathrm{F}$, and match the 2-torsion structures defined by τ and $\widetilde{\mathrm{W}}_\mathrm{F}$. The identification follows from the isomorphism (2.9). The matching of 2-torsion structures follows from an explicit calculation, as we now perform.

Multiplication by 2 on $\widetilde{\mathrm{F}}^\times_{\mathrm{sgn}}$ factors through an isomorphism

$$\mathbf{C}^\times \sqcup_{\mathbf{C}^\times} \widetilde{\mathrm{F}}^\times_{\mathrm{sgn}} \xrightarrow{\simeq} \widetilde{\mathrm{F}}^\times_{\mathrm{sgn}} \times_{\mathrm{F}^\times} \mathrm{F}^\times \tag{2.25}$$

where the push-out is along $(\cdot)^2 : \mathbf{C}^\times \to \mathbf{C}^\times$ and the pullback is along $(\cdot)^2 : \mathrm{F}^\times \to \mathrm{F}^\times$. Using the isomorphism (2.9), we may represent an element of $\widetilde{\mathrm{F}}^\times_{\mathrm{sgn}}$ by a pair (a, z) with $a \in \mathrm{F}^\times$ and $z \in \mathbf{C}^\times$. Its image under (2.25) is

$$((a^2, \{a, a\} z^2), a)$$

which equals the product of z^2 with the $(\tau(a), a)$. Hence, splitting of the pushout induced from τ sends (a, z) to z^2. The kernel of this map is the extension (1.29), as desired. $\square$

2.2.15. Given any F-torus T equipped with a character $\epsilon : \Lambda \to \mathbf{Z}/2$, where Λ is the sheaf of cocharacters of T, we have a commutative diagram

$$\begin{array}{ccc} \mathsf{Z}^2(\mathrm{W}_\mathrm{F}, \{\pm 1\}) & \xrightarrow{\mathsf{L}_{\mathsf{B}\{\pm 1\}}} & \mathsf{E}^1(\mathrm{F}^\times_{/2}, \mathbf{C}^\times) \\ \downarrow{\scriptstyle \epsilon^\vee} & & \downarrow{\scriptstyle \epsilon} \\ \mathsf{Z}^2(\mathrm{W}_\mathrm{F}, \check{\mathrm{T}}(\mathbf{C})) & \xrightarrow{\mathsf{L}_\mathrm{T}} & \mathsf{E}^1(\mathrm{T}(\mathrm{F}), \mathbf{C}^\times) \end{array} \tag{2.26}$$

Denote by μ the $\mathbb{E}_\infty$-monoidal morphism $\mathsf{B}\mathrm{T} \to \mathsf{B}^4\{\pm 1\}^{\otimes 2}$ obtained by composing sgn with $\epsilon \otimes \mathsf{B}\Psi$ (*cf.* (1.23)) and by $\widetilde{\mathrm{T}}$ the induced commutative extension of $\mathrm{T}(\mathrm{F})$ by $\mathbf{C}^\times$.

Denote by $\widetilde{\check{\mathrm{T}}}$ the $\check{\mathrm{T}}(\mathbf{C})$-gerbe over the pro-space $*/\mathrm{W}_\mathrm{F}$ induced from $\widetilde{\mathrm{W}}_\mathrm{F}$ along the dual character $\epsilon^\vee : \{\pm 1\} \to \check{\mathrm{T}}(\mathbf{C})$.

Corollary 2.2.16. *There is a canonical isomorphism in* $\mathsf{E}^1(\mathrm{T}(\mathrm{F}), \mathbf{C}^\times)$ *functorial in* (T, ϵ)*:*

$$\mathsf{L}_\mathrm{T}(\widetilde{\check{\mathrm{T}}}) \xrightarrow{\simeq} \widetilde{\mathrm{T}}. \tag{2.27}$$

Proof. The isomorphism (2.27) is defined as the image of (2.23) under the right vertical functor of (2.26), using the commutativity of the latter. ☐

2.3. The Z-linear component.

2.3.1. Let A be a finite abelian group of order invertible in F equipped with an injective character $\zeta : \mathrm{A} \to \mathbf{C}^\times$. Let T be an F-torus and μ be a $\mathbf{Z}$-linear morphism $\mathsf{B}\mathrm{T} \to \mathsf{B}^4\mathrm{A}(1)$.

The first goal of this subsection is to construct the isomorphism (2.6) for (T, μ), *i.e.* we shall produce an isomorphism in $\mathsf{E}^1(\mathrm{T}(\mathrm{F}), \mathbf{C}^\times)$ functorial in (T, μ):

$$\mathsf{L}_\mathrm{T}(\widetilde{\check{\mathrm{T}}}) \xrightarrow{\simeq} \widetilde{\mathrm{T}}_\zeta. \tag{2.28}$$

Afterwards, we will combine (2.27) and (2.28) to prove Theorem 2.1.6.

2.3.2. Denote by Λ the étale sheaf of cocharacters of T. Recall that tensor product with $\mathsf{B}\Psi$ defines an equivalence

$$\mathrm{Hom}_\mathbf{Z}(\Lambda, \mathsf{B}^2\mathrm{A}) \xrightarrow{\simeq} \mathrm{Hom}_\mathbf{Z}(\mathsf{B}\mathrm{T}, \mathsf{B}^4\mathrm{A}(1)). \tag{2.29}$$

Thus, μ corresponds under (2.29) to a $\mathbf{Z}$-linear morphism $\Lambda \to \mathsf{B}^2\mathrm{A}$. Inducing the latter along $\zeta : \mathrm{A} \to \mathbf{C}^\times$ and passing to adjoints, we obtain the $\check{\mathrm{T}}(\mathbf{C})$-gerbe $\widetilde{\check{\mathrm{T}}}$.

2.3.3. *Split tori.* Let us first construct (2.28) in the special case where T is split. We view Λ as an abelian group. The functor L_T renders the following diagram commute:

$$\begin{array}{ccc} \mathsf{Z}^2(\mathrm{W}_\mathrm{F}, \check{\mathrm{T}}(\mathbf{C})) & \xrightarrow{\simeq} & \mathrm{Hom}(\Lambda, \mathsf{Z}^2(\mathrm{W}_\mathrm{F}, \mathbf{C}^\times)) \\ \downarrow{\scriptstyle \mathsf{L}_\mathrm{T}} & & \downarrow{\scriptstyle \mathrm{Hom}(\Lambda, \mathsf{L}_{\mathbb{G}_m})} \\ \mathsf{E}^1(\mathrm{T}(\mathrm{F}), \mathbf{C}^\times) & \xrightarrow{\simeq} & \mathrm{Hom}(\Lambda, \mathsf{E}^1(\mathrm{F}^\times, \mathbf{C}^\times)) \end{array}$$

Here, the horizontal isomorphisms are induced from $\check{\mathrm{T}}(\mathbf{C}) \cong \check{\Lambda} \otimes \mathbf{C}^\times$ and $\mathrm{T}(\mathrm{F}) \cong \Lambda \otimes \mathrm{F}^\times$.

Since μ is the tensor product of $\mathsf{B}\Psi$ with a $\mathbf{Z}$-linear morphism $\Lambda \to \mathsf{B}^2\mathrm{A}$, the construction of (2.28) reduces to the case $\mathrm{T} = \mathbb{G}_m$, where μ corresponds to a section of $\mathsf{B}^2\mathrm{A}$. It remains to identify the composition

$$\Gamma(\mathrm{Spec}\,\mathrm{F}, \mathsf{B}^2\mathrm{A}) \xrightarrow{\simeq} \mathsf{Z}^2(\mathrm{W}_\mathrm{F}, \mathrm{A}) \xrightarrow{\zeta} \mathsf{Z}^2(\mathrm{W}_\mathrm{F}, \mathbf{C}^\times) \xrightarrow{\mathsf{L}_{\mathbb{G}_m}} \mathsf{E}^1(\mathrm{F}^\times, \mathbf{C}^\times) \tag{2.30}$$

with the composition

$$\Gamma(\mathrm{Spec}\,\mathrm{F}, \mathsf{B}^2\mathrm{A}) \xrightarrow{\otimes \mathsf{B}\Psi} \mathrm{Hom}_\mathbf{Z}(\mathsf{B}\mathbb{G}_m, \mathsf{B}^4\mathrm{A}(1)) \xrightarrow{\int_\mathrm{F}} \mathsf{E}^1(\mathrm{F}^\times, \mathrm{A}) \xrightarrow{\zeta} \mathsf{E}^1(\mathrm{F}^\times, \mathbf{C}^\times). \tag{2.31}$$

Lemma 2.3.4. *The maps* (2.30) *and* (2.31) *are canonically isomorphic.*

Proof. Both maps are natural in the finite subgroup A of $\mathbf{C}^\times$. Thus they factor through the colimit of $\Gamma(\mathrm{Spec}\,\mathrm{F}, \mathsf{B}^2\mathrm{A}_1)$, taken over subgroups A_1 of $\mathbf{C}^\times$ containing A. Since the colimit of $\mathrm{H}^2(\mathrm{Spec}\,\mathrm{F}, \mathrm{A}_1)$ vanishes, we have an isomorphism

$$\operatorname*{colim}_{\mathrm{A}_1} */\Gamma(\mathrm{Spec}\,\mathrm{F}, \mathsf{B}\mathrm{A}_1) \xrightarrow{\simeq} \operatorname*{colim}_{\mathrm{A}_1} \Gamma(\mathrm{Spec}\,\mathrm{F}, \mathsf{B}^2\mathrm{A}_1).$$

Thus, it suffices to identify (2.30) and (2.31) over the neutral component of $\Gamma(\operatorname{Spec} F, \mathsf{B}^2 A)$ for every finite subgroup A of $\mathbf{C}^\times$, functorially in A.

By taking loop spaces, this reduces to the commutativity of the diagram

$$\begin{array}{ccc} \Gamma(\operatorname{Spec} F, \mathsf{B}A) & \xrightarrow{\simeq} & \mathsf{Z}^1(W_F, A) \\ \downarrow{\scriptstyle \otimes\Psi} & & \downarrow{\scriptstyle \text{Artin}} \\ \operatorname{Hom}_{\mathbf{Z}}(\mathbb{G}_m, \mathsf{B}^2 A(1)) & \to & \operatorname{Hom}(F^\times, A) \end{array} \tag{2.32}$$

where the right vertical map is Artin reciprocity and the lower horizontal map is the evaluation at Spec F followed by Tate duality $H^2(\operatorname{Spec} F, A(1)) \cong A$. The commutativity of (2.32) amounts to expressing Artin reciprocity as adjoint to the pairing

$$\begin{aligned} H^1(\operatorname{Spec} F, A) \otimes F^\times &\xrightarrow{\mathrm{id}\otimes\Psi} H^1(\operatorname{Spec} F, A) \otimes H^1(\operatorname{Spec} F, \hat{\mathbf{Z}}(1)) \\ &\xrightarrow{\cup} H^2(\operatorname{Spec} F, A(1)) \xrightarrow{\simeq} A, \end{aligned}$$

which is essentially its definition. □

2.3.5. *Induced tori.* Suppose that T is the Weil restriction of a split F_1-torus T_1 for a finite Galois extension $F \subset F_1$. The definition of L_T renders the following diagram commute

$$\begin{array}{ccc} \mathsf{Z}^2(W_F, \check{T}(\mathbf{C})) & \xrightarrow{\mathsf{L}_T} & \mathsf{E}^1(T(F), \mathbf{C}^\times) \\ \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} \\ \mathsf{Z}^2(W_{F_1}, \check{T}_1(\mathbf{C})) & \xrightarrow{\mathsf{L}_{T_1}} & \mathsf{E}^1(T_1(F_1), \mathbf{C}^\times) \end{array} \tag{2.33}$$

Here, the right vertical isomorphism is induced from the identification $T(F) \cong T_1(F_1)$ and the left vertical isomorphism is induced from the identification between $\check{T}(\mathbf{C})$ and the push-forward of $\check{T}_1(\mathbf{C})$ along $*/W_{F_1} \to */W_F$.

Denote by $\nu : \operatorname{Spec} F_1 \to \operatorname{Spec} F$ the natural map. The étale sheaf Λ coincides with $\nu_* \Lambda_1$, where Λ_1 is the (constant) sheaf of cocharacters of Λ_1. The adjunction between ν_* and $\nu^!$ as functors on étale sheaves yields an isomorphism

$$\begin{aligned} \operatorname{Hom}_{\mathbf{Z}}(\Lambda, A[2]) &\xrightarrow{\simeq} \operatorname{Hom}_{\mathbf{Z}}(\nu_*\Lambda_1, A[2]) \\ &\xrightarrow{\simeq} \operatorname{Hom}_{\mathbf{Z}}(\Lambda_1, \nu^! A[2]) \xrightarrow{\simeq} \operatorname{Hom}_{\mathbf{Z}}(\Lambda_1, A[2]), \end{aligned} \tag{2.34}$$

where the last isomorphism comes from the identification $\nu^! A \cong A$.

Under the equivalences (2.32) and (2.29), the $\mathbf{Z}$-linear morphism $\mu : \mathsf{B}T \to \mathsf{B}^4 A(1)$ corresponds to a $\mathbf{Z}$-linear morphism $\mu_1 : \mathsf{B}T_1 \to \mathsf{B}^4 A(1)$ over $\operatorname{Spec} F_1$. Furthermore, (2.34) is compatible with the vertical isomorphisms of (2.33), so the desired isomorphism (2.28) for T follows from the one for T_1 (*cf.* §2.3.3).

2.3.6. *General tori.* We turn to the case where T is any F-torus. Choose a finite Galois extension $F \subset F_1$ such that $T_1 := T \times_{\operatorname{Spec} F} \operatorname{Spec} F_1$ splits. Denote by T' the Weil restriction of T_1 to Spec F, so we have an injection $T \to T'$ of F-tori.

Note that μ extends to a $\mathbf{Z}$-linear morphism $\mu' : \mathsf{B}T' \to \mathsf{B}^4 A(1)$: By (2.29), it suffices to prove that any $\mathbf{Z}$-linear morphism $\Lambda \to \mathsf{B}^2 A$ extends to a $\mathbf{Z}$-linear morphism $\Lambda' \to \mathsf{B}^2 A$. The obstruction lies in the cohomology group

$$H^3(\operatorname{Spec} F, (\Lambda'/\Lambda)^\vee \otimes A)$$

which vanishes because Λ'/Λ is torsion-free and Spec F has cohomological dimension 2.

The desired isomorphism (2.28) for T thus follows from the one for T' by functoriality along the map of F-tori $\mathrm{T} \to \mathrm{T}'$. We omit the verification that this isomorphism is independent of the choice of F_1 and the extension μ'.

Remark 2.3.7. The cover $\widetilde{\mathrm{T}}$ associated to a **Z**-linear morphism $\mu : \mathsf{B}\mathrm{T} \to \mathsf{B}^4\mathrm{A}(1)$ has been constructed by Kaletha (*cf.* [Kal22, §2.2]). However, his construction is effectively the left-hand-side of (2.28).[12] Therefore, one may also interpret (2.28) as the comparison between our construction of $\widetilde{\mathrm{T}}$, which does not invoke Langlands duality, with Kaletha's.

2.3.8. Let us now gather all ingredients to prove Theorem 2.1.6.

Proof of Theorem 2.1.6. Consider the decomposition (1.27) for μ:

$$\mu \xrightarrow{\simeq} \mu^{(1)} + \mu^{(2)},$$

where $\mu^{(1)}$ is defined by a character $\epsilon : \Lambda \to \mathbf{Z}/2$ and $\mu^{(2)}$ is **Z**-linear. (By convention, $\mu^{(1)}$ is trivial unless $|\mathrm{A}|$ is even.)

Denote by $\widetilde{\mathrm{T}}_\zeta^{(1)}$ and $\widetilde{\mathrm{T}}_\zeta^{(2)}$ the extensions of $\mathrm{T}(\mathrm{F})$ by $\mathbf{C}^\times$ induced from $\mu^{(1)}$, and by $\widetilde{\check{\mathrm{T}}}^{(1)}$ and $\widetilde{\check{\mathrm{T}}}^{(2)}$ the associated $\check{\mathrm{T}}(\mathbf{C})$-gerbes over $*/\mathrm{W}_\mathrm{F}$. The isomorphism (2.27) applied to (T, ϵ) and the isomorphism (2.28) applied to $(\mathrm{T}, \mu^{(2)})$ yield isomorphisms

$$\mathsf{L}_\mathrm{T}(\widetilde{\check{\mathrm{T}}}^{(1)}) \xrightarrow{\simeq} \widetilde{\mathrm{T}}_\zeta^{(1)},$$
$$\mathsf{L}_\mathrm{T}(\widetilde{\check{\mathrm{T}}}^{(2)}) \xrightarrow{\simeq} \widetilde{\mathrm{T}}_\zeta^{(2)}.$$

We sum them using the **Z**-linearity of L_T:

$$\begin{aligned}\mathsf{L}_\mathrm{T}(\widetilde{\check{\mathrm{T}}}) &\xrightarrow{\simeq} \mathsf{L}_\mathrm{T}(\widetilde{\check{\mathrm{T}}}^{(1)} + \widetilde{\check{\mathrm{T}}}^{(2)}) \\ &\xrightarrow{\simeq} \widetilde{\mathrm{T}}_\zeta^{(1)} + \widetilde{\mathrm{T}}_\zeta^{(2)} \xrightarrow{\simeq} \widetilde{\mathrm{T}}_\zeta.\end{aligned}$$

This is the desired isomorphism (2.6). □

2.4. Duality for the center.

2.4.1. Let A be a finite abelian group of order invertible in F, equipped with an injective character $\zeta : \mathrm{A} \to \mathbf{C}^\times$. Let G be a reductive group F-scheme and $\mu : \mathsf{B}\mathrm{G}_{\mathrm{ab}} \to \mathsf{B}^4\mathrm{A}(1)$ be an $\mathbb{E}_\infty$-monoidal morphism. (The results of this subsection will be applied to the pair $(\mathrm{G}^\sharp, \mu_{\mathrm{G}^\sharp_{\mathrm{ab}}})$ defined by a general rigidified morphism $\mathsf{B}\mathrm{G} \to \mathsf{B}^4\mathrm{A}(1)$, *cf.* §1.4.3.) We also use μ to denote its pullback to $\mathsf{B}\mathrm{G}$, viewed as a pointed morphism.

Denote by $\widetilde{\mathrm{G}}$ the image of μ under the construction functor (1.6). Its pullback along $\mathrm{Z}(\mathrm{F}) \to \mathrm{G}(\mathrm{F})$ is a commutative extension

$$1 \to \mathrm{A} \to \widetilde{\mathrm{Z}} \to \mathrm{Z}(\mathrm{F}) \to 1. \tag{2.35}$$

Denote by $\widetilde{\mathrm{Z}}_\zeta$ extension of $\mathrm{Z}(\mathrm{F})$ by $\mathbf{C}^\times$ induced from (2.35) along ζ.

In this subsection, we shall construct L-parameters for the set $\Pi(\widetilde{\mathrm{Z}})$ of ζ-genuine smooth characters of $\widetilde{\mathrm{Z}}$ using Theorem 2.1.6.

2.4.2. Let us first fix notation for data on the Galois side. The dual group H of (G, μ) is the Langlands dual of G and we have an object $\widetilde{\mathrm{Z}}_\mathrm{H}$ of $\mathsf{Z}^2(\mathrm{W}_\mathrm{F}, \mathrm{Z}_\mathrm{H}(\mathbf{C}))$.

[12] For this reason, the construction of $\widetilde{\mathrm{G}}$ for a **Z**-linear morphism $\mu : \mathsf{B}\mathrm{G}_{\mathrm{ab}} \to \mathsf{B}^4\mathrm{A}(1)$ given in *op.cit.* requires G to be quasi-split.

We also slightly extend the formalism of §1.4.7: Given a pro-group $\Sigma = \lim_{i\in \mathrm{I}} \Sigma_i$ and a *complex* of sheaves of abelian groups $\mathscr{A}$ over $*/\Sigma_i$, we write $\mathsf{Z}^n(\Sigma, \mathscr{A})$ for the $\mathbf{Z}$-linear space underlying

$$\Gamma(*/\Sigma, \mathscr{A}[n]) \coloneqq \operatorname*{colim}_{j\in \mathrm{I}_{/i}} \Gamma(*/\Sigma_j, \mathscr{A}[n]).$$

In other words, $\mathsf{Z}^n(\Sigma, \mathscr{A})$ is the space of *hyper*cocycles of degree n. We will still refer to objects of $\mathsf{Z}^2(\Sigma, \mathscr{A})$ as $\mathscr{A}$-gerbes over $*/\Sigma$.

Denote by $\widetilde{\mathrm{H}}_{\mathrm{ab}}$ the object of $\mathsf{Z}^2(\mathrm{W_F}, \mathrm{H_{ab}}(\mathbf{C}))$ induced from $\widetilde{\mathrm{Z}}_{\mathrm{H}}$ by functoriality along the map of complexes $\mathrm{Z_H}(\mathbf{C}) \to \mathrm{H_{ab}}(\mathbf{C})$. Denote by $\Phi(\widetilde{\mathrm{H}}_{\mathrm{ab}})$ the set of isomorphism classes of trivializations of $\widetilde{\mathrm{H}}_{\mathrm{ab}}$.

Remark 2.4.3. Note that the total space of the $\mathrm{H_{ab}}(\mathbf{C})$-gerbe $\widetilde{\mathrm{H}}_{\mathrm{ab}}$ is a pro-space over $*/\mathrm{W_F}$ and $\Phi(\widetilde{\mathrm{H}}_{\mathrm{ab}})$ is the set of isomorphism classes of its sections. In particular, functoriality with respect to $\widetilde{\mathrm{H}} \to \widetilde{\mathrm{H}}_{\mathrm{ab}}$ defines a map

$$\Phi(\widetilde{\mathrm{H}}) \to \Phi(\widetilde{\mathrm{H}}_{\mathrm{ab}}). \tag{2.36}$$

2.4.4. Denote by $\mathrm{T_{H,sc}}$ the Langlands dual torus of $\mathrm{T_{ad}}$. (It coincides with the maximal torus of $\mathrm{H_{sc}}$ induced from $\mathrm{T_H}$.) The functor (2.1) for T and $\mathrm{T_{ad}}$ fits into a commutative diagram

$$\begin{array}{ccc} \mathsf{Z}^2(\mathrm{W_F}, \mathrm{T_{H,sc}}(\mathbf{C})) & \to & \mathsf{Z}^2(\mathrm{W_F}, \mathrm{T_H}(\mathbf{C})) \\ \downarrow{\scriptstyle \mathsf{L}_{\mathrm{T_{ad}}}} & & \downarrow{\scriptstyle \mathsf{L}_{\mathrm{T}}} \\ \mathsf{E}^1(\mathrm{T_{ad}}(\mathrm{F}), \mathbf{C}^\times) & \longrightarrow & \mathsf{E}^1(\mathrm{T}(\mathrm{F}), \mathbf{C}^\times) \end{array} \tag{2.37}$$

Since $\mathrm{H}^3(\mathrm{W_F}, \mathrm{T_{H,sc}}(\mathbf{C}))$ vanishes, the cofiber of the top row of (2.37) is identified with $\mathsf{Z}^2(\mathrm{W_F}, \mathrm{H_{ab}}(\mathbf{C}))$. Therefore, (2.37) induces a functor

$$\mathsf{L}_{\mathrm{Z}} : \mathsf{Z}^2(\mathrm{W_F}, \mathrm{H_{ab}}(\mathbf{C})) \to \mathsf{E}^1(\mathrm{Z}(\mathrm{F}), \mathbf{C}^\times). \tag{2.38}$$

Since $\widetilde{\mathrm{H}}_{\mathrm{ab}}$ comes from an object of $\mathsf{Z}^2(\mathrm{W_F}, \mathrm{T_H}(\mathbf{C}))$, the isomorphism (2.6) induces an isomorphism in $\mathsf{E}^1(\mathrm{Z}(\mathrm{F}), \mathbf{C}^\times)$:

$$\mathsf{L}_{\mathrm{Z}}(\widetilde{\mathrm{H}}_{\mathrm{ab}}) \xrightarrow{\simeq} \widetilde{\mathrm{Z}}_\zeta. \tag{2.39}$$

2.4.5. We shall use (2.38) and (2.39) to construct the local Langlands correspondence for $\widetilde{\mathrm{Z}}$, in analogy with §2.1.8:

$$\mathrm{LLC} : \Pi(\widetilde{\mathrm{Z}}) \xrightarrow{\simeq} \Phi(\widetilde{\mathrm{H}}_{\mathrm{ab}}). \tag{2.40}$$

Indeed, the functor L_{Z} carries trivializations of $\widetilde{\mathrm{H}}_{\mathrm{ab}}$ to trivializations of $\widetilde{\mathrm{Z}}_\zeta$, which are in bijection with ζ-genuine characters of $\widetilde{\mathrm{Z}}$. This map intertwines the $\mathsf{Z}^1(\mathrm{W_F}, \mathrm{H_{ab}}(\mathbf{C}))$-action on trivializations of $\widetilde{\mathrm{H}}_{\mathrm{ab}}$ with the $\mathrm{Hom}(\mathrm{Z}(\mathrm{F}), \mathbf{C}^\times)$-action on trivializations of $\widetilde{\mathrm{Z}}_\zeta$, via the L_{Z}-action on loop spaces. We shall argue that the latter induces a bijection on π_0: Indeed, it occurs as the last vertical arrow in the commutative diagram

$$\begin{array}{ccccccc} \mathrm{H}^1(\mathrm{W_F}, \mathrm{T_{H,sc}}(\mathbf{C})) & \longrightarrow & \mathrm{H}^1(\mathrm{W_F}, \mathrm{T_H}(\mathbf{C})) & \longrightarrow & \mathrm{H}^1(\mathrm{W_F}, \mathrm{H_{ab}}(\mathbf{C})) & \longrightarrow & 1 \\ \downarrow{\scriptstyle \pi_1 \mathsf{L}_{\mathrm{T_{ad}}}} & & \downarrow{\scriptstyle \pi_1 \mathsf{L}_{\mathrm{T}}} & & \downarrow{\scriptstyle \pi_1 \mathsf{L}_{\mathrm{Z}}} & & \\ \mathrm{Hom}(\mathrm{T_{ad}}(\mathrm{F}), \mathbf{C}^\times) & \longrightarrow & \mathrm{Hom}(\mathrm{T}(\mathrm{F}), \mathbf{C}^\times) & \longrightarrow & \mathrm{Hom}(\mathrm{Z}(\mathrm{F}), \mathbf{C}^\times) & \longrightarrow & 1 \end{array}$$

Here, the top row is exact because $\mathrm{H}^2(\mathrm{W_F}, \mathrm{T_{H,sc}}(\mathbf{C}))$ vanishes (*cf.* [Kar13, Theorem 3.2.2]) and the bottom row is exact because $\mathbf{C}^\times$ is divisible. Since $\pi_1 \mathsf{L}_{\mathrm{T_{ad}}}$ and $\pi_1 \mathsf{L}_{\mathrm{T}}$ are isomorphisms (*cf.* Remark 2.1.3), so is $\pi_1 \mathsf{L}_{\mathrm{Z}}$.

It follows that the action of L_{Z} on trivializations of $\widetilde{\mathrm{H}}_{\mathrm{ab}}$ defines a bijection

$$\Phi(\widetilde{\mathrm{H}}_{\mathrm{ab}}) \xrightarrow{\simeq} \Pi(\widetilde{\mathrm{Z}}). \tag{2.41}$$

We define (2.40) to be the inverse to (2.41).

2.5. Duality for the cocenter.

2.5.1. We keep the notation of §2.4.1 and §2.4.2. The goal of this subsection is to address the following question: When does $\widetilde{\mathrm{G}}$ admit a ζ-genuine character?

We shall only answer this question when G is quasi-split. For the remainder of this subsection, we fix a Borel subgroup B and a section of the projection $\mathrm{B} \to \mathrm{T}$, realizing T as a subgroup of G.

The results of this subsection will not be used in the sequel. We include them because they support the philosophy that the local Langlands correspondence for covers defined by $\mathbb{E}_\infty$-monoidal morphisms $\mu : \mathsf{B}\mathrm{G}_{\mathrm{ab}} \to \mathsf{B}^4\mathrm{A}(1)$ is "not too far" from the local Langlands correspondence for linear algebraic groups. The case where μ is $\mathbf{Z}$-linear is due to Kaletha (*cf.* [Kal22, §2]) and no new idea is needed to treat the $\mathbb{E}_\infty$-monoidal case.

2.5.2. Under the split fiber sequence (1.22) (*cf.* §1.4.4), the $\mathbb{E}_\infty$-monoidal morphism μ defines a character $\pi_1\mathrm{G} \to \mathbf{Z}/2$ (trivial unless $|\mathrm{A}|$ is even) and $\mathbf{Z}$-linear morphism $\pi_1\mathrm{G} \to \mathsf{B}^2\mathrm{A}$.

Denote by $\pi_1^t\mathrm{G}$ the torsion subgroup of $\pi_1\mathrm{G}$. We restrict the two maps above to $\pi_1^t\mathrm{G}$ and apply one unit of Tate twist. This gives us two $\mathbf{Z}$-linear maps

$$\epsilon : \pi_1^t\mathrm{G}(1) \to \mu_2,$$
$$f : \pi_1^t\mathrm{G}(1) \to \mathsf{B}^2\mathrm{A}(1).$$

For any section $\mathscr{G}$ of $\mathsf{B}^2\mathrm{A}(1)$ over $\operatorname{Spec}\mathrm{F}$, we denote by $[\mathscr{G}] \in \mathrm{A}$ the image of its isomorphism class under Tate duality $\mathrm{H}^2(\operatorname{Spec}\mathrm{F}, \mathrm{A}(1)) \cong \mathrm{A}$. Write $\{\cdot,\cdot\}$ for the quadratic Hilbert symbol—we view it as valued in the subgroup $\{\pm 1\} \subset \mathrm{A}$ if $|\mathrm{A}|$ is even and trivial if $|\mathrm{A}|$ is odd. Then we may form the character

$$\pi_1^t\mathrm{G}(1)(\mathrm{F}) \to \mathrm{A}, \quad \theta \mapsto \{\epsilon(\theta), \epsilon(\theta)\} \cdot [f(\theta)]. \tag{2.42}$$

The following result is an analogue of [Kal22, Proposition 2.4.7].

Proposition 2.5.3. *The following statements are equivalent.*

1. *the class of* $\widetilde{\mathrm{Z}}_{\mathrm{H}}$ *in* $\mathrm{H}^2(\mathrm{W}_{\mathrm{F}}, \mathrm{Z}_{\mathrm{H}}(\mathbf{C}))$ *vanishes;*
2. *the homomorphism* (2.42) *vanishes;*
3. $\widetilde{\mathrm{G}}$ *admits a* ζ*-genuine character.*

2.5.4. Let us begin with an elementary observation: Quasi-splitness of G implies that T_{sc} is the Weil restriction of a split torus, so $\mathrm{H}^1(\operatorname{Spec}\mathrm{F}, \mathrm{T}_{\mathrm{sc}})$ vanishes. The groupoid of F-points of the cocenter G_{ab} can thus be identified as

$$\mathrm{G}_{\mathrm{ab}}(\mathrm{F}) \xrightarrow{\simeq} \mathrm{T}(\mathrm{F})/\mathrm{T}_{\mathrm{sc}}(\mathrm{F}) \xrightarrow{\simeq} \mathrm{G}(\mathrm{F})/\mathrm{G}_{\mathrm{sc}}(\mathrm{F}), \tag{2.43}$$

where the quotients are taken in the sense of groupoids.

In particular, the π_1 of $\mathrm{G}_{\mathrm{ab}}(\mathrm{F})$ is identified with $\pi_1^t\mathrm{G}(1)(\mathrm{F})$, while its π_0 is identified with the cokernel of $\mathrm{T}_{\mathrm{sc}}(\mathrm{F}) \to \mathrm{T}(\mathrm{F})$, as well as the cokernel of $\mathrm{G}_{\mathrm{sc}}(\mathrm{F}) \to \mathrm{G}(\mathrm{F})$.

2.5.5. Let us write $\mathsf{E}^1(\mathrm{G}_{\mathrm{ab}}(\mathrm{F}), \mathrm{A})$ for the fiber of

$$\mathsf{E}^1(\mathrm{T}(\mathrm{F}), \mathrm{A}) \to \mathsf{E}^1(\mathrm{T}_{\mathrm{sc}}(\mathrm{F}), \mathrm{A}),$$

so an object $\widetilde{\mathrm{G}}_{\mathrm{ab}}$ of $\mathsf{E}^1(\mathrm{G}_{\mathrm{ab}}(\mathrm{F}), \mathrm{A})$ can be thought of as a commutative extension $\widetilde{\mathrm{T}}$ of $\mathrm{T}(\mathrm{F})$ equipped with a splitting over $\mathrm{T}_{\mathrm{sc}}(\mathrm{F})$. (The case where G_{ab} is replaced by $\mathsf{B}\{\pm 1\}$ has already appeared in §2.2.13.)

The $\mathbb{E}_\infty$-monoidal morphism μ defines an object $\widetilde{\mathrm{G}}_{\mathrm{ab}}$ of $\mathsf{E}^1(\mathrm{G}_{\mathrm{ab}}(\mathrm{F}), \mathrm{A})$. Restricting its splitting $\mathrm{T}_{\mathrm{sc}}(\mathrm{F}) \to \widetilde{\mathrm{T}}$ to $\pi_1^t\mathrm{G}(1)(\mathrm{F})$, we obtain a character

$$\pi_1^t\mathrm{G}(1)(\mathrm{F}) \to \mathrm{A}, \tag{2.44}$$

which vanishes if and only if $\widetilde{\mathrm{G}}_{\mathrm{ab}}$ is the pullback of a commutative extension of $\pi_0(\mathrm{G}_{\mathrm{ab}}(\mathrm{F}))$ by A. Let us calculate (2.44).

Lemma 2.5.6. *The character* (2.44) *equals* (2.42).

Proof. We perform the decomposition (1.27): $\mu \cong \mu^{(1)} + \mu^{(2)}$, where $\mu^{(1)}$ is defined by ϵ and $\mu^{(2)}$ is $\mathbf{Z}$-linear. The character (2.44) attached to $\mu^{(1)}$ is given by $\theta \mapsto \{\epsilon(\theta), \epsilon(\theta)\}$ (*cf.* Proposition 2.2.3). It remains to identify the character (2.44) attached to $\mu^{(2)}$ with $\theta \mapsto [f(\theta)]$. Therefore, we may assume that μ is $\mathbf{Z}$-linear in what follows.

In this case, μ is the tensor product of $\mathsf{B}\Psi : \mathsf{B}\mathbb{G}_m \to \mathsf{B}^2\hat{\mathbf{Z}}(1)$ with the $\mathbf{Z}$-linear morphism $\pi_1\mathrm{G} \to \mathsf{B}^2\mathrm{A}$ which defines f. Applying the loop space functor to μ and evaluating at $\operatorname{Spec}\mathrm{F}$, we obtain a map of spaces

$$\mathrm{G}_{\mathrm{ab}}(\mathrm{F}) \to \Gamma(\operatorname{Spec}\mathrm{F}, \mathsf{B}^3\mathrm{A}(1)). \tag{2.45}$$

The character (2.44) is obtained from (2.45) by taking π_1 and identifying $\mathrm{H}^2(\operatorname{Spec}\mathrm{F}, \mathrm{A}(1))$ with A under Tate duality. This yields the character $\theta \mapsto [f(\theta)]$. □

2.5.7. Let us consider the induced cover $\widetilde{\mathrm{G}}_{\mathrm{ab},\zeta} \in \mathsf{E}^1(\mathrm{G}_{\mathrm{ab}}(\mathrm{F}), \mathbf{C}^\times)$ of $\widetilde{\mathrm{G}}_{\mathrm{ab}}$ (*cf.* §2.5.5). Define the functor $\mathsf{L}_{\mathrm{G}_{\mathrm{ab}}}$ by the diagram of fiber sequences

$$\begin{array}{ccccc}
\mathsf{Z}^2(\mathrm{W}_\mathrm{F}, \mathrm{Z}_\mathrm{H}(\mathbf{C})) & \to & \mathsf{Z}^2(\mathrm{W}_\mathrm{F}, \mathrm{T}_\mathrm{H}(\mathbf{C})) & \to & \mathsf{Z}^2(\mathrm{W}_\mathrm{F}, \mathrm{T}_{\mathrm{H},\mathrm{ad}}(\mathbf{C})) \\
\downarrow{\scriptstyle \mathsf{L}_{\mathrm{G}_{\mathrm{ab}}}} & & \downarrow{\scriptstyle \mathsf{L}_{\mathrm{T}}} & & \downarrow{\scriptstyle \mathsf{L}_{\mathrm{T}_{\mathrm{sc}}}} \\
\mathsf{E}^1(\mathrm{G}_{\mathrm{ab}}(\mathrm{F}), \mathbf{C}^\times) & \longrightarrow & \mathsf{E}^1(\mathrm{T}(\mathrm{F}), \mathbf{C}^\times) & \longrightarrow & \mathsf{E}^1(\mathrm{T}_{\mathrm{sc}}(\mathrm{F}), \mathbf{C}^\times)
\end{array}$$

The isomorphism (2.6) for T and T_{sc} yields an isomorphism

$$\mathsf{L}_{\mathrm{G}_{\mathrm{ab}}}(\widetilde{\mathrm{Z}}_\mathrm{H}) \xrightarrow{\cong} \widetilde{\mathrm{G}}_{\mathrm{ab},\zeta}. \tag{2.46}$$

2.5.8. We now prove Proposition 2.5.3.

Proof of Proposition 2.5.3. (1) ⇔ (2). According to Lemma 2.5.6, the commutative extension $\widetilde{\mathrm{G}}_{\mathrm{ab},\zeta}$ splits if and only if the character (2.42) vanishes. On the other hand, the functor $\mathsf{L}_{\mathrm{G}_{\mathrm{ab}}}$ induces an isomorphism on the set of isomorphism classes

$$\mathrm{H}^2(\mathrm{W}_\mathrm{F}, \mathrm{Z}_\mathrm{H}(\mathbf{C})) \xrightarrow{\cong} \operatorname{Hom}(\pi_1^t\mathrm{G}(1)(\mathrm{F}), \mathbf{C}^\times),$$

so the equivalence of (1) and (2) follows from (2.46).

(2) ⇒ (3). A splitting of $\widetilde{\mathrm{G}}_{\mathrm{ab},\zeta}$ induces a ζ-genuine character of $\widetilde{\mathrm{G}}$.

(3) ⇒ (2). Since the cover $\widetilde{\mathrm{G}}$ is induced from $\widetilde{\mathrm{G}}_{\mathrm{ab}}$, we have a canonical section $\mathrm{G}_{\mathrm{sc}}(\mathrm{F}) \to \widetilde{\mathrm{G}}$, whose restriction to $\pi_1^t\mathrm{G}(1)(\mathrm{F}) \cong \operatorname{Ker}(\mathrm{G}_{\mathrm{sc}}(\mathrm{F}) \to \mathrm{G}(\mathrm{F}))$ is the character (2.42) (*cf.* Lemma 2.5.6). If $\widetilde{\mathrm{G}}$ admits a ζ-genuine character, its restriction to $\mathrm{G}_{\mathrm{sc}}(\mathrm{F})$ must vanish because $\mathrm{G}_{\mathrm{sc}}(\mathrm{F})$ is perfect by Platonov's theorem (*cf.* [PR94, §7.2]). It follows that the character (2.42) must also vanish. □

Remark 2.5.9. By Proposition 2.5.3, the obstruction to the existence of a ζ-genuine character of $\widetilde{\mathrm{G}}$ only has to do with the torsion subgroup $\pi_1^t\mathrm{G}$ of $\pi_1\mathrm{G}$.

By taking a z-extension $\mathrm{G}' \to \mathrm{G}$, one can thus find a ζ-genuine character of the induced cover $\widetilde{\mathrm{G}}'$ of $\mathrm{G}'(\mathrm{F})$ and effectively reduces the local Langlands correspondence for $\widetilde{\mathrm{G}}$ to that for $\mathrm{G}'(\mathrm{F})$. This is explained in [Kal22, Theorem 2.6.2], so we shall not repeat it.

3. Structures on μ

In this section, we work over an arbitrary base scheme S and let G be a reductive group S-scheme. We adopt the notation of §1.1.5 for objects associated to G. Let A be a finite abelian group whose order is invertible over S.

Fix a rigidified morphism $\mu : \mathsf{B}\mathrm{G} \to \mathsf{B}^4\mathrm{A}(1)$. Write Q for its associated quadratic form and b, b_1, b_2 for the induced pairings (*cf.* §1.3.5).

The goal of this section is construct the "canonical quadratic structure" on μ with respect to the $\mathsf{B}\mathrm{Z}$-action on $\mathsf{B}\mathrm{G}$ (*cf.* Proposition 3.1.3). This provides the key technical ingredient in our calculation of Weissman's obstruction in §4.

3.1. The canonical quadratic structure.

3.1.1. Consider the self-tensor product $\mathsf{B}\Psi^{\otimes 2} : \mathsf{B}\mathbb{G}_m \otimes \mathsf{B}\mathbb{G}_m \to \mathsf{B}^4\hat{\mathbf{Z}}(2)$ of the delooped Kummer map $\mathsf{B}\Psi$. Tensoring it with the pairing b_2 yields a bilinear pairing

$$b_2 \otimes \mathsf{B}\Psi^{\otimes 2} : \mathsf{B}\mathrm{G}_{\mathrm{ab}} \otimes \mathsf{B}\mathrm{Z} \to \mathsf{B}^4\mathrm{A}(1). \tag{3.1}$$

We shall use the same notation $b_2 \otimes \mathsf{B}\Psi^{\otimes 2}$ to denote the pullback of (3.1) to $\mathsf{B}\mathrm{G} \times \mathsf{B}\mathrm{Z}$. It is *bi-rigidified* in the sense that it is equipped with trivializations over $e \times \mathsf{B}\mathrm{Z}$ and $\mathsf{B}\mathrm{G} \times e$ which are compatible over $e \times e$.

3.1.2. Consider the morphisms p_1, p_2, a in the diagram

$$\begin{array}{ccccc} & & \mathsf{B}\mathrm{G} \times \mathsf{B}\mathrm{Z} & \xrightarrow{a} & \mathsf{B}\mathrm{G} \\ & {\scriptstyle p_1}\swarrow & & \searrow{\scriptstyle p_2} & \\ \mathsf{B}\mathrm{G} & & & & \mathsf{B}\mathrm{Z} \end{array} \tag{3.2}$$

which are, respectively, projections onto the first and the second factors and the action map. Denote by μ_{Z} the restriction of μ to $\mathsf{B}\mathrm{Z}$.

Proposition 3.1.3. *In reference to* (3.2)*, there is a canonical isomorphism of bi-rigidified morphisms* $\mathsf{B}\mathrm{G} \times \mathsf{B}\mathrm{Z} \to \mathsf{B}^4\mathrm{A}(1)$*:*

$$a^*\mu - (p_1)^*\mu - (p_2)^*\mu_{\mathrm{Z}} \xrightarrow{\simeq} b_2 \otimes \mathsf{B}\Psi^{\otimes 2}. \tag{3.3}$$

3.1.4. The proof of Proposition 3.1.3 will appear in §3.2.5. Let us make some preliminary remarks about its statement.

First, (3.3) is supposed to be an isomorphism in a 1-groupoid. Namely, the space of bi-rigidified morphisms $\mathsf{B}\mathrm{G} \times \mathsf{B}\mathrm{Z} \to \mathsf{B}^4\mathrm{A}(1)$ is 1-truncated.

To see this, we note that the space of bi-rigidified morphisms $\mathsf{B}\mathrm{G} \times \mathsf{B}\mathrm{Z} \to \mathsf{B}^4\mathrm{A}(1)$ is equivalent to that of pointed morphisms $\mathsf{B}\mathrm{Z} \to \mathscr{M}aps_e(\mathsf{B}\mathrm{G}, \mathsf{B}^4\mathrm{A}(1))$. Because the third term in (1.19) is discrete, such morphisms factor through $\mathscr{H}om_{\mathbf{Z}}(\pi_1\mathrm{G}, \mathsf{B}^2\mathrm{A})$, so they correspond to monoidal morphisms

$$\mathrm{Z} \to \mathscr{H}om_{\mathbf{Z}}(\pi_1\mathrm{G}, \mathsf{B}\mathrm{A}),$$

which form a 1-groupoid. (Moreover, this shows that any bi-rigidified morphism $\mathsf{B}\mathrm{G}_{\mathrm{sc}} \times \mathsf{B}\mathrm{Z} \to \mathsf{B}^4\mathrm{A}(1)$ is canonically trivial.)

3.1.5. Next, we shall state a cocycle condition satisfied by (3.3). Given a G-torsor $\mathscr{E}$ and a Z-torsor $\mathscr{Z}$ over an S-scheme, (3.3) supplies a functorial isomorphism of sections of $\mathsf{B}^4\mathrm{A}(1)$

$$\mu(\mathscr{E}\otimes\mathscr{Z})-\mu(\mathscr{E})-\mu(\mathscr{Z})\xrightarrow{\simeq}(b_2\otimes\mathsf{B}\Psi^{\otimes 2})(\mathscr{E},\mathscr{Z}). \tag{3.4}$$

Furthermore, the isomorphism (3.4) is compatible with the natural trivializations of the two sides, when either $\mathscr{E}$ or $\mathscr{Z}$ is the trivial torsor.

Now, given $\mathscr{E}$ along with two Z-torsors $\mathscr{Z}_1$, $\mathscr{Z}_2$ over an S-scheme, there are two isomorphisms between $\mu(\mathscr{E}\otimes\mathscr{Z}_1\otimes\mathscr{Z}_2)$ and

$$\mu(\mathscr{E})+\mu(\mathscr{Z}_1)+\mu(\mathscr{Z}_2) \\ +(b_2\otimes\mathsf{B}\Psi^{\otimes 2})(\mathscr{E},\mathscr{Z}_1)+(b_2\otimes\mathsf{B}\Psi^{\otimes 2})(\mathscr{E},\mathscr{Z}_2)+(b_2\otimes\mathsf{B}\Psi^{\otimes 2})(\mathscr{Z}_1,\mathscr{Z}_2),$$

given by iteratively applying (3.4) in different orders. The cocycle condition states that these two isomorphisms are canonically identified. (We omit drawing this rather large commutative diagram.)

Note that this is indeed a *condition* and not additional structure, because the space of pointed morphisms $\mathsf{BZ}\times\mathsf{BZ}\to\mathscr{M}aps_e(\mathsf{BG},\mathsf{B}^4\mathrm{A}(1))$ is 1-truncated (*cf.* §3.1.4).

3.2. Construction of (3.3).

3.2.1. We shall first construct (3.3) in the case where G is split and equipped with a Killing pair $\mathrm{T}\subset\mathrm{B}\subset\mathrm{G}$.

Recall that any bi-rigidified morphism $\mathsf{BG}\times\mathsf{BZ}\to\mathsf{B}^4\mathrm{A}(1)$ is canonically trivialized as such over $\mathsf{BG}_{\mathrm{sc}}\times\mathsf{BZ}$ (*cf.* §3.1.4).

By restrictions along $\mathsf{BT}\to\mathsf{BG}$ and $\mathsf{BT}_{\mathrm{sc}}\to\mathsf{BG}_{\mathrm{sc}}$, the bi-rigidified morphism $a^*\mu-(p_1)^*\mu-(p_2)^*\mu_{\mathrm{Z_G}}$ defines a bi-rigidified morphism

$$\mathsf{BT}\times\mathsf{BZ}\to\mathsf{B}^4\mathrm{A}(1), \tag{3.5}$$

equipped with a trivialization τ as such over $\mathsf{BT}_{\mathrm{sc}}\times\mathsf{BZ}$.

3.2.2. The bi-rigidified morphism (3.5) extends to the bi-rigidified morphism

$$m^*\mu-(p_1)^*\mu-(p_2)^*\mu:\mathrm{BT}\times\mathrm{BT}\to\mathsf{B}^4\mathrm{A}(1), \tag{3.6}$$

where m, p_1, p_2 are the multiplication and projection morphisms from $\mathsf{BT}\times\mathsf{BT}$ to BT.

By [Zha22, Proposition 4.7.3], the bi-rigidified morphism (3.6) is identified with $b\otimes\mathsf{B}\Psi^{\otimes 2}$, where b is the symmetric form attached to Q. By restricting to $\mathsf{BT}\times\mathsf{BZ}$, we obtain an isomorphism of bi-rigidified morphisms

$$a^*\mu-(p_1)^*\mu-(p_2)^*\mu_{\mathrm{Z}}\xrightarrow{\simeq}b\otimes\mathsf{B}\Psi^{\otimes 2} \tag{3.7}$$

from $\mathsf{BT}\times\mathsf{BZ}$ to $\mathsf{B}^4\mathrm{A}(1)$.

3.2.3. Since the restriction of b to $\Lambda_{\mathrm{sc}}\otimes\Lambda$ extends to $\Lambda_{\mathrm{sc}}\otimes\Lambda_{\mathrm{ad}}$ as the bilinear pairing b_1, the restriction of $b\otimes\mathsf{B}\Psi^{\otimes 2}$ to $\mathsf{BT}_{\mathrm{sc}}\times\mathsf{BT}$ likewise extends to $\mathsf{BT}_{\mathrm{sc}}\times\mathsf{BT}_{\mathrm{ad}}$ as a bi-rigidified morphism. This endows $b\otimes\mathsf{B}\Psi^{\otimes 2}$ with a trivialization τ_1 over $\mathsf{BT}_{\mathrm{sc}}\times\mathsf{BZ}$.

We shall prove that the trivializations τ, τ_1 are intertwined by the isomorphism (3.7). More precisely, consider the diagram of bi-rigidified morphisms $\mathsf{BT}_{\mathrm{sc}}\times\mathsf{BZ}\to\mathsf{B}^4\mathrm{A}(1)$:

$$\begin{array}{ccc} a^*\mu-(p_1)^*\mu-(p_2)^*\mu_{\mathrm{Z}}|_{\mathsf{BT}_{\mathrm{sc}}\times\mathsf{BZ}} & & \\ {\scriptstyle(3.7)}\downarrow & \overset{\tau}{\searrow} & \\ & & 0 \\ & \underset{\tau_1}{\nearrow} & \\ b\otimes\mathsf{B}\Psi^{\otimes 2}|_{\mathsf{BT}_{\mathrm{sc}}\times\mathsf{BZ}} & & \end{array} \tag{3.8}$$

Lemma 3.2.4. *The diagram* (3.8) *commutes.*

Proof. We first observe that this assertion involves no additional structure. Indeed, bi-rigidified morphisms $\mathsf{B}\mathrm{T}_{\mathrm{sc}} \times \mathsf{B}\mathrm{Z} \to \mathsf{B}^4\mathrm{A}(1)$ are classified by pointed morphisms

$$\mathsf{B}\mathrm{Z} \to \mathscr{H}om_{\mathbf{Z}}(\Lambda_{\mathrm{sc}}, \mathsf{B}^2\mathrm{A}), \tag{3.9}$$

which form a 1-groupoid (*cf.* §3.1.4).

Next, we reduce the commutativity of (3.8) to its pullback along $\mathsf{B}\mathrm{T}_{\mathrm{sc}} \times \mathrm{T}_{\mathrm{ad}} \to \mathsf{B}\mathrm{T}_{\mathrm{sc}} \times \mathsf{B}\mathrm{Z}$. Indeed, bi-rigidified morphisms $\mathsf{B}\mathrm{T}_{\mathrm{sc}} \times \mathrm{T}_{\mathrm{ad}} \to \mathsf{B}^4\mathrm{A}(1)$ also form a 1-groupoid, isomorphic to $\mathrm{Maps}_e(\mathrm{T}_{\mathrm{ad}}, \mathscr{H}om(\Lambda_{\mathrm{sc}}, \mathsf{B}^2\mathrm{A}))$. We need to show that the pullback functor

$$\mathrm{Maps}_e(\mathsf{B}\mathrm{Z}, \mathscr{H}om(\Lambda_{\mathrm{sc}}, \mathsf{B}^2\mathrm{A})) \to \mathrm{Maps}_e(\mathrm{T}_{\mathrm{ad}}, \mathscr{H}om(\Lambda_{\mathrm{sc}}, \mathsf{B}^2\mathrm{A})) \tag{3.10}$$

is faithful. Since (3.10) is a functor of Picard groupoids, it suffices to prove that its induced map on π_1 is injective. The latter occurs as the bottom horizontal arrow of the following commutative square

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbf{Z}}(\mathsf{B}\mathrm{Z}, \mathscr{H}om_{\mathbf{Z}}(\Lambda_{\mathrm{sc}}, \mathsf{B}\mathrm{A})) & \longrightarrow & \mathrm{Hom}_{\mathbf{Z}}(\mathrm{T}_{\mathrm{ad}}, \mathscr{H}om_{\mathbf{Z}}(\Lambda_{\mathrm{sc}}, \mathsf{B}\mathrm{A})) \\ \downarrow \simeq & & \downarrow \simeq \\ \mathrm{Maps}_e(\mathsf{B}\mathrm{Z}, \mathscr{H}om_{\mathbf{Z}}(\Lambda_{\mathrm{sc}}, \mathsf{B}\mathrm{A})) & \longrightarrow & \mathrm{Maps}_e(\mathrm{T}_{\mathrm{ad}}, \mathscr{H}om_{\mathbf{Z}}(\Lambda_{\mathrm{sc}}, \mathsf{B}\mathrm{A})) \end{array} \tag{3.11}$$

Here, the vertical functors are the forgetful ones: The left one is an isomorphism for degree reasons, and the right one is an isomorphism by the étale cohomology of T_{ad}. The kernel of the top horizontal arrow of (3.11) is identified with

$$\mathrm{Hom}_{\mathbf{Z}}(\mathsf{B}\mathrm{T}, \mathscr{H}om_{\mathbf{Z}}(\Lambda_{\mathrm{sc}}, \mathsf{B}\mathrm{A})) \xrightarrow{\simeq} \mathrm{Hom}_{\mathbf{Z}}(\mathrm{T}, \mathscr{H}om_{\mathbf{Z}}(\Lambda_{\mathrm{sc}}, \mathrm{A}))$$

which vanishes because $\mathscr{H}om_{\mathbf{Z}}(\Lambda_{\mathrm{sc}}, \mathrm{A})$ is discrete. We may now prove the commutativity of (3.8) after replacing $\mathsf{B}\mathrm{T}_{\mathrm{sc}} \times \mathsf{B}\mathrm{Z}$ by $\mathsf{B}\mathrm{T}_{\mathrm{sc}} \times \mathrm{T}_{\mathrm{ad}}$.

Along the composite $\mathsf{B}\mathrm{T}_{\mathrm{sc}} \times \mathrm{T}_{\mathrm{ad}} \to \mathsf{B}\mathrm{T}_{\mathrm{sc}} \times \mathsf{B}\mathrm{Z} \to \mathsf{B}\mathrm{T} \times \mathsf{B}\mathrm{T}$, the restrictions of m and p_1 coincide and the restriction of p_2 is trivial. This endows $a^*\mu - (p_1)^*\mu - (p_2)^*\mu_{\mathrm{Z}}$ with a trivialization over $\mathsf{B}\mathrm{T}_{\mathrm{sc}} \times \mathrm{T}_{\mathrm{ad}}$. Since (3.7) is compatible with the trivializations over $\mathsf{B}\mathrm{T} \times e$, it intertwines this trivialization with the one of $b \otimes \mathsf{B}\Psi^{\otimes 2}|_{\mathsf{B}\mathrm{T}_{\mathrm{sc}} \times \mathrm{T}_{\mathrm{ad}}}$ induced from $b(\cdot, 0) = 0$.

Using these two trivializations, the restriction of (3.8) to $\mathsf{B}\mathrm{T}_{\mathrm{sc}} \times \mathrm{T}_{\mathrm{ad}}$ reads as follows:

$$\begin{array}{ccc} 0 & & \\ & \searrow^{\tau} & \\ \downarrow{\scriptstyle \mathrm{id}} & & 0 \\ & \nearrow_{\tau_1} & \\ 0 & & \end{array} \tag{3.12}$$

Here, τ is induced from the G_{ad}-equivariance structure of the restriction of μ to $\mathsf{B}\mathrm{G}_{\mathrm{sc}}$ and τ_1 is the map $\mathsf{B}\mathrm{T}_{\mathrm{sc}} \times \mathrm{T}_{\mathrm{ad}} \to \mathsf{B}^3\mathrm{A}(1)$ given by applying the loop space functor to the second factor in $b_1 \otimes \mathsf{B}\Psi^{\otimes 2}$. The commutativity of (3.12) is precisely [Zha22, Proposition 5.5.4]. □

3.2.5. We are now ready to construct the isomorphism (3.3).

Proof of Proposition 3.1.3. Consider the monoidal morphism

$$\mathrm{Z} \to \mathscr{H}om_{\mathbf{Z}}(\pi_1\mathrm{G}, \mathsf{B}\mathrm{A}) \tag{3.13}$$

classifying the bi-rigidified morphism $a^*\mu - (p_1)^*\mu - (p_2)^*\mu_{\mathrm{Z}}$ (*cf.* §3.1.4). We need to construct an isomorphism between (3.13) and the adjoint of the pairing

$$b_2 \otimes \Psi : \pi_1\mathrm{G} \otimes \mathrm{Z} \to \mathsf{B}\mathrm{A} \tag{3.14}$$

defined by tensoring with $\Psi : \mathbb{G}_m \to \mathsf{B}\hat{\mathbf{Z}}(1)$ along the second factor of b_2.

Suppose first that G is split and equipped with a Killing pair $\mathrm{T} \subset \mathrm{B} \subset \mathrm{G}$. In this case, we have identified (3.5) with $b \otimes \mathsf{B}\Psi^{\otimes 2}|_{\mathsf{BT}\times\mathsf{BZ}}$ via the isomorphism (3.7) and proved that the trivialization τ corresponds to the trivialization of $b \otimes \mathsf{B}\Psi^{\otimes 2}|_{\mathsf{BT}_{\mathrm{sc}}\times\mathsf{BZ}}$ defined by $b_1 \otimes \mathsf{B}\Psi^{\otimes 2}$ (*cf.* Lemma 3.2.4). This yields a morphism of fiber sequences

$$\begin{array}{ccc} \mathrm{Z} & \xrightarrow{(3.13)} & \mathscr{H}om_{\mathbf{Z}}(\pi_1\mathrm{G}, \mathsf{B}\mathrm{A}) \\ \downarrow & & \downarrow \\ \mathrm{T} & \xrightarrow{b\otimes\Psi} & \mathscr{H}om_{\mathbf{Z}}(\Lambda, \mathsf{B}\mathrm{A}) \\ \downarrow & & \downarrow \\ \mathrm{T}_{\mathrm{ad}} & \xrightarrow{b_1\otimes\Psi} & \mathscr{H}om_{\mathbf{Z}}(\Lambda_{\mathrm{sc}}, \mathsf{B}\mathrm{A}) \end{array} \tag{3.15}$$

which gives an isomorphism between (3.13) and the adjoint of (3.14).

We shall argue that this isomorphism is independent of the choice of the Killing pair $\mathrm{T} \subset \mathrm{B} \subset \mathrm{G}$. For this, it suffices to show that the commutativity witness of the top square in (3.15) is independent of the choice of the Killing pair. Given another Killing pair $\mathrm{T}' \subset \mathrm{B}' \subset \mathrm{G}$, we need to show that the canonical identification $\mathsf{BT} \cong \mathsf{BT}'$ intertwines the isomorphism (3.7) defined for BT, respectively BT'. This follows because (3.7) is the restriction of an isomorphism between rigidified morphisms $\mathsf{BT}\times\mathsf{BT} \to \mathsf{B}^4\mathrm{A}(1)$, and the latter form a discrete space classified by bilinear pairings $\Lambda \otimes \Lambda \to \mathrm{A}(-1)$.

Since the isomorphism between (3.13) and the adjoint of (3.14) is constructed for any split G without additional choices, the case for any reductive G follows by étale descent. □

3.3. $\mathsf{BZ}^\sharp$-equivariance.

3.3.1. Recall the reductive group S-scheme $\mathrm{G}^\sharp$ and its center $\mathrm{Z}^\sharp$ (*cf.* §1.3.4). There is a natural map of group S-schemes of multiplicative type $\mathrm{Z}^\sharp \to \mathrm{Z}$. The BZ-action on BG restricts to a $\mathsf{BZ}^\sharp$-action, which we record in the diagram

$$\begin{array}{ccccc} & & \mathsf{BG}\times\mathsf{BZ}^\sharp & \xrightarrow{a^\sharp} & \mathsf{BG} \\ & \swarrow^{p_1} & & \searrow^{p_2} & \\ \mathsf{BG} & & & & \mathsf{BZ}^\sharp \end{array} \tag{3.16}$$

Denote by $\mu_{\mathrm{Z}^\sharp}$ the restriction of μ to $\mathsf{BZ}^\sharp$. Recall that $\mu_{\mathrm{Z}^\sharp}$ has a canonical $\mathbb{E}_\infty$-monoidal structure (*cf.* Proposition 1.3.6).

Corollary 3.3.2. *In reference to* (3.16)*, there is a canonical isomorphism of bi-rigidified morphisms* $\mathsf{BG}\times\mathsf{BZ}^\sharp \to \mathsf{B}^4\mathrm{A}(1)$*:*

$$(a^\sharp)^*\mu - (p_1)^*\mu - (p_2)^*\mu_{\mathrm{Z}^\sharp} \xrightarrow{\simeq} 0. \tag{3.17}$$

Proof. The bilinear pairing b_2 (*cf.* (1.17)) restricts to the trivial pairing

$$\pi_1\mathrm{G} \otimes \mathrm{Fib}(\Lambda^\sharp \to \Lambda^\sharp_{\mathrm{ad}}) \to \mathrm{A}(-1),$$

because the horizontal arrows of (1.15) vanish over $\Lambda^\sharp$, respectively $\Lambda^\sharp_{\mathrm{ad}}$. This induces a trivialization of the restriction of $b_2 \otimes \mathsf{B}\Psi^{\otimes 2}$ to $\mathsf{BG}_{\mathrm{ab}} \otimes \mathsf{BZ}^\sharp$.

The isomorphism (3.17) is the restriction of (3.3) to $\mathsf{BG}\times\mathsf{BZ}^\sharp$, composed with the trivialization of the right-hand-side defined above. □

3.3.3. The isomorphism (3.17) induces an isomorphism of rigidified (*not* bi-rigidified) morphisms $\mathsf{BG} \times \mathsf{BZ}^\sharp \to \mathsf{B}^4\mathrm{A}(1)$:

$$(a^\sharp)^*\mu \xrightarrow{\simeq} (p_1)^*\mu + (p_2)^*\mu_{\mathrm{Z}^\sharp}, \tag{3.18}$$

which may be regarded as the part of a $\mathsf{BZ}^\sharp$-equivariance structure on μ "against $\mu_{\mathrm{Z}^\sharp}$". The restrictions of (3.18) to $\mathsf{BG} \times e$ and $e \times \mathsf{BZ}^\sharp$ are induced from the equality of maps $a^\sharp = p_1$, $a^\sharp = p_2$ over these loci.

The isomorphism (3.18) is equipped with cocycle data. To be more transparent, let us formulate it in functorial terms: Given a G-torsor $\mathscr{E}$ and $\mathrm{Z}^\sharp$-torsors $\mathscr{Z}_1$, $\mathscr{Z}_2$ over an S-scheme, the diagram of sections of $\mathsf{B}^4\mathrm{A}(1)$ commute

$$\begin{array}{ccc} \mu(\mathscr{E} \otimes \mathscr{Z}_1 \otimes \mathscr{Z}_2) & \xrightarrow{\simeq} & \mu(\mathscr{E}) + \mu_{\mathrm{Z}^\sharp}(\mathscr{Z}_1 \otimes \mathscr{Z}_2) \\ \downarrow \simeq & & \downarrow \simeq \\ \mu(\mathscr{E} \otimes \mathscr{Z}_1) + \mu_{\mathrm{Z}^\sharp}(\mathscr{Z}_2) & \xrightarrow{\simeq} & \mu(\mathscr{E}) + \mu_{\mathrm{Z}^\sharp}(\mathscr{Z}_1) + \mu_{\mathrm{Z}^\sharp}(\mathscr{Z}_2) \end{array} \tag{3.19}$$

Here, the right vertical arrow appeals to the monoidal structure on $\mu_{\mathrm{Z}^\sharp}$ and the remaining arrows are instances of (3.18).

The commutativity of (3.19) follows from the cocycle condition on the canonical quadratic structure (*cf.* §3.1.5). Similarly to the latter, it is a *condition* and not additional structure. Likewise, higher coherence (for triples of $\mathrm{Z}^\sharp$-torsors, *etc.*) is trivially satisfied.

4. Weissman's obstruction

Let F be a local field with a fixed algebraic closure $\bar{\mathrm{F}}$. Let G be a reductive group F-scheme. Let A be a finite abelian group with order invertible in F, equipped with an injective character $\zeta : \mathrm{A} \to \mathbf{C}^\times$. Let μ be an A-valued étale metaplectic cover of G.

In this section, we define Weissman's obstruction $\Omega_\beta(\sigma)$, starting with the case $\Omega(\sigma)$ for the trivial G-isocrystal, and explain why it obstructs the existence of fibers of the conjectural map LLC_β (*cf.* Conjecture 1.4.16) at σ. Being conjectural, we need to assume something about LLC_β to make this precise: This is the compatibility with central core characters (*cf.* Lemma 4.3.11) which requires Theorem 2.1.6 to state. Then we express $\Omega_\beta(\sigma)$ in terms of $\Omega(\sigma)$ and the Kottwitz invariant of β (*cf.* Theorem 4.3.9, Corollary 4.3.12).

The last two subsections, §4.4 and §4.5, can be considered supplements to the article. In §4.4, we prove that for tori, the vanishing of $\Omega_\beta(\sigma)$ is necessary and sufficient for $\mathrm{LLC}_\beta^{-1}(\sigma)$. In §4.5, we prove a "dual version" of one of the ingredients in Theorem 4.3.9: It identifies the cover $\widetilde{\mathrm{G}}_\beta$ when G_β is isomorphic to G, *i.e.* when β comes from a Z-isocrystal.

4.1. The case for $\widetilde{\mathrm{G}}$.

4.1.1. We shall associated to G and μ a finite abelian group K and a map

$$\Omega : \Phi(\widetilde{\mathrm{H}}) \to \mathrm{Hom}(\mathrm{K}, \mathbf{C}^\times). \tag{4.1}$$

For any $\sigma \in \Phi(\widetilde{\mathrm{H}})$, we shall refer to $\Omega(\sigma)$ as *Weissman's obstruction* of σ. It has the property that $\Omega(\sigma) \neq 1$ implies that the fiber of the conjectural local Langlands correspondence (1.32) at σ is empty, assuming "compatibility with central core characters".

The obstruction Ω was first observed by Weissman when G is a torus (*cf.* [Wei09, §4], [GG18, §8.3]).

4.1.2. *Definition of* K. We let Q be the quadratic form associated to μ (*cf.* §1.3.5) and consider the induced étale sheaves $\Lambda^\sharp$, $\Lambda^\sharp_{\mathrm{sc}}$, $\Lambda^\sharp_{\mathrm{ad}}$ (*cf.* §1.3.4). Tensoring with $\mathbb{G}_m$, we obtain F-tori $\mathrm{T}^\sharp$, $\mathrm{T}^\sharp_{\mathrm{sc}}$, $\mathrm{T}^\sharp_{\mathrm{ad}}$ fitting into a commutative diagram of isogenies

$$\begin{array}{ccccc} \mathrm{T}^\sharp_{\mathrm{sc}} & \longrightarrow & \mathrm{T}^\sharp & \longrightarrow & \mathrm{T}^\sharp_{\mathrm{ad}} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{T}_{\mathrm{sc}} & \longrightarrow & \mathrm{T} & \longrightarrow & \mathrm{T}_{\mathrm{ad}} \end{array} \tag{4.2}$$

Denote by $\mathrm{Z}^\sharp$ the kernel of $\mathrm{T}^\sharp \to \mathrm{T}^\sharp_{\mathrm{ad}}$, so we have a natural map $\mathrm{Z}^\sharp \to \mathrm{Z}$. Define

$$\mathrm{K} := \mathrm{Ker}(\mathrm{Z}^\sharp(\mathrm{F}) \to \mathrm{Z}(\mathrm{F})).$$

4.1.3. Denote by $\widetilde{\mathrm{G}}$ the image of μ under (1.6). Write $\widetilde{\mathrm{Z}}$ for its pullback along $\mathrm{Z}(\mathrm{F}) \to \mathrm{G}(\mathrm{F})$ and $\widetilde{\mathrm{Z}}^\sharp$ for its further pullback to $\mathrm{Z}^\sharp(\mathrm{F})$.

The subgroup $\mathrm{Ker}(\widetilde{\mathrm{Z}}^\sharp \to \widetilde{\mathrm{Z}})$ of $\widetilde{\mathrm{Z}}^\sharp$ is identified with K via the projection onto $\mathrm{Z}^\sharp(\mathrm{F})$. Thus, we obtain a map

$$i : \mathrm{K} \to \widetilde{\mathrm{Z}}^\sharp. \tag{4.3}$$

Lemma 4.1.4. *The group* $\widetilde{\mathrm{Z}}^\sharp$ *is commutative and its image in* $\widetilde{\mathrm{G}}$ *is central.*

Proof. The commutativity of $\widetilde{\mathrm{Z}}^\sharp$ follows from the fact that $\mu_{\mathrm{Z}^\sharp}$ is $\mathbb{E}_\infty$-monoidal (*cf.* Proposition 1.3.6). It remains to prove that the image of $\widetilde{\mathrm{Z}}^\sharp$ in $\widetilde{\mathrm{G}}$ is central.

Denote by $a^\sharp : \mathrm{G}(\mathrm{F}) \times \mathrm{Z}^\sharp(\mathrm{F}) \to \mathrm{G}(\mathrm{F})$ the multiplication map. Applying the construction functor (1.6), with $\mathrm{G} \times \mathrm{Z}^\sharp$ playing the role of G, to the isomorphism (3.18), we obtain an isomorphism of covers of $\mathrm{G}(\mathrm{F}) \times \mathrm{Z}^\sharp(\mathrm{F})$:

$$(a^\sharp)^*\widetilde{\mathrm{G}} \xrightarrow{\simeq} (p_1)^*\widetilde{\mathrm{G}} + (p_2)^*\widetilde{\mathrm{Z}}^\sharp, \tag{4.4}$$

whose restrictions to $\mathrm{G}(\mathrm{F}) \times e$ and $e \times \mathrm{Z}^\sharp(\mathrm{F})$ are induced from the equality of maps $a^\sharp = p_1$, respectively $a^\sharp = p_2$ over these subgroups.

Equivalently, one may express (4.4) as a morphism of short exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{A} \times \mathrm{A} & \longrightarrow & \widetilde{\mathrm{G}} \times \widetilde{\mathrm{Z}}^\sharp & \longrightarrow & \mathrm{G}(\mathrm{F}) \times \mathrm{Z}^\sharp(\mathrm{F}) & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \Sigma} & & \downarrow{\scriptstyle \tilde{a}^\sharp} & & \downarrow{\scriptstyle a^\sharp} & & \\ 1 & \longrightarrow & \mathrm{A} & \longrightarrow & \widetilde{\mathrm{G}} & \longrightarrow & \mathrm{G}(\mathrm{F}) & \longrightarrow & 1 \end{array}$$

where $\tilde{a}^\sharp$ restricts to the identity on $\widetilde{\mathrm{G}} \times e$ and the natural map on $e \times \widetilde{\mathrm{Z}}^\sharp$. By expressing an element $(\tilde{g}, \tilde{z}) \in \widetilde{\mathrm{G}} \times \widetilde{\mathrm{Z}}^\sharp$ as $(\tilde{g}, 1) \cdot (1, \tilde{z})$, respectively $(1, \tilde{z}) \cdot (\tilde{g}, 1)$, and using the fact that $\tilde{a}^\sharp$ is a group homomorphism, we see that $\tilde{g}$ commutes with the image of $\tilde{z}$. □

4.1.5. By Lemma 4.1.4 and Schur's lemma, $\widetilde{\mathrm{Z}}^\sharp$ acts by a character χ on any irreducible ζ-genuine smooth representation V. The association of χ to [V] defines a map

$$\Pi(\widetilde{\mathrm{G}}) \to \Pi(\widetilde{\mathrm{Z}}^\sharp), \tag{4.5}$$

where the target stands for the set of ζ-genuine smooth characters $\widetilde{\mathrm{Z}}^\sharp \to \mathbf{C}^\times$.

We refer to the image of $[\mathrm{V}] \in \Pi(\widetilde{\mathrm{G}})$ under (4.5) as the *central core character* of [V].

Remark 4.1.6. Our notion of the "central core character" is different from Weissman's (*cf.* [Wei18, §6.3]). Namely, the notion of *op.cit.* concerns only the maximal torus of $\mathrm{Z}^\sharp$ whereas ours concerns the entire $\mathrm{Z}^\sharp$.

4.1.7. *Definition of* Ω. The map (4.1) is defined to be the composition of (2.36) with the inverse of (2.40) and the restriction along (4.3):

$$\Omega : \Phi(\widetilde{\mathrm{H}}) \to \Phi(\widetilde{\mathrm{H}}_{\mathrm{ab}}) \xrightarrow{\simeq} \Pi(\widetilde{\mathrm{Z}}^\sharp) \xrightarrow{i^*} \mathrm{Hom}(\mathrm{K}, \mathbf{C}^\times).$$

Lemma 4.1.8 (Weissman)**.** *Suppose that there is a map* $\mathrm{LLC} : \Pi(\widetilde{\mathrm{G}}) \to \Phi(\widetilde{\mathrm{H}})$ *satisfying the following compatibility with central core characters: It renders the diagram*

$$\begin{array}{ccc} \Pi(\widetilde{\mathrm{G}}) & \xrightarrow{\mathrm{LLC}} & \Phi(\widetilde{\mathrm{H}}) \\ \downarrow{\scriptstyle (4.5)} & & \downarrow{\scriptstyle (2.36)} \\ \Pi(\widetilde{\mathrm{Z}}^\sharp) & \xrightarrow{(2.40)} & \Phi(\widetilde{\mathrm{H}}_{\mathrm{ab}}) \end{array} \tag{4.6}$$

commutative. Then for any $\sigma \in \Phi(\widetilde{\mathrm{H}})$ *with* $\Omega(\sigma) \neq 1$*, the set* $\mathrm{LLC}^{-1}(\sigma)$ *is empty.*

Proof. Fix $\sigma \in \Phi(\widetilde{\mathrm{H}})$ and let V be an irreducible ζ-genuine smooth representation of $\widetilde{\mathrm{G}}$ belonging to fiber of LLC at σ.

Suppose that $\widetilde{\mathrm{Z}}^\sharp$ acts on V via some character χ. By the commutativity of (4.6), the subgroup K of $\widetilde{\mathrm{Z}}^\sharp$ acts by the character $\Omega(\sigma)$. However, since the image of K in $\widetilde{\mathrm{G}}$ is trivial, this implies that $\Omega(\sigma) = 1$. □

Remark 4.1.9. Every character $\chi : \mathrm{K} \to \mathbf{C}^\times$ occurs as $\Omega(\sigma)$ for some $\sigma \in \Phi(\widetilde{\mathrm{H}})$.

Indeed, one may first extend (ζ, χ) along the inclusion $\mathrm{A} \times \mathrm{K} \subset \widetilde{\mathrm{T}}^\sharp$ to obtain a ζ-genuine character of $\widetilde{\mathrm{T}}^\sharp$. Under the local Langlands correspondence for $(\mathrm{T}^\sharp, \mu_{\mathrm{T}^\sharp})$ (*cf.* §2.1.8), the latter defines an L-parameter $\sigma_{\mathrm{T}} \in \Phi(\widetilde{\mathrm{T}}_{\mathrm{H}})$ with respect to the canonical maximal torus T_{H} of H. The image $\sigma \in \Phi(\widetilde{\mathrm{H}})$ of σ_{T} satisfies $\Omega(\sigma) = \chi$, by construction of (2.40).

4.2. The Pontryagin dual of K.

4.2.1. Recall the finite abelian group K associated to G and μ (*cf.* §4.1.2). In this subsection, we shall construct a surjective map

$$\gamma : (\pi_1 \mathrm{G})_{\mathrm{Gal_F}} \to \mathrm{Hom}(\mathrm{K}, \mathbf{C}^\times). \tag{4.7}$$

Let us note a consequence of Pontryagin duality.

Lemma 4.2.2. *Let* Λ_1, Λ_2 *be étale sheaves of finite free* $\mathbf{Z}$*-modules over* Spec F *equipped with a pairing* $c : \Lambda_1 \otimes \Lambda_2 \to \mathrm{A}$*. Denote by* $\Lambda_1^\sharp \subset \Lambda_1$*,* $\Lambda_2^\sharp \subset \Lambda_2$ *the kernels of* c*. Then the adjoint of* c *factors through an isomorphism of étale sheaves*

$$\Lambda_1/\Lambda_1^\sharp \xrightarrow{\simeq} \mathscr{H}om(\Lambda_2/\Lambda_2^\sharp, \mathrm{A}). \tag{4.8}$$

Proof. It suffices to check that (4.8) is an isomorphism over a separable closure of F, so we may assume that Λ_1, Λ_2 are finite free $\mathbf{Z}$-modules rather than sheaves of such.

Since $\Lambda_2/\Lambda_2^\sharp$ is N-torsion for $\mathrm{N} := |\mathrm{A}|$, ζ induces an isomorphism

$$\mathscr{H}om(\Lambda_2/\Lambda_2^\sharp, \mathrm{A}) \xrightarrow{\simeq} (\Lambda_2/\Lambda_2^\sharp)^\vee,$$

where $(\cdot)^\vee$ denotes Pontryagin dual, *i.e.* continuous homomorphisms into the topological group U_1 of unit complex numbers.

We view the composite $\zeta \cdot c$ as a U_1-valued pairing and consider its adjoint

$$\Lambda_1 \to (\Lambda_2)^\vee. \tag{4.9}$$

The kernel of (4.9) equals $\Lambda_1^\sharp$. We claim that its cokernel is identified with $(\Lambda_2^\sharp)^\vee$. Indeed, since Pontryagin duality is an exact involution, the dual of (4.9) is $\Lambda_2 \to (\Lambda_1)^\vee$, which has kernel $\Lambda_2^\sharp$. The isomorphism (4.8) follows. □

4.2.3. We shall apply Lemma 4.2.2 to the pairings b and b_1 associated to Q. More precisely, applying a Tate twist to (1.15) and using Lemma 4.2.2, we find a commutative square

$$\begin{array}{ccc} (\Lambda/\Lambda^\sharp)(1) & \xrightarrow{\simeq} & \mathscr{H}om(\Lambda/\Lambda^\sharp, \mathrm{A}) \\ \downarrow & & \downarrow \\ (\Lambda_{\mathrm{ad}}/\Lambda_{\mathrm{ad}}^\sharp)(1) & \xrightarrow{\simeq} & \mathscr{H}om(\Lambda_{\mathrm{sc}}/\Lambda_{\mathrm{sc}}^\sharp, \mathrm{A}) \end{array} \tag{4.10}$$

where the horizontal maps are isomorphisms.

4.2.4. *Construction of* γ. Taking global sections of (4.10) over Spec F, we obtain the commutative square

$$\begin{array}{ccc} \mathrm{Ker}(\mathrm{T}^\sharp \to \mathrm{T})(\mathrm{F}) & \xrightarrow{\simeq} & \mathrm{Hom}((\Lambda/\Lambda^\sharp)_{\mathrm{Gal_F}}, \mathrm{A}) \\ \downarrow & & \downarrow \\ \mathrm{Ker}(\mathrm{T}_{\mathrm{ad}}^\sharp \to \mathrm{T}_{\mathrm{ad}})(\mathrm{F}) & \xrightarrow{\simeq} & \mathrm{Hom}((\Lambda_{\mathrm{sc}}/\Lambda_{\mathrm{sc}}^\sharp)_{\mathrm{Gal_F}}, \mathrm{A}) \end{array}$$

Taking kernels of the vertical maps, we obtain an isomorphism

$$\mathrm{K} \xrightarrow{\simeq} \mathrm{Hom}((\pi_1\mathrm{G})_{\mathrm{Gal_F}}/(\pi_1\mathrm{G}^\sharp)_{\mathrm{Gal_F}}, \mathrm{A}). \tag{4.11}$$

Since $(\pi_1\mathrm{G})_{\mathrm{Gal_F}}/(\pi_1\mathrm{G}^\sharp)_{\mathrm{Gal_F}}$ is N-torsion (for $\mathrm{N} := |\mathrm{A}|$), its Pontryagin dual is identified with K under (4.11). Applying bi-duality yields a short exact sequence

$$(\pi_1\mathrm{G}^\sharp)_{\mathrm{Gal_F}} \to (\pi_1\mathrm{G})_{\mathrm{Gal_F}} \xrightarrow{\gamma} \mathrm{Hom}(\mathrm{K}, \mathbf{C}^\times) \to 1. \tag{4.12}$$

The map (4.7) is defined as the second map displayed in this short exact sequence.

4.3. The case for $\widetilde{\mathrm{G}}_\beta$.

4.3.1. For each $\beta \in \mathsf{Isoc}_{\mathrm{G}}$, we have a morphism of group F-schemes

$$\mathrm{Z} \to \mathrm{G}_\beta, \tag{4.13}$$

sending an R-point z of Z to the automorphism of the pullback of β to $\mathrm{X} \times \mathrm{Spec}\,\mathrm{R}$ given by acting by z. The image of (4.13) is central in G_β.

Let $\widetilde{\mathrm{G}}_\beta$ denote the image μ under the construction functor (1.8). Denote by $\widetilde{\mathrm{Z}}_\beta$ the pullback of $\widetilde{\mathrm{G}}_\beta$ along the map $\mathrm{Z}(\mathrm{F}) \to \mathrm{G}_\beta(\mathrm{F})$ induced from (4.13) and by $\widetilde{\mathrm{Z}}_\beta^\sharp$ its further pullback to $\mathrm{Z}^\sharp(\mathrm{F})$. Thus $\mathrm{Ker}(\widetilde{\mathrm{Z}}_\beta^\sharp \to \widetilde{\mathrm{Z}}_\beta)$ is identified with K along the projection onto $\mathrm{Z}^\sharp(\mathrm{F})$. This yields an injection

$$i_\beta : \mathrm{K} \to \widetilde{\mathrm{Z}}_\beta^\sharp. \tag{4.14}$$

This map specializes to (4.3) when β is the trivial G-isocrystal.

4.3.2. Let us now identify $\widetilde{\mathrm{Z}}_\beta$ for any $\beta \in \mathsf{Isoc}_{\mathrm{G}}$.

Recall the bilinear pairing $b_2 \otimes \mathsf{B}\Psi^{\otimes 2} : \mathsf{B}\mathrm{G}_{\mathrm{ab}} \otimes \mathsf{B}\mathrm{Z} \to \mathsf{B}^4\mathrm{A}(1)$ (*cf.* §3.1.1). Evaluating at the G_{ab}-isocrystal defined by β, we find a rigidified morphism

$$(b_2 \otimes \mathsf{B}\Psi^{\otimes 2})(\beta, \cdot) : \mathrm{X} \times \mathsf{B}\mathrm{Z} \to \mathsf{B}^4\mathrm{A}(1),$$

whch is canonically the pullback of a rigidified morphism $\mathsf{B}\mathrm{Z} \to \mathsf{B}^4\mathrm{A}(1)$ (*cf.* Lemma 1.2.5), to be denoted using the same expression.

The following result is a consequence of the canonical quadratic structure (*cf.* Proposition 3.1.3). Its statement invokes the construction functor (1.6) for Z.

Proposition 4.3.3. *For any $\beta \in \mathsf{Isoc}_{\mathrm{G}}$, there is a canonical isomorphism of covers of* $\mathrm{Z}(\mathrm{F})$*:*

$$\widetilde{\mathrm{Z}}_\beta \xrightarrow{\simeq} \widetilde{\mathrm{Z}} + \int_{\mathrm{F}} (b_2 \otimes \mathsf{B}\Psi^{\otimes 2})(\beta, \cdot). \tag{4.15}$$

Proof. Consider the action map $a : \mathsf{BG} \times \mathsf{BZ} \to \mathsf{BG}$. Its restriction along the G-isocrystal $\beta : \mathrm{X} \to \mathsf{BG}$ yields a morphism

$$a_\beta : \mathrm{X} \times \mathsf{BZ} \to \mathsf{BG}. \tag{4.16}$$

The loop space functor applied to (4.16) recovers (4.13). More precisely, taking fiber product of X with itself over the two stacks in (4.16), we obtain a morphism from $\mathrm{X} \times \mathrm{Z}$ to the group X-sheaf of automorphisms of β, which is adjoint to (4.13).

Let us pull back μ along the composition of the projection $p : \mathrm{X} \times \mathsf{BZ} \to \mathrm{X}$ and $\mu : \mathrm{X} \to \mathsf{BG}$. By construction, we have an identification of covers of $\mathrm{Z}(\mathrm{F})$

$$\widetilde{\mathrm{Z}}_\beta \xrightarrow{\simeq} \int_{\mathrm{F}} (a_\beta)^* \mu - p^* \beta^* \mu. \tag{4.17}$$

It remains to identify the right-hand-sides of (4.15) and (4.17). We shall do so by identifying the "integrands", *i.e.* providing an isomorphism

$$(a_\beta)^* \mu - p^* \beta^* \mu \xrightarrow{\simeq} \mu_{\mathrm{Z}} + (b_2 \otimes \mathsf{B}\Psi^{\otimes 2})(\beta, \cdot) \tag{4.18}$$

of rigidified morphisms $\mathrm{X} \times \mathsf{BZ} \to \mathsf{B}^4\mathrm{A}(1)$, where μ_{Z} denotes the restriction of μ along the composition $\mathrm{X} \times \mathsf{BZ} \to \mathsf{BZ} \to \mathsf{BG}$.

The isomorphism (4.18) is the restriction of (3.3) along $(\beta, \mathrm{id}) : \mathrm{X} \times \mathsf{BZ} \to \mathsf{BG} \times \mathsf{BZ}$. □

Remark 4.3.4. It follows from Proposition 4.3.3 that the cover $\widetilde{\mathrm{Z}}_\beta$ depends only on the G_{ab}-isocrystal induced from β.

4.3.5. Note that pairing $b_2 \otimes \mathsf{B}\Psi^{\otimes 2}$ is canonically trivialized over $\mathsf{BG}_{\mathrm{ab}} \otimes \mathsf{BZ}^\sharp$ (*cf.* the proof of Corollary 3.3.2). In particular, the pullback of (4.15) along $\mathrm{Z}^\sharp(\mathrm{F}) \to \mathrm{Z}(\mathrm{F})$ yields an isomorphism of covers of $\mathrm{Z}^\sharp(\mathrm{F})$:

$$\widetilde{\mathrm{Z}}^\sharp_\beta \xrightarrow{\simeq} \widetilde{\mathrm{Z}}^\sharp. \tag{4.19}$$

Let us compose the inverse of (4.19) with the natural map $\widetilde{\mathrm{Z}}^\sharp_\beta \to \widetilde{\mathrm{G}}_\beta$ to obtain a map:

$$\widetilde{\mathrm{Z}}^\sharp \to \widetilde{\mathrm{G}}_\beta, \tag{4.20}$$

Lemma 4.3.6. *The image of* (4.20) *is central in* $\widetilde{\mathrm{G}}_\beta$.

Proof. Denote by $\mu_{\mathrm{G}_\beta} := \mathrm{T}_\beta(\mu)$ the translation of μ by β (*cf.* §1.2.3). Consider the $\mathsf{BZ}^\sharp$-action on BG_β via the inclusion (4.13).

By the proof of Lemma 4.1.4, it suffices to show that μ_{G_β} is $\mathsf{BZ}^\sharp$-equivariant against $\mu_{\mathrm{Z}^\sharp} : \mathsf{BZ}^\sharp \to \mathsf{B}^4\mathrm{A}(1)$ in the sense of §3.3.3 and that, upon acting on the neutral point of BG_β, this equivariance structure reduces to the isomorphism

$$\mu_{\mathrm{G}_\beta}|_{\mathsf{BZ}^\sharp} \xrightarrow{\simeq} \mu_{\mathrm{Z}^\sharp} \tag{4.21}$$

induced from (4.18) and the trivialization of $b_2 \otimes \mathsf{B}\Psi^{\otimes 2}$ over $\mathsf{BG}_{\mathrm{ab}} \times \mathsf{BZ}^\sharp$.

By Lemma 1.2.5, it suffices to construct the $\mathsf{BZ}^\sharp$-equivariance structure after base change along $\mathrm{X} \to \operatorname{Spec} \mathrm{F}$. The base change of μ_{G_β} to $\mathrm{X} \times \mathsf{BG}_\beta$ is the pullback of μ along (1.9) minus the constant section $p^* \beta^* \mu$. However, (1.9) is $\mathsf{BZ}^\sharp$-equivariant, so the desired $\mathsf{BZ}^\sharp$-equivariance structure on μ_{G_β} follows from that of μ (*cf.* §3.3.3). The fact that acting on the neutral point of BG_β recovers the isomorphism (4.21) is a consequence of the construction of (3.18) (which uses the trivialization of $b_2 \otimes \mathsf{B}\Psi^{\otimes 2}$ over $\mathsf{BG} \times \mathsf{BZ}^\sharp$). □

4.3.7. By Lemma 4.3.6 and Schur's lemma, we have a map

$$\Pi(\widetilde{\mathrm{G}}_\beta) \to \Pi(\widetilde{\mathrm{Z}}^\sharp) \tag{4.22}$$

sending the isomorphism class [V] of an irreducible ζ-genuine smooth representation V of $\widetilde{\mathrm{G}}_\beta$ to the character of $\widetilde{\mathrm{Z}}^\sharp$ by which it acts on V through (4.20).

The image of [V] under (4.22) can be viewed as the "central core character" of [V], generalizing the construction of §4.1.5.

4.3.8. We now arrive at a crucial point: The isomorphism (4.19) is generally *incompatible* with the inclusions of K via i and i_β (*cf.* (4.3), (4.14)). In other words, the quotient i_β/i factors through a character

$$\mathrm{K} \to \mathrm{A}. \tag{4.23}$$

We express this character in terms of the Kottwitz invariant of β (*cf.* §1.1.6).

Theorem 4.3.9. *The character* (4.23) *equals the image of* $\mathrm{Kott}(\beta)$ *under* (4.7)*,* i.e.

$$\frac{i_\beta}{i} = \gamma(\mathrm{Kott}(\beta)). \tag{4.24}$$

Proof. Using the isomorphism (4.15) and the $\mathbf{Z}$-linear structure on $\mathsf{Cov}(\mathrm{Z}(\mathrm{F}), \mathrm{A})$, we may express (4.23) as follows: Consider the cover

$$\int_{\mathrm{F}} (b_2 \otimes \mathsf{B}\Psi^{\otimes 2})(\beta, \cdot) \in \mathsf{Cov}(\mathrm{Z}(\mathrm{F}), \mathrm{A})$$

equipped with the splitting over $\mathrm{Z}^\sharp(\mathrm{F})$ defined by the trivialization of $(b_2 \otimes \mathsf{B}\Psi^{\otimes 2})(\beta, \cdot)$ over $\mathsf{B}\mathrm{Z}^\sharp$. The restriction of this splitting to K is the character (4.23).

The $\mathbf{Z}$-linear morphism $b_2 \otimes \Psi^{\otimes 2} : \mathsf{B}\mathrm{G}_{\mathrm{ab}} \otimes \mathsf{B}\mathrm{Z} \to \mathsf{B}^4\mathrm{A}(1)$ is trivialized over $\mathsf{B}\mathrm{G}_{\mathrm{ab}} \otimes \mathsf{B}\mathrm{Z}^\sharp$, so by taking fibers, we obtain a pairing

$$\langle \cdot, \cdot \rangle : \mathsf{B}\mathrm{G}_{\mathrm{ab}} \otimes \mathrm{Fib}(\mathrm{Z}^\sharp \to \mathrm{Z}) \to \mathsf{B}^2\mathrm{A}(1).$$

This pairing encodes the character (4.23) in the following manner: Given $\beta : \mathrm{X} \to \mathsf{B}\mathrm{G}_{\mathrm{ab}}$ and $a \in \mathrm{K} \cong \mathrm{H}^0(\operatorname{Spec}\mathrm{F}, \mathrm{Fib}(\mathrm{Z}^\sharp \to \mathrm{Z}))$, the class of $\langle \beta, a \rangle$ in

$$\mathrm{H}^2(\mathrm{X}, \mathrm{A}(1)) \cong \mathrm{H}^2(\operatorname{Spec}\mathrm{F}, \mathrm{A}(1)) \cong \mathrm{A} \tag{4.25}$$

is the image of a under (4.23). Here, the isomorphisms are given by pullback along $\mathrm{X} \to \operatorname{Spec}\mathrm{F}$ (*cf.* Lemma 1.2.5) and Tate duality.

On the other hand, $\langle \cdot, a \rangle : \mathsf{B}\mathrm{G}_{\mathrm{ab}} \to \mathsf{B}^2\mathrm{A}(1)$ is the tensor product of a $\mathbf{Z}$-linear map $\pi_1\mathrm{G} \to \mathrm{A}$ with the Kummer map. By construction, this $\mathbf{Z}$-linear map is the image of $a \in \mathrm{K}$ under (4.11). The desired equality (4.24) thus reduces to the following compatibility between Kottwitz invariant (1.3) and Tate duality: Given any map of étale sheaves $f : \pi_1\mathrm{G} \to \mathrm{A}$, the following diagram commutes

$$\begin{array}{ccc} \pi_0\mathsf{Isoc}_{\mathrm{G}_{\mathrm{ab}}} & \xrightarrow{\mathrm{Kott}} & (\pi_1\mathrm{G})_{\mathrm{Gal}_{\mathrm{F}}} \\ \big\downarrow{\scriptstyle f\otimes\Psi} & & \big\downarrow{\scriptstyle f} \\ \mathrm{H}^2(\mathrm{X}, \mathrm{A}(1)) & \xrightarrow{(4.25)} & \mathrm{A} \end{array} \tag{4.26}$$

Here, the left vertical arrow is induced from $f \otimes \Psi : \mathrm{G}_{\mathrm{ab}} \to \mathsf{B}\mathrm{A}(1)$. The commutativity of (4.26) reduces to the case where G is a torus by construction (*cf.* §1.1.6), then to the case where $\mathrm{G} = \mathbb{G}_m$ by functoriality (*cf.* [Kot85, §2]), where it follows from the definition. ☐

4.3.10. Finally, let us define (the generalized) Weissman's obstruction

$$\Omega_\beta : \Phi(\widetilde{\mathrm{H}}) \to \mathrm{Hom}(\mathrm{K}, \mathbf{C}^\times)$$

for an arbitrary G-isocrystal β.

We set Ω_β to be the composition

$$\begin{aligned}\Omega_\beta : \Phi(\widetilde{\mathrm{H}}) &\to \Phi(\widetilde{\mathrm{H}}_{\mathrm{ab}}) \\ &\xrightarrow{\simeq} \Pi(\widetilde{\mathrm{Z}}^\sharp) \xrightarrow{i_\beta^*} \mathrm{Hom}(\mathrm{K}, \mathbf{C}^\times).\end{aligned}$$

The proof of Lemma 4.1.8 also yields the following result.

Lemma 4.3.11. *Suppose that there is a map* $\mathrm{LLC}_\beta : \Pi(\widetilde{\mathrm{G}}_\beta) \to \Phi(\widetilde{\mathrm{H}})$ *satisfying the following compatibility with central core characters: It renders the diagram*

$$\begin{array}{ccc} \Pi(\widetilde{\mathrm{G}}_\beta) & \xrightarrow{\mathrm{LLC}_\beta} & \Phi(\widetilde{\mathrm{H}}) \\ \downarrow{\scriptstyle (4.22)} & & \downarrow{\scriptstyle (2.36)} \\ \Pi(\widetilde{\mathrm{Z}}^\sharp) & \xrightarrow{(2.40)} & \Phi(\widetilde{\mathrm{H}}_{\mathrm{ab}}) \end{array} \tag{4.27}$$

commutative. Then for any $\sigma \in \Phi(\widetilde{\mathrm{H}})$ *with* $\Omega_\beta(\sigma) \neq 1$*, the set* $\mathrm{LLC}_\beta^{-1}(\sigma)$ *is empty.* □

Corollary 4.3.12. *For each* $\sigma \in \Phi(\widetilde{\mathrm{H}})$ *and* $\beta \in \mathsf{Isoc}_{\mathrm{G}}$*, the character* $\Omega_\beta(\sigma)$ *vanishes if and only if*

$$\gamma(\mathrm{Kott}(\beta)) = \Omega(\sigma)^{-1}. \tag{4.28}$$

Proof. The quotient $\Omega_\beta(\sigma)/\Omega(\sigma)$ is the character $i_\beta/i : \mathrm{K} \to \mathbf{C}^\times$. By Theorem 4.3.9, the latter equals $\gamma(\mathrm{Kott}(\beta))$. Hence the equality (4.28) holds if and only if $\Omega_\beta(\sigma) = 1$. □

4.3.13. To conclude, let us rewrite the short exact sequence (4.12) using (1.5):

$$\pi_0(\mathsf{Basic}_{\mathrm{G}^\sharp}) \to \pi_0(\mathsf{Basic}_{\mathrm{G}}) \to \mathrm{Hom}(\mathrm{K}, \mathbf{C}^\times) \to 1.$$

Here, the middle arrow sends β to $\gamma(\mathrm{Kott}(\beta))$. The group structure on $\pi_0(\mathsf{Basic}_{\mathrm{G}})$ is induced from that on $(\pi_1\mathrm{G})_{\mathrm{Gal}_{\mathrm{F}}}$, and similarly for $\pi_0(\mathsf{Basic}_{\mathrm{G}^\sharp})$.

Given $\sigma \in \Phi(\widetilde{\mathrm{H}})$, Corollary 4.3.12 shows that there exists a basic G-isocrystal β for which (4.28) holds. Furthermore, the subset of $\pi_0(\mathsf{Basic}_{\mathrm{G}})$ consisting of isomorphism classes of such β forms a torsor under the image of $\pi_0(\mathsf{Basic}_{\mathrm{G}^\sharp})$.

By Remark 4.3.12, the character $\Omega(\sigma)^{-1}$ of K is arbitrary as σ varies. This means that to guarantee the equality (4.28), one really needs to consider basic G-isocrystals spanning a full set of representatives of $\pi_0(\mathsf{Basic}_{\mathrm{G}})/\pi_0(\mathsf{Basic}_{\mathrm{G}^\sharp})$.

4.4. Example: tori.

4.4.1. In this subsection, we specialize to the case G = T is an F-torus. We shall construct the local Langlands correspondence (*cf.* Conjecture 1.4.16) for T.

More precisely, for each $\beta \in \mathsf{Isoc}_{\mathrm{T}}$, we shall construct a map

$$\mathrm{LLC}_\beta : \Pi(\widetilde{\mathrm{T}}_\beta) \to \Phi(\widetilde{\mathrm{H}}). \tag{4.29}$$

In fact, LLC_β is uniquely determined by the compatibility diagram (4.27) since (2.36) becomes an isomorphism in this case, so let us turn this into a definition.

4.4.2. *Construction of* LLC_β. The map (4.20) specializes to a map

$$\widetilde{\mathrm{T}}^\sharp \to \widetilde{\mathrm{T}}_\beta \tag{4.30}$$

whose image is central (*cf.* Lemma 4.3.6). Thus, given any irreducible ζ-genuine smooth representation V of $\widetilde{\mathrm{T}}_\beta$, the action of $\widetilde{\mathrm{T}}^\sharp$ on V through (4.30) is a ζ-genuine smooth character. This defines a map

$$\Pi(\widetilde{\mathrm{T}}_\beta) \to \Pi(\widetilde{\mathrm{T}}^\sharp). \tag{4.31}$$

The map LLC_β is the composition of (4.31) with the local Langlands correspondence for $\mathrm{T}^\sharp$ equipped with the restriction $\mu_{\mathrm{T}^\sharp}$ of μ (*cf.* §2.1.8).

4.4.3. The following description of $\mathrm{LLC}_\beta^{-1}(\sigma)$ generalizes Weissman's result for the trivial T-isocrystal β (*cf.* [Wei16, Theorem 2.12]), with the same proof.

Proposition 4.4.4. *Given* $\beta \in \mathsf{Isoc}_{\mathrm{T}}$ *and* $\sigma \in \Phi(\widetilde{\mathrm{H}})$*, the set* $\mathrm{LLC}_\beta^{-1}(\sigma)$ *is finite and nonempty if and only if* (4.28) *holds.*

Proof. Lemma 4.3.11 and Corollary 4.3.12 together imply that $\mathrm{LLC}_\beta^{-1}(\sigma)$ is empty when (4.28) fails. It remains to prove that when (4.28) holds, $\mathrm{LLC}_\beta^{-1}(\sigma)$ is nonempty and finite.

The L-parameter σ corresponds, under the local Langlands correspondence for $(\mathrm{T}^\sharp, \mu^\sharp)$ (*cf.* §2.1.8), to a ζ-genuine smooth character χ_σ of $\widetilde{\mathrm{T}}^\sharp$. By construction, a ζ-genuine smooth representation V of $\widetilde{\mathrm{T}}_{\beta,\zeta}$ has L-parameter σ if and only if $\widetilde{\mathrm{T}}^\sharp$ acts on V via χ_σ. Since (4.28) holds, χ_σ annihilates the kernel of (4.30), so it factors through a character

$$\bar{\chi}_\sigma : \widetilde{\mathrm{T}}^\sharp/\mathrm{K} \to \mathbf{C}^\times,$$

where $\widetilde{\mathrm{T}}^\sharp/\mathrm{K}$ is identified with a subgroup of the center $\widetilde{\mathrm{C}}_\beta$ of $\widetilde{\mathrm{T}}_\beta$ (*cf.* Lemma 4.3.6).

By the Stone–von Neumann theorem, $\Pi(\widetilde{\mathrm{T}}_\beta)$ is in bijection with genuine characters of $\widetilde{\mathrm{C}}_\beta$. Hence $\mathrm{LLC}_\beta^{-1}(\sigma)$ is in bijection with extensions of $\bar{\chi}_\sigma$ along the inclusion

$$\widetilde{\mathrm{T}}^\sharp/\mathrm{K} \subset \widetilde{\mathrm{C}}_\beta,$$

which is of finite index. This implies that $\mathrm{LLC}_\beta^{-1}(\sigma)$ is nonempty and finite. □

4.5. Z**-isocrystals.**

4.5.1. We return to the context where G is a reductive group F-scheme. Given a Z-isocrystal β, we may consider the induced G-isocrystal, hence the group F-scheme G_β. There is a canonical isomorphism of group F-schemes

$$\mathrm{G} \xrightarrow{\simeq} \mathrm{G}_\beta, \tag{4.32}$$

defined as follows: Restricting the action map $a : \mathsf{BG} \times \mathsf{BZ} \to \mathsf{BG}$ along the Z-isocrystal $\beta : \mathrm{X} \to \mathsf{BZ}$ yields a morphism $a_\beta : \mathsf{BG} \times \mathrm{X} \to \mathsf{BG}$, which induces (4.32) on loop spaces.

In this subsection, we express the pullback of $\widetilde{\mathrm{G}}_\beta$ along (4.32) in terms of the cover $\widetilde{\mathrm{G}}$.

4.5.2. Evaluating the bilinear pairing $b_2 \otimes \mathsf{B}\Psi^{\otimes 2} : \mathsf{BG}_{\mathrm{ab}} \otimes \mathsf{BZ} \to \mathsf{B}^4\mathrm{A}(1)$ at $\beta : \mathrm{X} \to \mathsf{BZ}$ and descending along $\mathrm{X} \to \operatorname{Spec} \mathrm{F}$ (*cf.* Lemma 1.2.5), we obtain a $\mathbf{Z}$-linear morphism

$$b_2 \otimes \mathsf{B}\Psi^{\otimes 2}(\cdot, \beta) : \mathsf{BG}_{\mathrm{ab}} \to \mathsf{B}^4\mathrm{A}(1), \tag{4.33}$$

which defines a rigidified morphism $\mathsf{BG} \to \mathsf{B}^4\mathrm{A}(1)$ that we denote by the same expression.

The following result can be thought of as a "dual version" of Proposition 4.3.3.

Proposition 4.5.3. *For any $\beta \in \mathsf{Isoc}_Z$, there is a canonical isomorphism of covers of* $G(F)$ *with regard to the identification* (4.32)*:*

$$\widetilde{G}_\beta \xrightarrow{\simeq} \widetilde{G} + \int_F (b_2 \otimes \mathsf{B}\Psi^{\otimes 2})(\cdot, \beta). \tag{4.34}$$

Proof. Consider the pullback $p^*\beta^*\mu$ of μ along the projection $p : \mathsf{B}G \times X \to X$ and the G-isocrystal $\beta : X \to \mathsf{B}G$.

The isomorphism (3.3), restricted along $(\mathrm{id}, \beta) : \mathsf{B}G \times X \to \mathsf{B}G \times \mathsf{B}Z$, yields an isomorphism of rigidified section of $\mathsf{B}^4 A(1)$ over $\mathsf{B}G \times X$:

$$(a_\beta)^*\mu - p^*\beta^*\mu \xrightarrow{\simeq} \mu + (b_2 \otimes \mathsf{B}\Psi^{\otimes 2})(\cdot, \beta), \tag{4.35}$$

or equivalently, over $\mathsf{B}G$ (*cf.* Lemma 1.2.5).

The isomorphism (4.34) is the image of (4.35) under (1.6). □

Remark 4.5.4. Proposition 4.5.3 expresses $\widetilde{G}_\beta$ as the image of the rigidified morphism $\mu + (b_2 \otimes \mathsf{B}\Psi^{\otimes 2})(\cdot, \beta)$ under (1.6).

If μ is the étale realization of a central extension of G by K_2 (*cf.* [Zha22, §2.3]), one may wonder whether the rigidified morphism $\mu + (b_2 \otimes \mathsf{B}\Psi^{\otimes 2})(\cdot, \beta)$ also comes from étale realization. This is generally *not* the case.

For a "naturally occurring" example, let us take $G := GL_2$ endowed with the Kazhdan–Patterson cover, viewed as a central extension E of G by K_2 (*cf.* [GG18, §13.2]). Assuming $\operatorname{char} F \neq 2$, the latter defines a rigidified morphism $\mu : \mathsf{B}G \to \mathsf{B}^4\{\pm 1\}^{\otimes 2}$ under étale realization. Identifying both $\pi_1 G$ and $\mathrm{Fib}(\Lambda \to \Lambda_{\mathrm{ad}})$ with $\mathbf{Z}$, the bilinear form b_2 is given by

$$b_2 : \mathbf{Z} \otimes \mathbf{Z} \to \mathbf{Z}/2, \quad 1 \otimes 1 \mapsto 1.$$

We argue that $(b_2 \otimes \mathsf{B}\Psi^{\otimes 2})(\cdot, \beta)$ (hence its sum with μ) does not lift to a central extension of G by K_2, unless β is the trivial Z-isocrystal. Indeed, $(b_2 \otimes \mathsf{B}\Psi^{\otimes 2})(\cdot, \beta)$ arises as the tensor product of Ψ with a $\mathbf{Z}$-linear morphism

$$\pi_1 G \to \mathsf{B}^2\{\pm 1\}, \tag{4.36}$$

which sends the generator of $\pi_1 G \cong \mathbf{Z}$ to the Kummer gerbe $\mathsf{B}\Psi(\beta)$ of β—the latter represents a nontrivial class in $H^2(\operatorname{Spec} F, \{\pm 1\})$ when β is nontrivial. If $(b_2 \otimes \mathsf{B}\Psi^{\otimes 2})(\cdot, \beta)$ lifts to a central extension of G by K_2, then the $\mathbb{E}_1$-monoidal morphism $\Lambda \to \mathsf{B}^2\{\pm 1\}$, obtained by pre-composing (4.36) with the projection $\Lambda \twoheadrightarrow \pi_1 G$, can be expressed in terms of the second Brylinski–Deligne invariant of E, *i.e.* it factors as a monoidal morphism

$$\Lambda \to \mathsf{B}\mathbb{G}_m \xrightarrow{\mathsf{B}\Psi} \mathsf{B}^2\{\pm 1\}.$$

This implies that $\mathsf{B}\Psi(\beta)$ lifts to a section of $\mathsf{B}\mathbb{G}_m$ over $\operatorname{Spec} F$, hence trivial by Hilbert 90.

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