SCHEME THEORY I

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These are lecture notes for a first-semester algebraic geometry course taught at the University of Münster, Fall 2024. Please send corrections to yifei.zhao.93@gmail.com.

Contents

You must go where I cannot,
Pangur Bán, Pangur Bán.
Níl sa saol seo ach ceo.
Is ní bheimid beo,
ach seal beag gearr.
(This world is nothing but mist. And
we will live but a little while.)
—Aisling from *the Secret of Kells*

1. Defining schemes

Schemes are globalizations of rings, in the same way that topological manifolds are globalizations of Euclidean spaces.

Given a topological manifold X, we can cover it by open subspaces $X_i \to X$ ($i \in I$), where each X_i is homeomorphic to a Euclidean space. Then we can cover the intersections $X_{ij} := X_i \cap X_j$ ($i, j \in I$) by open subspaces $X_{ijk} \to X_{ij}$ ($k \in K_{ij}$) homeomorphic to Euclidean spaces. This gives a presentation of each topological manifold X as "glued" from copies of Euclidean spaces. Formally, X is a coequalizer in the category Top of topological spaces:

$$\bigsqcup_{\substack{i,j\in I\\ k\in K_{ij}}} X_{ijk} \rightrightarrows \bigsqcup_{i\in I} X_i \to X.$$

We wish to do the same for "spaces" defined by general rings. But, what is a suitable category of "spaces" in which we can do this?

The first idea is Top itself, where each ring A corresponds to its set of primes $|\mathrm{Spec}(A)|$ equipped with the Zariski topology. But this is too coarse: the passage from A to $|\mathrm{Spec}(A)|$ loses all information about nilpotents in A, for instance. Instead, we shall perform this gluing in a more sophisticated category Shv of "Zariski sheaves".

1.1. Functors of points.

1.1.1. Let Ring denote the category of commutative, unital rings (henceforth referred to simply as *rings*). Define the category of *presheaves* to be the functor category:

$$\mathsf{PShv} := \mathsf{Fun}(\mathsf{Ring}, \mathsf{Set}),$$

where Set stands for the category of sets.[1]

Given a presheaf X and a ring R, elements of the set X(R) are called R-*points of* X. For this reason, presheaves are also called "functors of points".

Remark 1.1.2. We think of presheaves as "spaces" in algebraic geometry. Let me illustrate this intuition with an example.

Consider the ring $A := \mathbf{Z}[x, y]/(x^2 + y^2 + 1)$ and the presheaf $X := \mathrm{Hom}_{\mathsf{Ring}}(A, -)$ corepresented by it. We think of X as the "space" defined by $x^2 + y^2 + 1 = 0$ in some abstract sense. It follows from the definition that an R-point of X consists of a pair $(a, b) \in R^{\times 2}$ satisfying $a^2 + b^2 + 1 = 0$, *i.e.* we are indeed solving the equation $x^2 + y^2 + 1 = 0$, but in the ring R. For example, our X has no $\mathbf{R}$-points but plenty of $\mathbf{C}$-points.

[1] There are size issues in the formation of PShv. Namely, because Ring is a large category, "Hom sets" in PShv are proper classes. To avoid proper classes, what one can do is to fix a universe U (which is a set closed under certain set-theoretic operations) and define Set to consist only of those elements of U and Ring for those rings whose underlying sets lie in Set. If we admit Grothendieck's axiom that every set belongs to a universe, then objects and Hom-sets of PShv really are sets (but belonging to a larger universe). Going forward, we will only point out size issues when absolutely necessary.

By definition, a presheaf is determined by its values on all $\mathrm{R} \in \mathsf{Ring}$ together with their functoriality. This is one sense in which our "spaces" are defined by their "points".

1.1.3. An object of PShv is called an *affine scheme* if it is representable. Denote by $\mathsf{Sch}^{\mathrm{aff}}$ the category of affine schemes, so there is a fully faithful functor:

$$\mathsf{Sch}^{\mathrm{aff}} \to \mathsf{PShv}. \tag{1.1}$$

By the Yoneda lemma, $\mathsf{Sch}^{\mathrm{aff}}$ is equivalent to $\mathsf{Ring}^{\mathrm{op}}$. For a ring A, we denote by Spec(A) the corresponding object of $\mathsf{Sch}^{\mathrm{aff}}$ and call it the *spectrum* of A. Thus, for every $\mathrm{R} \in \mathsf{Ring}$, the R-points of Spec(A) are precisely the ring homomorphisms $\mathrm{A} \to \mathrm{R}$.

The category Ring contains all (small) limits and colimits. Hence the same holds for $\mathsf{Sch}^{\mathrm{aff}}$. The category of presheaves PShv also contains all limits and colimits, which are computed pointwise (as in any functor category). The functor (1.1) preserves limits (as Yoneda embeddings do) but does not preserve colimits in general.

Remark 1.1.4. The initial object of PShv is the presheaf taking value $\varnothing$ at every ring; note that this is *not* the affine scheme $\varnothing := \mathrm{Spec}(0)$—the latter evaluates to a singleton on itself. The terminal object of PShv is the presheaf taking value the singleton at every ring; it is representable by $\mathrm{Spec}(\mathbf{Z})$.

Example 1.1.5. The functor $\mathsf{Ring} \to \mathsf{Set}$ sending a ring R to its underlying set is representable by the affine scheme $\mathbb{A}^1_{\mathbf{Z}} := \mathrm{Spec}(\mathbf{Z}[x])$. This affine scheme is called the *affine line* over $\mathbf{Z}$. Likewise, for any $\mathrm{A} \in \mathsf{Ring}$, we write $\mathbb{A}^1_{\mathrm{A}} := \mathrm{Spec}(\mathrm{A}[x])$.

Note that $\mathbb{A}^1_{\mathbf{Z}}$ is a ring object in the category of affine schemes: The addition $\mathbb{A}^1_{\mathbf{Z}} \times \mathbb{A}^1_{\mathbf{Z}} \to \mathbb{A}^1_{\mathbf{Z}}$ is defined by $\mathbf{Z}[x] \to \mathbf{Z}[x_1, x_2]$, $x \mapsto x_1 + x_2$ and the multiplication $\mathbb{A}^1_{\mathbf{Z}} \times \mathbb{A}^1_{\mathbf{Z}} \to \mathbb{A}^1_{\mathbf{Z}}$ is defined by $\mathbf{Z}[x] \to \mathbf{Z}[x_1, x_2]$, $x \mapsto x_1 x_2$. The additive and multiplicative units $\mathrm{Spec}(\mathbf{Z}) \to \mathbb{A}^1_{\mathbf{Z}}$ are defined by the maps from $\mathbf{Z}[x]$ to $\mathbf{Z}$ carrying x to 0, respectively 1. This ring structure induces the ring structure on R for any R-point of $\mathbb{A}^1_{\mathbf{Z}}$.

Example 1.1.6. Since the functor (1.1) preserves limits and push-outs in Ring are given by tensor products, we see that the following diagram of affine schemes:

$$\begin{array}{ccc} \mathrm{Spec}(\mathrm{A}_1 \otimes_{\mathrm{A}} \mathrm{A}_2) & \rightarrow & \mathrm{Spec}(\mathrm{A}_1) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathrm{A}_2) & \longrightarrow & \mathrm{Spec}(\mathrm{A}) \end{array} \tag{1.2}$$

is Cartesian in PShv. In other words, the fiber product of affine schemes is the affine scheme associated to the tensor product of the corresponding rings.

In particular, for every set I, the product $\mathbb{A}^{\mathrm{I}}_{\mathbf{Z}} := \prod_{i \in \mathrm{I}} \mathbb{A}^1_{\mathbf{Z}}$ is isomorphic to $\mathrm{Spec}(\mathbf{Z}[x_i]_{i \in \mathrm{I}})$, where $\mathbf{Z}[x_i]_{i \in \mathrm{I}}$ is the polynomial ring on generators x_i ($i \in \mathrm{I}$). The affine scheme $\mathbb{A}^{\mathrm{I}}_{\mathbf{Z}}$ is called an *affine space*. If $\mathrm{I} = \{1, \cdots, n\}$, we write $\mathbb{A}^n_{\mathbf{Z}} := \mathbb{A}^{\mathrm{I}}_{\mathbf{Z}}$.

Remark 1.1.7. Given any $\mathrm{A} \in \mathsf{Ring}$, we may choose a set x_i ($i \in \mathrm{I}$) of generators of A to obtain a surjection $\mathbf{Z}[x_i]_{i \in \mathrm{I}} \twoheadrightarrow \mathrm{A}$. Then we may choose a set of ideal generators y_j ($j \in \mathrm{J}$) of its kernel $\mathrm{I} \subset \mathbf{Z}[x_i]_{i \in \mathrm{I}}$ to present A as a tensor product $\mathrm{A} \cong \mathbf{Z}[x_i]_{i \in \mathrm{I}} \otimes_{\mathbf{Z}[y_j]_{j \in \mathrm{J}}} \mathbf{Z}$. The affine scheme Spec(A) is then the fiber product:

$$\begin{array}{ccc} \mathrm{Spec}(\mathrm{A}) & \rightarrow & \mathbb{A}^{\mathrm{I}}_{\mathbf{Z}} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbf{Z}) & \rightarrow & \mathbb{A}^{\mathrm{J}}_{\mathbf{Z}} \end{array} \tag{1.3}$$

In particular, every affine scheme is constructed from $\mathbb{A}^1_{\mathbf{Z}}$ by iterated limits. Combined with Example 1.1.6, this shows that $\mathsf{Sch}^{\mathrm{aff}}$ is precisely the smallest full subcategory of PShv which is closed under limits and contains $\mathbb{A}^1_{\mathbf{Z}}$.

1.2. Zariski sheaves.

1.2.1. A morphism $\mathrm{Spec}(\mathrm{R}') \to \mathrm{Spec}(\mathrm{R})$ of affine schemes is called a *standard open*[2] if there exists some $f \in \mathrm{R}$ and an isomorphism $\mathrm{R}' \cong \mathrm{R}_f$ of R-algebras; here, R_f stands for the localization of R at the multiplicative subset generated by f. (Caution: the choice of $f \in \mathrm{R}$ which exhibits R' as the localization R_f is generally not unique.)

A collection of morphisms $\mathrm{Spec}(\mathrm{R}_i) \to \mathrm{Spec}(\mathrm{R})$ $(i \in \mathrm{I})$ of affine schemes is called a *standard open cover* if I is finite, each $\mathrm{Spec}(\mathrm{R}_i) \to \mathrm{Spec}(\mathrm{R})$ is a standard open, and any field-valued point of $\mathrm{Spec}(\mathrm{R})$ factors through $\mathrm{Spec}(\mathrm{R}_i) \to \mathrm{Spec}(\mathrm{R})$ for some $i \in \mathrm{I}$.

The condition on field-valued points can be characterized in ring-theoretic terms.

Lemma 1.2.2. *Let* $\mathrm{Spec}(\mathrm{R}_i) \to \mathrm{Spec}(\mathrm{R})$ *(*$i \in \mathrm{I}$*) be a family of standard opens, with each* $\mathrm{R}_i \cong \mathrm{R}_{f_i}$ *for some* $f_i \in \mathrm{R}$*. Then the following are equivalent:*

(1) any field-valued point of $\mathrm{Spec}(\mathrm{R})$ *factors through* $\mathrm{Spec}(\mathrm{R}_i)$ *for some* $i \in \mathrm{I}$*;*
(2) the ideal generated by f_i *(*$i \in \mathrm{I}$*) equals* R*.*

Proof. Statement (1) says that every map $\mathrm{R} \to \mathrm{K}$, where K is a field, sends one of the f_i's to a nonzero element.

By taking kernels, this is equivalent to the assertion that every prime $\mathfrak{p}$ of R does *not* contain f_i for some $i \in \mathrm{I}$, which is equivalent to the assertion that $\mathrm{R}/(f_i)_{i \in \mathrm{I}}$ contains no primes, *i.e.* it is the zero ring. This is statement (2). ☐

Remark 1.2.3. In particular, if $\mathrm{Spec}(\mathrm{R}_i) \to \mathrm{Spec}(\mathrm{R})$ $(i \in \mathrm{I})$ is a family of standard opens satisfying the equivalent conditions of Lemma 1.2.2, then the same holds after replacing I by a finit subset.

Indeed, this follows by writing $1 \in \mathrm{R}$ as an R-linear combination of f_i $(i \in \mathrm{I})$, if $\mathrm{R}_i \cong \mathrm{R}_{f_i}$, which involves finitely many terms.

1.2.4. Given any $\mathrm{X} \in \mathsf{PShv}$ and any morphism $\mathrm{Spec}(\mathrm{R}') \to \mathrm{Spec}(\mathrm{R})$ of affine schemes with $x \in \mathrm{X}(\mathrm{R})$, we shall use the notation $x|_{\mathrm{R}'}$ for its image along $\mathrm{X}(\mathrm{R}) \to \mathrm{X}(\mathrm{R}')$. We think of $x|_{\mathrm{R}'}$ as the "restriction" of the R-point x to R'.

Given any $\mathrm{X} \in \mathsf{PShv}$ and any standard open cover $\mathrm{Spec}(\mathrm{R}_i) \to \mathrm{Spec}(\mathrm{R})$, we may form a diagram in Set:

$$\mathrm{X}(\mathrm{R}) \to \textstyle\prod_{i \in \mathrm{I}} \mathrm{X}(\mathrm{R}_i) \rightrightarrows \prod_{i,j \in \mathrm{I}} \mathrm{X}(\mathrm{R}_{ij}). \tag{1.4}$$

Here, $\mathrm{R}_{ij} := \mathrm{R}_i \otimes_{\mathrm{R}} \mathrm{R}_j$ and the two parallel morphisms from $\prod_{i \in \mathrm{I}} \mathrm{X}(\mathrm{R}_i)$ to $\prod_{i,j} \mathrm{X}(\mathrm{R}_{ij})$ sends an I-indexed family of points $x_i \in \mathrm{X}(\mathrm{R}_i)$ to the $\mathrm{I} \times \mathrm{I}$-indexed family $x_{ij} := x_i|_{\mathrm{R}_{ij}} \in \mathrm{X}(\mathrm{R}_{ij})$, respectively $x_{ij} := x_j|_{\mathrm{R}_{ij}} \in \mathrm{X}(\mathrm{R}_{ij})$.

Note that the first morphism $\mathrm{X}(\mathrm{R}) \to \prod_{i \in \mathrm{I}} \mathrm{X}(\mathrm{R}_i)$ equalizes the two parallel morphisms. Indeed, this follows by functoriality of X, applied to the commutative diagram below for each $i, j \in \mathrm{I}$:

$$\begin{array}{ccc} \mathrm{R} & \longrightarrow & \mathrm{R}_i \\ \downarrow & & \downarrow \\ \mathrm{R}_j & \longrightarrow & \mathrm{R}_{ij} \end{array}$$

[2] Some authors call it a "distinguished open". We generally try to follow the terminology of the Stacks project [Sta18].

1.2.5. An object $X \in \mathsf{PShv}$ is called a *Zariski sheaf* if (1.4) is an equalizer for every standard open cover $\mathrm{Spec}(R_i) \to \mathrm{Spec}(R)$ $(i \in I)$.

Concretely, this means that elements of $X(R)$ are in bijection with I-indexed families of elements $x_i \in X(R_i)$ such that $x_i|_{R_{ij}} = x_j|_{R_{ij}}$ for every pair of elements $i, j \in I$.

Denote by Shv the full subcategory of PShv consisting of Zariski sheaves. This gives us a fully faithful functor:

$$\mathsf{Shv} \to \mathsf{PShv}, \tag{1.5}$$

which preserves all limits.

Remark 1.2.6. Let $\varnothing$ denote the initial object of Shv. We claim that $\varnothing$ is represented by $\mathrm{Spec}(0)$, where 0 stands for the zero ring.

Indeed, it is enough to prove that for any Zariski sheaf X, the value $X(0)$ is a singleton. To do this, we consider the standard open cover of $\mathrm{Spec}(0)$ indexed by the empty set $I := \varnothing$. Then the products in (1.4) are singleton sets, so the same holds for $X(0)$.

Proposition 1.2.7. *Every affine scheme is a Zariski sheaf.*

1.2.8. We shall deduce Proposition 1.2.7 from an assertion about modules which will be useful later.

Namely, given a standard open cover $\mathrm{Spec}(R_i) \to \mathrm{Spec}(R)$ $(i \in I)$ and an R-module M, we may set $M_i := M \otimes_R R_i$ and $M_{ij} := M \otimes_R R_{ij}$ for $i, j \in I$. We have morphisms of R-modules induced from the ring maps $R \to R_i$ and $R_i \to R_{ij} \leftarrow R_j$:

$$M \to \textstyle\bigoplus_{i \in I} M_i \rightrightarrows \bigoplus_{i,j \in I} M_{ij}. \tag{1.6}$$

Lemma 1.2.9. *Diagram* (1.6) *is an equalizer in* Mod_R.

Proof. The assertion that (1.6) is an equalizer can be verified after localizing at every prime of R. Indeed, by taking the kernel and cokernel of the map from M to the equalizer of the two parallel arrows in (1.6), this reduces to the following statement: if an R-module N satisfies $N_{\mathfrak{p}} = 0$ for every prime $\mathfrak{p}$ of R, then $N = 0$. (Let us recall how this goes: take any $x \in N$ and consider its annihilator $\mathrm{Ann}(x) \subset A$. The assumption shows that for every prime $\mathfrak{p}$, there exists some $f \in \mathrm{Ann}(x)$, $f \notin \mathfrak{p}$. Hence $\mathrm{Ann}(x) = A$ and $x = 0$.)

Since direct sums commute with localization, we may assume that R is local with maximal ideal $\mathfrak{m}$. Write $R_i = R_{f_i}$ for elements $f_i \in R$ $(i \in I)$. Since they generate R as an ideal (Lemma 1.2.2), we have $f_i \notin \mathfrak{m}$ for some $i \in I$. Fix an element $1 \in I$ with $f_1 \notin \mathfrak{m}$, so it is a unit. Then the natural map $M \to M_1$ is an isomorphism. The map from M to the equalizer of the two parallel arrows in (1.6) thus has inverse given by projection onto $M_1 \cong M$. □

Proof of Proposition 1.2.7. We first prove that $\mathbb{A}^1_{\mathbf{Z}}$ is a Zariski sheaf. Unwinding the definitions, this means that for every standard open cover $\mathrm{Spec}(R_i) \to \mathrm{Spec}(R)$ $(i \in I$ finite$)$, with $R_{ij} := R_i \otimes_R R_j$, the following diagram in Set is an equalizer:

$$R \to \textstyle\prod_{i \in I} R_i \rightrightarrows \prod_{i,j \in I} R_{ij}. \tag{1.7}$$

Since I is finite, this is diagram (1.6) for the special case $M = R$, so the assertion follows from Lemma 1.2.9.

To prove that each affine scheme is a Zariski sheaf, we observe that affine schemes are obtained from $\mathbb{A}^1_{\mathbf{Z}}$ via iterated limits (Remark 1.1.7), while the property of being a Zariski sheaf is preserved under limits. □

1.3. Sheaves on sites.

1.3.1. We shall now look more closely at how the category Shv of Zariski sheaves behaves as a full subcategory of PShv.

In fact, we shall prove our assertions about Shv in greater generality, as there is nothing really special about standard open covers.

1.3.2. A *site* consists of a category $\mathcal{C}$ together with a set $\mathrm{Cov}(c)$ of families of morphisms $c_i \to c$ $(i \in \mathrm{I})$ for each object $c \in \mathcal{C}$, satisfying the following conditions:

(1) every isomorphism $c' \xrightarrow{\simeq} c$ belongs to $\mathrm{Cov}(c)$;[3]
(2) if $c_i \to c$ $(i \in \mathrm{I})$ belongs to $\mathrm{Cov}(c)$, and for each $i \in \mathrm{I}$, we have a family $c_{ij} \to c_i$ $(j \in \mathrm{J}_i)$ in $\mathrm{Cov}(c_i)$, then the family $c_{ij} \to c_i \to c$ $(i \in \mathrm{I},\ j \in \mathrm{J}_i)$ belongs to $\mathrm{Cov}(c)$;
(3) if $c_i \to c$ $(i \in \mathrm{I})$ belongs to $\mathrm{Cov}(c)$, then for every morphism $d \to c$, the fiber product $d_i := c_i \times_c d$ exists for all $i \in \mathrm{I}$, and the family $d_i \to d$ $(i \in \mathrm{I})$ belongs to $\mathrm{Cov}(d)$.

Each family of morphisms $c_i \to c$ $(i \in \mathrm{I})$ in $\mathrm{Cov}(c)$ is called a *cover* of c. We will refer to the three axioms above as (1) *identity*, (2) *locality*, and (3) *base change*. ("Base change" generally refers to fiber products in categorical contexts: we think of the operation $\times_c d$ as changing the base from c to d.)

1.3.3. Given a site $\mathcal{C}$, we write $\mathsf{PShv}(\mathcal{C}) := \mathsf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathsf{Set})$ and call its objects *presheaves on* $\mathcal{C}$. Note that this category does not depend on the chosen covers in $\mathcal{C}$. For $\mathrm{X} \in \mathsf{PShv}(\mathcal{C})$ and $c \in \mathcal{C}$, elements of $\mathrm{X}(c)$ are also called *sections* of X over c. Given a morphism $c' \to c$ in $\mathcal{C}$, we write $x|_{c'}$ for the image of $x \in \mathrm{X}(c)$ in $\mathrm{X}(c')$ and call it the *restriction*.

A *sheaf on* $\mathcal{C}$ is a presheaf $\mathrm{X} \in \mathsf{PShv}(\mathcal{C})$ for which the diagram:

$$\mathrm{X}(c) \to \textstyle\prod_{i\in\mathrm{I}} \mathrm{X}(c_i) \rightrightarrows \prod_{i,j\in\mathrm{I}} \mathrm{X}(c_{ij}) \tag{1.8}$$

is an equalizer for every family $c_i \to c$ in $\mathrm{Cov}(c)$; here $c_{ij} := c_i \times_c c_j$ and two parallel morphisms in (1.8) are defined by restrictions along the first, respectively the second factor (*cf.* §1.2.4).

Example 1.3.4. Let $\mathcal{C}$ be a site whose covers consist only of isomorphisms. Then every object of $\mathsf{PShv}(\mathcal{C})$ is a sheaf.

Example 1.3.5 (Sites defined by topological spaces)**.** Let T be a topological space. Let $\mathcal{C}$ be the poset of open subsets of T, *i.e.* an object of $\mathcal{C}$ is an open subset $\mathrm{U} \subset \mathrm{T}$ and a morphism of $\mathcal{C}$ is an inclusion of open subsets $\mathrm{U}_1 \to \mathrm{U}_2$. For each $\mathrm{U} \in \mathcal{C}$, we define $\mathrm{Cov}(\mathrm{U})$ to be the set of families $\mathrm{U}_i \to \mathrm{U}$ $(i \in \mathrm{I})$ which cover U as a set. This choice of covers turns $\mathcal{C}$ into a site. (Pre)sheaves on the site $\mathcal{C}$ are typically called (pre)sheaves *on* T.

The presheaf on T assigning to each $\mathrm{U} \subset \mathrm{T}$ the set of continuous real-valued functions is a sheaf. The presheaf on T assigning to each $\mathrm{U} \subset \mathrm{T}$ the set of bounded real-valued functions is generally *not* a sheaf.

Example 1.3.6 (The standard Zariski site)**.** The example we are interested in is $\mathcal{C} := \mathsf{Sch}^{\mathrm{aff}}$ where covers of an affine scheme $\mathrm{Spec}(\mathrm{R})$ are the standard open covers (*cf.* §1.2.1). Sheaves on this site are precisely the Zariski sheaves defined earlier.

1.3.7. Sheaves on $\mathcal{C}$ form a full subcategory $\mathsf{Shv}(\mathcal{C})$ of $\mathsf{PShv}(\mathcal{C})$. In other words, we have a fully faithful functor:

$$\mathsf{Shv}(\mathcal{C}) \to \mathsf{PShv}(\mathcal{C}). \tag{1.9}$$

We shall explicitly construct a left adjoint of (1.9).

[3]More precisely, the singleton $\{c' \xrightarrow{\simeq} c\}$, for any isomorphism $c' \xrightarrow{\simeq} c$, is an element of $\mathrm{Cov}(c)$.

1.3.8. First, let us turn the set of covers $\mathrm{Cov}(c)$ of an object $c \in \mathcal{C}$ into a category, whose morphisms are given by "refinements".

Given two covers $c_i \to c$ $(i \in \mathrm{I})$ and $c_j \to c$ $(j \in \mathrm{J})$, a *refinement* $\{c_j\} \to \{c_i\}$ consists of a map $\varphi : \mathrm{J} \to \mathrm{I}$ together with morphisms $c_j \to c_{\varphi(j)}$ over c, for each $j \in \mathrm{J}$.

A refinement where φ is injective and each $c_j \to c_{\varphi(j)}$ is an isomorphism is also called a *subcover*.

1.3.9. Let X be a presheaf on $\mathcal{C}$. For each family $c_i \to c$ $(i \in \mathrm{I})$ in $\mathrm{Cov}(c)$, we write $\check{\mathrm{H}}^0(\{c_i\}, \mathrm{X})$ for the equalizer of the two parallel morphisms in (1.8):

$$\check{\mathrm{H}}^0(\{c_i\}, \mathrm{X}) := \lim\big(\textstyle\prod_{i\in\mathrm{I}} \mathrm{X}(c_i) \rightrightarrows \prod_{i,j\in\mathrm{I}} \mathrm{X}(c_{ij}) \big).$$

Explicitly, an element of $\check{\mathrm{H}}^0(\{c_i\}, \mathrm{X})$ is an I-indexed tuple $x_i \in \mathrm{X}(c_i)$ such that for each $i, j \in \mathrm{I}$, there holds $x_i|_{c_{ij}} = x_j|_{c_{ij}}$.

The association $\{c_i\}_{i\in\mathrm{I}} \mapsto \check{\mathrm{H}}^0(\{c_i\}, \mathrm{X})$ is contravariantly functorial with respect to refinements. We set $\check{\mathrm{H}}^0(c, \mathrm{X})$ as the colimit over $\mathrm{Cov}(c)^{\mathrm{op}}$ with refinements as morphisms:

$$\check{\mathrm{H}}^0(c, \mathrm{X}) := \operatorname*{colim}_{\{c_i\}\in\mathrm{Cov}(c)} \check{\mathrm{H}}^0(\{c_i\}, \mathrm{X}). \tag{1.10}$$

This is a filtered colimit in view of the following observation.

Lemma 1.3.10. *The diagram* $\mathrm{Cov}(c)^{\mathrm{op}} \to \mathsf{Set}$ *assigning* $\check{\mathrm{H}}^0(\{c_i\}, \mathrm{X})$ *to* $\{c_i\}$ *is filtered.*[4]

Proof. Explicitly, we need to check that:

(1) $\mathrm{Cov}(c)$ is not empty;
(2) every pair of covers of c has a common refinement;
(3) for every pair of refinements in $\mathrm{Cov}(c)$:

$$\{c_j\} \rightrightarrows \{c_i\} \tag{1.11}$$

there exists a refinement $\{c_k\} \to \{c_j\}$ in $\mathrm{Cov}(c)$ such that the two induced maps from $\check{\mathrm{H}}^0(\{c_i\}, \mathrm{X})$ to $\check{\mathrm{H}}^0(\{c_k\}, \mathrm{X})$ coincide.

Statement (1) follows from the identity axiom.

Statement (2) follows from the base change and locality axioms. More precisely, given two covers $c_i \to c$ $(i \in \mathrm{I})$ and $c_j \to c$ $(j \in \mathrm{J})$, we form $c_{ij} := c_i \times_c c_j$. Then $c_{ij} \to c_j$ $(i \in \mathrm{I})$ is a cover for each $j \in \mathrm{J}$ by the base change axiom, so $c_{ij} \to c$ $(i \in \mathrm{I},\ j \in \mathrm{J})$ is a cover by the locality axiom.

For statement (3), one can prove directly a stronger assertion: the two induced maps from $\check{\mathrm{H}}^0(\{c_i\}, \mathrm{X})$ to $\check{\mathrm{H}}^0(\{c_j\}, \mathrm{X})$ already coincide. Indeed, given refinements $\varphi : \{c_j\} \to \{c_i\}$ and $\psi : \{c_j\} \to \{c_i\}$, then both maps $c_j \to c_{\varphi(j)}$, $c_j \to c_{\psi(j)}$ factor through $c_{\varphi(j)} \times_c c_{\psi(j)}$, where a tuple of elements $x_i \in \mathrm{X}(c_i)$ agreeing on overlaps is equalized. ☐

Remark 1.3.11. The fact that $\check{\mathrm{H}}^0(c, \mathrm{X})$ is a filtered colimit in Set gives rise to the following explicit description of it: any element of $\check{\mathrm{H}}^0(c, \mathrm{X})$ is represented by an element of $\check{\mathrm{H}}^0(\{c_i\}, \mathrm{X})$ for some cover $c_i \to c$ $(i \in \mathrm{I})$, *i.e.* an I-tuple $x_i \in \mathrm{X}(c_i)$ agreeing on overlaps, and two such tuples $\{x_i\}$, $\{x_j\}$ represent the same element in $\check{\mathrm{H}}^0(c, \mathrm{X})$ whenever there is a common refinement $\{c_k\}$ of the covers $\{c_i\}$, $\{c_j\}$ such that their images in $\check{\mathrm{H}}^0(\{c_k\}, \mathrm{X})$ coincide.

[4] Caution: the category $\mathrm{Cov}(c)^{\mathrm{op}}$ itself is not filtered in general, *i.e.* we are not always able to equalize the two parallel morphisms in (1.11) in $\mathrm{Cov}(c)$. For example, consider the discrete topological space $\{1,2,3\}$, covered by the two open subsets $\{1,2\}, \{2,3\}$. This cover receives two refinements from the cover by $\{1\}$, $\{2\}$, $\{3\}$ defined by sending $\{2\}$ into $\{1,2\}$, respectively $\{2,3\}$. These two refinements cannot be equalized.

1.3.12. The assignment $c \mapsto \check{\mathrm{H}}^0(c, \mathrm{X})$ is functorial, so it defines a presheaf X^+ on $\mathcal{C}$ with:

$$\mathrm{X}^+(c) := \check{\mathrm{H}}^0(c, \mathrm{X}).$$

Furthermore, the assignment $\mathrm{X} \mapsto \mathrm{X}^+$ is an endofunctor of $\mathsf{PShv}(\mathcal{C})$.

Note that there is a natural morphism of presheaves:

$$\mathrm{X} \to \mathrm{X}^+, \tag{1.12}$$

defined by the restriction map $\mathrm{X}(c) \to \check{\mathrm{H}}^0(\{c_i\}, \mathrm{X})$ for each cover $c_i \to c$ $(i \in \mathrm{I})$. If X is a sheaf, then (1.12) is an isomorphism.

Moreoever, sections of X^+ "locally lift to X" in the following sense: given any $c \in \mathcal{C}$ and $x^+ \in \mathrm{X}^+$, there exists a cover $c_i \to c$ $(i \in \mathrm{I})$ such that each $x^+|_{c_i} \in \mathrm{X}^+(c_i)$ is the image of an element $x_i \in \mathrm{X}(c_i)$ (*cf.* Remark 1.3.11).

Proposition 1.3.13. *For any presheaf* X *on* $\mathcal{C}$*, the presheaf* $(\mathrm{X}^+)^+$ *is a sheaf.*

Proof. In fact, a finer statement holds. Let us call a presheaf $\mathrm{X} \in \mathsf{Shv}(\mathcal{C})$ *separated* if every cover $c_i \to c$ $(i \in \mathrm{I})$, the induced map below is injective:

$$\mathrm{X}(c) \to \lim\Big(\prod_{i\in\mathrm{I}} \mathrm{X}(c_i) \rightrightarrows \prod_{i,j\in\mathrm{I}} \mathrm{X}(c_{ij}) \Big), \tag{1.13}$$

where $c_{ij} := c_i \times_c c_j$ is the overlap.

Since the colimit (1.10) is filtered, any section $x \in \mathrm{X}^+(c)$ is an equivalence class of I-tuples $x_i \in \mathrm{X}(c_i)$ agreeing on $\mathrm{X}(c_{ij})$, for some cover $c_i \in c$ $(i \in \mathrm{I})$ (*cf.* Remark 1.3.11). Using this description, we will prove the two claims below.

Claim: X^+ is separated for any $\mathrm{X} \in \mathsf{PShv}(\mathcal{C})$.

Namely, we need to prove that given a cover $c_i \to c$ $(i \in \mathrm{I})$ and two elements $x^+, y^+ \in \mathrm{X}^+(c)$ which agree in $\prod_{i\in\mathrm{I}} \mathrm{X}^+(c_i)$, there holds $x^+ = y^+$.

Taking common refinements, we can find a cover $c_j \to c$ $(j \in \mathrm{I})$ such that x^+ and y^+ are both represented by J-tuples of elements $x_j \in \mathrm{X}(c_j)$, $y_j \in \mathrm{X}(c_j)$.

For each $j \in \mathrm{J}$, the base change $c_{ij} := c_i \times_c c_j \to c_j$ $(i \in \mathrm{I})$ is a cover of c_j with the property that $x_j|_{c_{ij}}$ and $y_j|_{c_{ij}}$ have the same image in $\mathrm{X}^+(c_{ij})$ for each $i \in \mathrm{I}$. This means that for each pair $(i,j) \in \mathrm{I} \times \mathrm{J}$, there is a cover $c_{ijk} \to c_{ij}$ $(k \in \mathrm{K}_{ij})$ such that $x_j|_{c_{ijk}} = y_j|_{c_{ijk}}$ in $\mathrm{X}(c_{ijk})$. Thus, $c_{ijk} \to c_j$ $(i \in \mathrm{I}, k \in \mathrm{K}_{ij})$ is a refinement with the property that (x_j) and (y_j) have the same image in $\check{\mathrm{H}}^0(\{c_{ijk}\}, \mathrm{X})$. Thus $x^+ = y^+$.

Claim: X^+ is a sheaf if X is separated.

By (1), X^+ is already separated. It suffices to prove that given a cover $c_i \to c$ $(i \in \mathrm{I})$ and elements $x_i^+ \in \mathrm{X}^+(c_i)$ agreeing on overlaps, there exists an element $x^+ \in \mathrm{X}^+(c)$ restricting to $x_i^+ \in \mathrm{X}^+(c_i)$ for each $i \in \mathrm{I}$.

For each $i \in \mathrm{I}$, consider a cover $c_{ij} \to c_i$ $(j \in \mathrm{J}_i)$ such that $x_i^+|_{c_{ij}}$ is represented by a J_i-tuple of elements $x_{ij} \in \mathrm{X}(c_{ij})$. The hypothesis that the x_i^+'s agree on overlaps implies that $x_{ij}|_{c_{ij}\times_c c_{i'j'}}$ and $x_{i'j'}|_{c_{ij}\times_c c_{i'j'}}$ agree on a refinement of the cover $c_{ij} \times_c c_{i'j'} \to c_i \times_c c_{i'}$ $(j \in \mathrm{J}_i, j' \in \mathrm{J}_{i'})$. Since X is separated, this implies that the collection of elements $x_{ij} \in \mathrm{X}(c_{ij})$ $(i \in \mathrm{I}, j \in \mathrm{J}_i)$ agrees on overlaps. Hence it defines an element $x^+ \in \mathrm{X}^+(c)$.

To prove that $x^+|_{c_i} = x_i^+$ for each $i \in \mathrm{I}$, we use the fact that the $x^+|_{c_{ij}}$ is the image of $x_{ij} \in \mathrm{X}(c_{ij})$ for each $i \in \mathrm{I}$, $j \in \mathrm{J}_i$, and so is $x_i^+|_{c_{ij}}$. Thus $x^+|_{c_i}$ and x_i^+ coincide on the cover $c_{ij} \to c_i$ $(j \in \mathrm{J}_i)$. Since X^+ is separated by the first claim, we have $x^+|_{c_i} = x_i^+$. ☐

Proposition 1.3.14. *The functor* $\mathrm{X} \mapsto (\mathrm{X}^+)^+$ *provides a left adjoint of* (1.9).

Proof. It suffices to prove that for any $\mathrm{Y} \in \mathsf{Shv}(\mathcal{C})$, restriction along (1.12) defines a bijection:

$$\mathrm{Hom}_{\mathsf{PShv}(\mathcal{C})}(\mathrm{X}^+, \mathrm{Y}) \xrightarrow{\simeq} \mathrm{Hom}_{\mathsf{PShv}(\mathcal{C})}(\mathrm{X}, \mathrm{Y}). \tag{1.14}$$

The inverse map is constructed as follows: given any morphism $X \to Y$, we apply the functor $(\cdot)^+$ to obtain a morphism $X^+ \to Y^+$ and observe that Y^+ is naturally isomorphic to Y. To check that this is indeed inverse to (1.14), one may use the fact that sections of X^+ locally lift to X (*cf.* §1.3.12). □

1.3.15. We call the left adjoint of (1.9) (*cf.* Proposition 1.3.14) the *sheafification* functor:

$$\mathsf{PShv}(\mathcal{C}) \to \mathsf{Shv}(\mathcal{C}), \quad X \mapsto X^\dagger \cong (X^+)^+. \tag{1.15}$$

The unit of this adjunction coincides with the natural morphism of presheaves defined by an iteration of (1.12):

$$X \to X^+ \to (X^+)^+ \cong X^\dagger. \tag{1.16}$$

It is universal among morphisms from X to sheaves.

Remark 1.3.16. From the adjuction, it follows that (1.9) commutes with limits (which can also be verified directly) and that the colimits in $\mathsf{Shv}(\mathcal{C})$ are computed by sheafification of the colimits taken in $\mathsf{PShv}(\mathcal{C})$.

Remark 1.3.17. From the isomorphism $X^\dagger \cong (X^+)^+$, it follows that every section of $X^\dagger$ lifts locally to X, *i.e.* given $x^\dagger \in X^\dagger(c)$, there exists a cover $c_i \to c$ $(i \in I)$ such that each $x^\dagger|_{c_i}$ is the image of a section $x_i \in X(c_i)$ (*cf.* §1.3.12).

Remark 1.3.18 (Monomorphisms and epimorphisms in $\mathsf{Shv}(\mathcal{C})$)**.** Let $f : Y \to X$ be a morphism in $\mathsf{Shv}(\mathcal{C})$. Then:

(1) f is a monomorphism if and only if it is a subfunctor, *i.e.* $Y(c) \to X(c)$ is injective for every $c \in \mathcal{C}$;
(2) f is an epimorphism if and only if every section of X lifts locally to Y, *i.e.* for every $x \in X(c)$, there exists a cover $c_i \to c$ $(i \in I)$ such that each $x|_{c_i}$ is the image of a section $y_i \in Y(c_i)$; indeed, the "⇐" direction is clear and the "⇒" direction follows from considering the two maps from X to $X \sqcup_Y X$ and the description of $X \sqcup_Y X$ as the sheafification of the presheaf push-out;
(3) f is an isomorphism if and only if it is both a monomorphism and an epimorphism; indeed, it suffices to show that every section of X lifts to Y, which follows from taking a local lift using (2) and glue them using (1) and the sheaf axiom of Y.

Remark 1.3.19 (Local objects)**.** An object $c \in \mathcal{C}$ is *local* if every cover $c_i \to c$ $(i \in I)$ splits, *i.e.* there exists a morphism $c \to c_i$ for some $i \in I$ so that the composition $c \to c_i \to c$ is the identity. Then for any $X \in \mathsf{PShv}(\mathcal{C})$, the map induced from (1.16) is bijective: $X(c) \xrightarrow{\simeq} X^\dagger(c)$. Indeed, the injectivity is clear and the surjectivity follows from existence of local lifts (*cf.* Remark 1.3.17).

For the site of affine schemes $\mathsf{Sch}^{\mathrm{aff}}$ equipped with standard open covers, an object $\mathrm{Spec}(R)$ is local in the above sense if and only if R is a local ring.[5] This gives a site-theoretic meaning to the notion of local rings.

1.3.20. Finally, we note one important consequence of the fact that sheafification is given by the functor $X \mapsto (X^+)^+$.

Corollary 1.3.21. *The sheafification functor* (1.15) *commutes with finite limits.*

Proof. This is because the colimit in the formation of $\check{H}^0(c, X)$ (1.10) is filtered (*cf.* Lemma 1.3.10), and filtered colimits and finite limits in Set commute. □

[5]This is on the homework.

1.4. The category of sheaves.

1.4.1. Let $\mathcal{C}$ be a site. We have constructed an adjunction:

$$\mathsf{PShv}(\mathcal{C}) \rightleftarrows \mathsf{Shv}(\mathcal{C})$$

where the right adjoint (the forgetful functor) is fully faithful and the left adjoint preserves finite limits (*cf.* Corollary 1.3.21).

In this subsection, we explore some consequences of this structure. In fact, we shall see that $\mathsf{Shv}(\mathcal{C})$ shares many formal properties with the category Set (which is also the special case of $\mathsf{Shv}(\mathcal{C})$ for $\mathcal{C} = *$). We begin by proving that colimits in $\mathsf{Shv}(\mathcal{C})$ are "universal", *i.e.* their formation commutes with arbitrary base change.

Lemma 1.4.2 (Colimits are universal)**.** *For every (small) diagram* $\mathrm{I} \to \mathsf{Shv}(\mathcal{C})$, $i \mapsto \mathrm{X}_i$ *equipped with a morphism to the constant diagram on* $\mathrm{X} \in \mathsf{Shv}(\mathcal{C})$ *and any morphism* $\mathrm{Y} \to \mathrm{X}$, *the canonical map below is an isomorphism:*

$$\underset{i\in\mathrm{I}}{\mathrm{colim}}(\mathrm{X}_i \times_{\mathrm{X}} \mathrm{Y}) \xrightarrow{\simeq} \underset{i\in\mathrm{I}}{\mathrm{colim}}(\mathrm{X}_i) \times_{\mathrm{X}} \mathrm{Y}.$$

Proof. The analogous property holds for Set, so it holds for $\mathsf{PShv}(\mathcal{C})$. Then we use the fact that sheafification commutes with colimits (because it is a left adjoint) and finite limits (*cf.* Corollary 1.3.21). □

1.4.3. *Quotients.* Let X be a sheaf on $\mathcal{C}$. An *equivalence relation* on X is a subsheaf:

$$\mathrm{R} \subset \mathrm{X} \times \mathrm{X} \tag{1.17}$$

such that for each $c \in \mathcal{C}$, the image of $\mathrm{R}(c)$ in $\mathrm{X}(c) \times \mathrm{X}(c)$ is an equivalence relation on $\mathrm{X}(c)$. (This means that for any $x, y, z \in \mathrm{X}(c)$, we have $(x, x) \in \mathrm{R}(c)$, $(x, y) \in \mathrm{R}(c)$ implies $(y, x) \in \mathrm{R}(c)$, and $(x, y) \in \mathrm{R}(c)$, $(y, z) \in \mathrm{R}(c)$ implies $(x, z) \in \mathrm{R}(c)$.)

Given an equivalence relation $\mathrm{R} \subset \mathrm{X} \times \mathrm{X}$, its *quotient* X/R is the sheafification of the presheaf sending $c \in \mathcal{C}$ to the set of equivalence classes $\mathrm{X}(c)/\simeq_{\mathrm{R}(c)}$ of elements of $\mathrm{X}(c)$ with respect to $\mathrm{R}(c)$. In other words, X/R is the coequalizer in the category $\mathsf{Shv}(\mathcal{C})$:

$$\mathrm{R} \rightrightarrows \mathrm{X} \to \mathrm{X}/\mathrm{R}.$$

Remark 1.4.4. By Lemma 1.4.2, the formation of quotients in $\mathsf{Shv}(\mathcal{C})$ is universal, *i.e.* it commutes with base change. More precisely, given a morphism $\mathrm{Y} \to \mathrm{X}$ in $\mathsf{Shv}(\mathcal{C})$ and an equivalence relation R on Y *over* X, then for any morphism $\mathrm{X}' \to \mathrm{X}$, we obtain an equivalence relation $\mathrm{R}' \coloneqq \mathrm{R} \times_{\mathrm{X}} \mathrm{X}'$ on $\mathrm{Y}' \coloneqq \mathrm{Y} \times_{\mathrm{X}} \mathrm{X}'$ over X' and there is a canonical isomorphism:

$$\mathrm{Y}'/\mathrm{R}' \xrightarrow{\simeq} \mathrm{Y}/\mathrm{R} \times_{\mathrm{X}} \mathrm{X}'.$$

1.4.5. The natural map $\mathrm{X} \to \mathrm{X}/\mathrm{R}$ is an epimorphism and we have the following Cartesian diagram in $\mathsf{Shv}(\mathcal{C})$, as sheafification preserves finite limits (*cf.* Corollary 1.3.21):

$$\begin{array}{ccc} \mathrm{R} & \longrightarrow & \mathrm{X} \\ \downarrow & & \downarrow \\ \mathrm{X} & \longrightarrow & \mathrm{X}/\mathrm{R} \end{array} \tag{1.18}$$

The following lemma shows that every epimorphism in $\mathsf{Shv}(\mathcal{C})$ arises this way.

Lemma 1.4.6 (Epimorphisms are quotients)**.** *Let* $\mathrm{Y} \to \mathrm{X}$ *be an epimorphism in* $\mathsf{Shv}(\mathcal{C})$. *Then* X *is identified with the quotient of* Y *by the equivalence relation* $\mathrm{Y} \times_{\mathrm{X}} \mathrm{Y} \subset \mathrm{Y} \times \mathrm{Y}$.

Proof. The claim holds in Set, so it holds for $\mathsf{PShv}(\mathcal{C})$. Let $X' \subset X$ be the presheaf image of Y in X. Since the natural map $Y \times_{X'} Y \to Y \times_X Y$ is a bijection, we see that X' is the presheaf coequalizer of the two parallel morphisms in:

$$Y \times_X Y \rightrightarrows Y. \tag{1.19}$$

Since $Y \to X$ is an epimorphism, so is the induced morphism $(X')^\dagger \to X$. On the other hand, $(X')^\dagger \to X$ is a monomorphism since $X' \to X$ is a subfunctor and sheafification commutes with finite limits (*cf.* Corollary 1.3.21). Hence $(X')^\dagger \xrightarrow{\simeq} X$ (*cf.* Remark 1.3.18) and X is identified with the sheaf coequalizer of (1.19). □

1.4.7. Next, we show that certain fundamental properties of morphisms in $\mathsf{Shv}(\mathcal{C})$ are both "stable under base change" and "local on the target".

A property of morphisms P is *stable under base change* if given every morphism $f : Y \to X$ in $\mathsf{Shv}(\mathcal{C})$ satisfying P and any morphism $X' \to X$ in $\mathsf{Shv}(\mathcal{C})$, the base change $f' : Y' := Y \times_X X' \to X'$ also satisfies P.

A property of morphisms P is *local on the target* if given a morphism $f : Y \to X$ in $\mathsf{Shv}(\mathcal{C})$ and a collection of morphisms $X_i \to X$ ($i \in I$) in $\mathsf{Shv}(\mathcal{C})$ such that $\bigsqcup_{i\in I} X_i \to X$ is an epimorphism and each base change $f_i : Y_i := Y \times_X X_i \to X_i$ satisfies P, the morphism f also satisfies P.

Lemma 1.4.8. *The following properties* P *of morphisms in* $\mathsf{Shv}(\mathcal{C})$ *are stable under base change and local on the target:*

(1) $P =$ *"epimorphism";*
(2) $P =$ *"monomorphism";*
(3) $P =$ *"isomorphism".*

Proof. (1) To prove that the property $P =$ "epimorphism" is local on the target, we first observe that if each $f_i : Y_i \to X_i$ is an epimorphism, then so is the induced morphism $\bigsqcup_{i\in I} Y_i \to \bigsqcup_{i\in I} X_i$. Since the composition $\bigsqcup_{i\in I} Y_i \to \bigsqcup_{i\in I} X_i \to X$ factors through f, the former being an epimorphism implies that f is an epimorphism.

To prove that stability under base change, note that f is an epimorphism if and only if the natural map $X \sqcup_Y X \to X$ is an isomorphism. Since colimits are universal (*cf.* Lemma 1.4.2), the latter implies that $X' \sqcup_{Y'} X' \to X'$ is an isomorphism, *i.e.* f' is an epimorphism.

(2) The property $P =$ "monomorphism" is clearly stable under base change. To prove that it is local on the target, we first observe that if each $f_i : Y_i \to X_i$ is a monomorphism, then so is the induced morphism $\bigsqcup_{i\in I} Y_i \to \bigsqcup_{i\in I} X_i$. Indeed, this follows from checking that the diagonal is an isomorphism, which follows from universality of colimits (*cf.* Lemma 1.4.2).

Consider $Z \in \mathsf{Shv}(\mathcal{C})$ equipped with morphisms g_1, g_2 to Y with $f \cdot g_1 = f \cdot g_2$. Base change along $\bigsqcup_{i\in I} X_i \to X$, we obtain morphisms g_1', g_2' such that $f' \cdot g_1' = f' \cdot g_2'$:

$$\begin{array}{ccccc} Z' & \overset{g_1'}{\underset{g_2'}{\rightrightarrows}} & \bigsqcup_{i\in I} Y_i & \xrightarrow{f'} & \bigsqcup_{i\in I} X_i \\ \downarrow & & \downarrow & & \downarrow \\ Z & \overset{g_1}{\underset{g_2}{\rightrightarrows}} & Y & \xrightarrow{f} & X \end{array}$$

Assuming that f' is a monomorphism, we see that $g_1' = g_2'$. Thus g_1 and g_2 are equalized after composition with $Z' \to Z$, which we know is an epimorphism by (1). Hence $g_1 = g_2$.

(3) The fact that the property $P =$ "isomorphism" is stable under base change and local on the target follows from (1) and (2) (*cf.* Remark 1.3.18). □

1.4.9. *Gluing sheaves.* Finally, we prove that the category of sheaves itself is "local on the target", *i.e.* it can be glued along collections of morphisms $X_i \to X$ ($i \in I$) for which $\bigsqcup_{i\in I} X_i \to X$ is an epimorphism.

For each $X \in \mathsf{Shv}(\mathcal{C})$, we consider the category $\mathsf{Shv}(\mathcal{C})_{/X}$ of sheaves over X. The assignment $X \mapsto \mathsf{Shv}(\mathcal{C})_{/X}$ is functorial in the sense that it is supplied with the following data: given a morphism $f : X' \to X$, we have a functor defined by Cartesian product:

$$\mathsf{Shv}(\mathcal{C})_{/X} \to \mathsf{Shv}(\mathcal{C})_{/X'}, \quad Y \mapsto Y|_{X'} := Y \times_X X'.$$

1.4.10. Given a family of morphisms $X_i \to X$ ($i \in I$) in $\mathsf{Shv}(\mathcal{C})$, we shall define the category $\mathsf{Shv}(\mathcal{C})_{/\{X_i\}}$ of *descent data* with respect to the family $X_i \to X$ ($i \in I$). We write $X_{ij} := X_i \times_X X_j$ for each $i, j \in I$ and similarly for X_{ijk}.

An object of $\mathsf{Shv}(\mathcal{C})_{/\{X_i\}}$ consists of a sheaf $Y_i \in \mathsf{Shv}(\mathcal{C})_{/X_i}$ for each $i \in I$, together with an isomorphism in $\mathsf{Shv}(\mathcal{C})_{/X_{ij}}$:

$$\varphi_{ij} : Y_i|_{X_{ij}} \xrightarrow{\simeq} Y_j|_{X_{ij}} \tag{1.20}$$

for each $i, j \in I$, making the following diagram in $\mathsf{Shv}(\mathcal{C})_{/X_{ijk}}$ commute for each $i, j, k \in I$:

$$\begin{array}{ccc} Y_i|_{X_{ijk}} & \xrightarrow{p_{12}^*\varphi_{ij}} & Y_j|_{X_{ijk}} \\ & \searrow^{p_{13}^*\varphi_{ik}} & \downarrow{\scriptstyle p_{23}^*\varphi_{jk}} \\ & & Y_k|_{X_{ijk}} \end{array} \tag{1.21}$$

Here, $p_{12} : X_{ijk} \to X_{ij}$ is the projection onto the first two factors, and similarly for p_{23}, p_{13}. The commutativity of (1.21) is called the *cocycle condition* satisfied by the collection of isomorphisms φ_{ij}.

A morphism $(\{Y_i\}, \{\varphi_{ij}\}) \to (\{Z_i\}, \{\psi_{ij}\})$ in $\mathsf{Shv}(\mathcal{C})_{/\{X_i\}}$ is a family of morphisms $Y_i \to Z_i$ in $\mathsf{Shv}(\mathcal{C})_{/X_i}$ for each $i \in I$, making the following diagram commute for each $i, j \in I$:

$$\begin{array}{ccc} Y_i|_{X_{ij}} & \xrightarrow{\varphi_{ij}} & Y_j|_{X_{ij}} \\ \downarrow & & \downarrow \\ Z_i|_{X_{ij}} & \xrightarrow{\psi_{ij}} & Z_j|_{X_{ij}} \end{array}$$

Remark 1.4.11. Note that if X_{ij} is the *empty sheaf*, *i.e.* the initial object of $\mathsf{Shv}(\mathcal{C})$, then φ_{ij} (1.20) is the unique isomorphism $\varnothing \cong \varnothing$. Indeed, this is because the base change of any morphism $Y \to Z$ in $\mathsf{Shv}(\mathcal{C})$ along $\varnothing \to Z$ is isomorphic to $\varnothing$, as one verifies for $\mathsf{PShv}(\mathcal{C})$ and uses the fact that sheafification commutes with finite limits (*cf.* Corollary 1.3.21).

In particular, if $X_{ij} \cong \varnothing$ for each $i, j \in I$, then the category of descent data $\mathsf{Shv}(\mathcal{C})_{/\{X_i\}}$ is equivalent to $\prod_{i\in I} \mathsf{Shv}(\mathcal{C})_{/X_i}$.

1.4.12. Given a family of morphisms $X_i \to X$ ($i \in I$) in $\mathsf{Shv}(\mathcal{C})$, there is a functor:

$$\mathsf{Shv}(\mathcal{C})_{/X} \to \mathsf{Shv}(\mathcal{C})_{/\{X_i\}} \tag{1.22}$$

sending Y to the pair $(\{Y_i\}, \{\varphi_{ij}\})$ where $Y_i := Y|_{X_i}$ for each $i \in I$, and $\varphi_{ij} : Y_i|_{X_{ij}} \xrightarrow{\simeq} Y_j|_{X_{ij}}$ is the canonical isomorphism obtained from identifying both sides with $Y|_{X_{ij}}$.

Proposition 1.4.13 (Descent of Shv). *Given a family of morphisms* $X_i \to X$ *(*$i \in I$*) in* $\mathsf{Shv}(\mathcal{C})$ *such that* $\bigsqcup_{i\in I} X_i \to X$ *is an epimorphism, the induced functor* (1.22) *is an equivalence of categories.*

Proof. We shall construct the functor inverse to (1.22) as follows.

Given a descent datum $(\{Y_i\}, \{\varphi_{ij}\})$, we consider the pair of commutative squares:

$$\begin{array}{ccc} \bigsqcup_{i,j\in I} Y_i|_{X_{ij}} & \overset{p}{\underset{p\cdot\varphi_{ij}}{\rightrightarrows}} & \bigsqcup_{i\in I} Y_i \\ \downarrow & & \downarrow \\ \bigsqcup_{i,j\in I} X_{ij} & \overset{p_1}{\underset{p_2}{\rightrightarrows}} & \bigsqcup_{i\in I} X_i \end{array} \tag{1.23}$$

where the two bottom horizontal morphisms are defined by projections $p_1 : X_{ij} \to X_i$, $p_2 : X_{ij} \to X_j$ onto the first, respectively the second factor, and the two top horizontal morphisms are defined by the projection $p : Y_i|_{X_{ij}} \to Y_i$, respectively from the composition of $\varphi_{ij} : Y_i|_{X_{ij}} \xrightarrow{\simeq} Y_j|_{X_{ij}}$ with the projection $p : Y_j|_{X_{ij}} \to Y_j$.

It follows from the cocycle condition (1.21) that $R := \bigsqcup_{i,j\in I} Y_i|_{X_{ij}}$ is an equivalence relation on $\widetilde{Y} := \bigsqcup_{i\in I} Y_i$ with respect to the two morphisms in (1.23). We define $Y := \widetilde{Y}/R$. It admits a natural morphism to X, since the latter is identified with the quotient of $\bigsqcup_{i\in I} X_i$ by $\bigsqcup_{i,j\in I} X_{ij}$ (*cf.* Lemma 1.4.6). Thus we may view Y as an object of $\mathsf{Shv}(\mathcal{C})_{/X}$.

We claim that the assignment of Y to $(\{Y_i\}, \{\varphi_{ij}\})$ provides an inverse to the functor (1.22). The fact that any $Y \in \mathsf{Shv}(\mathcal{C})_{/X}$ is recovered from the composition follows from the fact that $\bigsqcup_{i\in I} Y_i \to Y$ is an epimorphism (*cf.* Lemma 1.4.8), hence a quotient 1.4.6). The fact that any descent datum $(\{Y_i\}, \{\varphi_{ij}\})$ is recovered from the composition will follow, once we show that the induced square:

$$\begin{array}{ccc} \bigsqcup_{i\in I} Y_i & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \bigsqcup_{i\in I} X_i & \longrightarrow & X \end{array} \tag{1.24}$$

is Cartesian. This statement can be checked upon base change along the epimorphism $\bigsqcup_{i\in I} X_i \to X$ (*cf.* Lemma 1.4.8). However, once we perform this base change, (1.24) becomes the top square in (1.23), which we know to be Cartesian. □

1.5. Open immersions.

1.5.1. Let us now return to the category Shv of Zariski sheaves, namely sheaves on the site $\mathcal{C} := \mathsf{Sch}^{\mathrm{aff}}$ of affine schemes whose covers are the standard open covers.

In this subsection, we shall define the notion of "open immersions" in Shv. We will first define this notion when the target is an affine scheme, and then generalize it to arbitrary morphisms in Shv using base change.

1.5.2. A morphism $f : Y \to \mathrm{Spec}(A)$ in Shv ($A \in \mathsf{Ring}$) is called an *open immersion* if it is a monomorphism and there exists a family of standard opens $\mathrm{Spec}(A_i) \to \mathrm{Spec}(A)$ ($i \in I$) factoring through Y, such that any field-valued point of Y factors through $\mathrm{Spec}(A_i)$ for some $i \in I$.

Note that if f is an open immersion and $A \to A'$ is a ring map, then the base change $f' : Y' := Y \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A') \to \mathrm{Spec}(A')$ is also an open immersion. Indeed, this follows from inspecting the Cartesian squares below for $A'_i := A' \otimes_A A_i$:

$$\begin{array}{ccccc} \mathrm{Spec}(A'_i) & \longrightarrow & Y' & \xrightarrow{f'} & \mathrm{Spec}(A') \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}(A_i) & \longrightarrow & Y & \xrightarrow{f} & \mathrm{Spec}(A) \end{array}$$

We define a morphism $f : \mathrm{Y} \to \mathrm{X}$ in Shv to be an *open immersion* if for any R-point $x : \mathrm{Spec}(\mathrm{R}) \to \mathrm{X}$ ($\mathrm{R} \in \mathsf{Ring}$), the base change $\mathrm{Y} \times_{\mathrm{X}} \mathrm{Spec}(\mathrm{R}) \to \mathrm{Spec}(\mathrm{R})$ is an open immersion in the sense above.

This generalizes the notion of open immersions when $\mathrm{X} = \mathrm{Spec}(\mathrm{A})$ is affine, oweing to the base change property observed above. It follows immediately from this definition that open immersions are stable under base change (*cf.* §1.4.7).

Lemma 1.5.3. *Open immersions in* Shv *are closed under compositions.*

Proof. Let $\mathrm{Z} \to \mathrm{Y}$ and $\mathrm{Y} \to \mathrm{X}$ be open immersions of Zariski sheaves. We want to prove that the composition $\mathrm{Z} \to \mathrm{Y} \to \mathrm{X}$ is still an open immersion. By base change, we may assume that $\mathrm{X} = \mathrm{Spec}(\mathrm{A})$ is an affine scheme.

Choose a family of standard opens $\mathrm{Spec}(\mathrm{A}_i) \to \mathrm{Spec}(\mathrm{A})$ ($i \in \mathrm{I}$) factoring through Y such that any field-valued point of Y factors through some $\mathrm{Spec}(\mathrm{A}_i)$. Base change $\mathrm{Z} \to \mathrm{Y}$ to each $\mathrm{Spec}(\mathrm{A}_i)$, we may also find a family of standard opens $\mathrm{Spec}(\mathrm{A}_{ij}) \to \mathrm{Spec}(\mathrm{A})$ ($j \in \mathrm{J}_i$) factoring through $\mathrm{Z} \times_{\mathrm{Y}} \mathrm{Spec}(\mathrm{A}_i)$ such that any field-valued point of the latter factors through some $\mathrm{Spec}(\mathrm{A}_{ij})$. But the family of morphisms:

$$\mathrm{Spec}(\mathrm{A}_{ij}) \to \mathrm{Spec}(\mathrm{A}) \quad (i \in \mathrm{I}, j \in \mathrm{J}_i)$$

has the desired property with respect to Z. □

Remark 1.5.4. Open immersions enjoy the following permanence property. Given a commutative diagram in Shv:

$$\begin{array}{ccc} \mathrm{Y} & \xrightarrow{f'} & \mathrm{X}' \\ & \searrow^{f} & \downarrow{\scriptstyle i} \\ & & \mathrm{X} \end{array}$$

where i is a monomorphism, if f is an open immersion, then so is f'. Indeed, this is because given an R-point of X' ($\mathrm{R} \in \mathsf{Ring}$), we have an isomorphism $\mathrm{Y} \times_{\mathrm{X}'} \mathrm{Spec}(\mathrm{R}) \xrightarrow{\simeq} \mathrm{Y} \times_{\mathrm{X}} \mathrm{Spec}(\mathrm{R})$ of Zariski sheaves over $\mathrm{Spec}(\mathrm{R})$.

1.5.5. *Open covers.* A family of open immersions $\mathrm{X}_i \to \mathrm{X}$ ($i \in \mathrm{I}$) in Shv is called an *open cover* if every field-valued point of X factors through X_i for some $i \in \mathrm{I}$.

For example, a standard open cover of an affine scheme is an open cover. By definition, every open immersion into an affine scheme admits an open cover by affine schemes. The following Lemma shows that every open cover of an affine scheme can be "refined" by a standard open cover.

Lemma 1.5.6. *Let* $\mathrm{A} \in \mathsf{Ring}$ *and* $\mathrm{X}_i \to \mathrm{Spec}(\mathrm{A})$ *(*$i \in \mathrm{I}$*) be an open cover in* Shv. *Then* $\mathrm{X}_i \to \mathrm{Spec}(\mathrm{A})$ *(*$i \in \mathrm{I}$*) admits a refinement by a standard open cover,* i.e. *there is a standard open cover* $\mathrm{Spec}(\mathrm{A}_j) \to \mathrm{Spec}(\mathrm{A})$ *(*$j \in \mathrm{J}$*) and a map* $\varphi : \mathrm{J} \to \mathrm{I}$ *such that each* $\mathrm{Spec}(\mathrm{A}_j) \to \mathrm{Spec}(\mathrm{A})$ *factors through* $\mathrm{X}_{\varphi(j)}$.

Proof. We may find standard opens $\mathrm{Spec}(\mathrm{A}_{ij}) \to \mathrm{Spec}(\mathrm{A})$ ($i \in \mathrm{I}$, $j \in \mathrm{J}_i$) such that each $\mathrm{Spec}(\mathrm{A}_{ij}) \to \mathrm{Spec}(\mathrm{A})$ factors through X_i and any field-valued point of X_i factors through some $\mathrm{Spec}(\mathrm{A}_{ij})$. It follows that every field-valued point of $\mathrm{Spec}(\mathrm{A})$ factors through some $\mathrm{Spec}(\mathrm{A}_{ij})$. One may then pass to a finite subset of $\bigsqcup_{i \in \mathrm{I}} \mathrm{J}_i$ to obtain the desired standard open cover of $\mathrm{Spec}(\mathrm{A})$ (*cf.* Remark 1.2.3). □

Corollary 1.5.7. *Let* $\mathrm{X}_i \to \mathrm{X}$ *(*$i \in \mathrm{I}$*) be an open cover in* Shv. *Then the induced morphism* $\bigsqcup_{i \in \mathrm{I}} \mathrm{X}_i \to \mathrm{X}$ *in* Shv *is an epimorphism.*

Proof. For any R-point x of X, the base change $\mathrm{Spec}(\mathrm{R})\times_{\mathrm{X}}\mathrm{X}_i\to\mathrm{Spec}(\mathrm{R})$ ($i\in\mathrm{I}$) is open cover, and any refinement of it by a standard open cover (which exists thanks to Lemma 1.5.6) provides local lifts of x to $\bigsqcup_{i\in\mathrm{I}}\mathrm{X}_i$. □

1.5.8. In the last part of this subsection, we shall prove that open immersions are local on the target (*cf.* §1.4.7). This requires some preliminary discussions.

Lemma 1.5.9. *Given an isomorphism* $\mathrm{X}_1\sqcup\mathrm{X}_2\cong\mathrm{Spec}(\mathrm{A})$ *for* $\mathrm{X}_1,\mathrm{X}_2\in\mathsf{Shv}$ *and* $\mathrm{A}\in\mathsf{Ring}$, X_1 *and* X_2 *are affine schemes and their inclusions into* $\mathrm{Spec}(\mathrm{A})$ *are open immersions.*

Proof. We will prove that X_1 is an affine scheme and $\mathrm{X}_1\to\mathrm{Spec}(\mathrm{A})$ is an open immersion.

Consider the isomorphism of rings induced from $\mathrm{X}_1\sqcup\mathrm{X}_2\cong\mathrm{Spec}(\mathrm{A})$:

$$\mathrm{A}\xrightarrow{\simeq}\mathrm{Hom}(\mathrm{Spec}(\mathrm{A}),\mathbb{A}^1_{\mathbf{Z}})\xrightarrow{\simeq}\mathrm{Hom}(\mathrm{X}_1,\mathbb{A}^1_{\mathbf{Z}})\times\mathrm{Hom}(\mathrm{X}_2,\mathbb{A}^1_{\mathbf{Z}}),$$

where the ring structures on the Hom-sets[6] are induced from the ring structure on $\mathbb{A}^1_{\mathbf{Z}}$ (*cf.* Example 1.1.5). For $i=1,2$, we write $\mathrm{A}_i:=\mathrm{Hom}(\mathrm{X}_i,\mathbb{A}^1_{\mathbf{Z}})$ and $e_1:=(1,0)$, $e_2:=(0,1)$ for the elements of A under the isomorphism $\mathrm{A}\cong\mathrm{A}_1\times\mathrm{A}_2$. Then $\mathrm{Spec}(\mathrm{A}_1)\to\mathrm{Spec}(\mathrm{A})$ is the standard open defined by localization at e_1. We shall prove that X_1 coincides with $\mathrm{Spec}(\mathrm{A}_1)$ as subfunctors of $\mathrm{Spec}(\mathrm{A})$.

Note that $\mathrm{X}_1\to\mathrm{Spec}(\mathrm{A})$ factors through $\mathrm{Spec}(\mathrm{A}_1)$ via the natural map $\mathrm{X}_1\to\mathrm{Spec}(\mathrm{A}_1)$. For the opposite inclusion, it suffices to prove that $\mathrm{Spec}(\mathrm{A}_1)\times_{\mathrm{Spec}(\mathrm{A})}\mathrm{X}_2$ is the empty sheaf, by universality of colimits (*cf.* Lemma 1.4.2). Now, an R-point ($\mathrm{R}\in\mathsf{Ring}$) of this fiber product defines a ring map $\mathrm{A}\to\mathrm{R}$ such that the image of e_1 is both invertible and zero. Such a map only exists if $\mathrm{R}\cong 0$, where it is unique. □

Lemma 1.5.10. *Given a family of morphisms* $\mathrm{X}_i\to\mathrm{X}$ *(*$i\in\mathrm{I}$*) in* Shv *such that* $\bigsqcup_{i\in\mathrm{I}}\mathrm{X}_i\to\mathrm{X}$ *is an epimorphism and an* R*-point* $x:\mathrm{Spec}(\mathrm{R})\to\mathrm{X}$ *(*$\mathrm{R}\in\mathsf{Ring}$*), there exists a standard open cover* $\mathrm{Spec}(\mathrm{R}_j)\to\mathrm{Spec}(\mathrm{R})$ *(*$j\in\mathrm{J}$*) such that for each* $j\in\mathrm{J}$, $x|_{\mathrm{R}_j}$ *lifts to* X_i *for some* $i\in\mathrm{I}$*:*

$$\begin{array}{ccc}\mathrm{Spec}(\mathrm{R}_j) & \to & \mathrm{X}_i\\ \downarrow & & \downarrow\\ \mathrm{Spec}(\mathrm{R}) & \xrightarrow{x} & \mathrm{X}\end{array}$$

Proof. Since $\widetilde{\mathrm{X}}:=\bigsqcup_{i\in\mathrm{I}}\mathrm{X}_i\to\mathrm{X}$ is an epimorphism, we may first find a standard open cover $\mathrm{Spec}(\mathrm{R}_k)\to\mathrm{Spec}(\mathrm{R})$ ($k\in\mathrm{K}$) such that each $x|_{\mathrm{R}_k}$ lifts to $\widetilde{\mathrm{X}}$. Using the morphism $\mathrm{Spec}(\mathrm{R}_k)\to\widetilde{\mathrm{X}}$ and universality of colimits (*cf.* Lemma 1.4.2), we find:

$$\mathrm{Spec}(\mathrm{R}_k)\cong\bigsqcup_{i\in\mathrm{I}}\mathrm{Spec}(\mathrm{R}_k)\times_{\widetilde{\mathrm{X}}}\mathrm{X}_i,\tag{1.25}$$

where the induced morphism $\mathrm{Spec}(\mathrm{R}_k)\times_{\widetilde{\mathrm{X}}}\mathrm{X}_i\to\widetilde{\mathrm{X}}$ factors through the projection onto X_i.

By Lemma 1.5.9, each term in (1.25) is an affine scheme $\mathrm{Spec}(\mathrm{R}_{ik})$ openly immersed in $\mathrm{Spec}(\mathrm{R}_k)$. Hence the collection $\mathrm{Spec}(\mathrm{R}_{ik})\to\mathrm{Spec}(\mathrm{R})$ ($i\in\mathrm{I}$, $k\in\mathrm{K}$) is an open cover such that $x|_{\mathrm{R}_{ik}}$ lifts to X_i. Taking a refinement by a standard open cover (*cf.* Corollary 1.5.7), we obtain the desired standard open cover $\mathrm{Spec}(\mathrm{R}_j)\to\mathrm{Spec}(\mathrm{R})$ ($j\in\mathrm{J}$). □

Corollary 1.5.11. *The property of being an open immersion is local on the target.*

Proof. Given a family of morphisms $\mathrm{X}_i\to\mathrm{X}$ ($i\in\mathrm{I}$) and a morphism $f:\mathrm{Y}\to\mathrm{X}$ in Shv, such that $\bigsqcup_{i\in\mathrm{I}}\mathrm{X}_i\to\mathrm{X}$ is an epimorphism and each base change $f_i:\mathrm{Y}_i:=\mathrm{Y}\times_{\mathrm{X}}\mathrm{X}_i\to\mathrm{X}_i$ is an open immersion, we need to prove that f is an open immersion.

[6] There are no size issues: $\mathrm{Hom}(\mathrm{X}_i,\mathbb{A}^1_{\mathbf{Z}})$ belongs to Set because it can be realized as a subset of A ($i=1,2$).

Since epimorphisms and open immersions are stable under base change, we reduce to the case where $X \cong \mathrm{Spec}(A)$ is affine. By Lemma 1.5.10, we may find a standard open cover of $\mathrm{Spec}(A)$ consisting of standard opens which factor through X_i for some $i \in I$. By base change f_i to these standard opens, we further reduce to the case where $X_i \cong \mathrm{Spec}(A_i) \to X \cong \mathrm{Spec}(A)$ ($i \in I$) is a standard open cover.

Since being a monomorphism is local on the target (*cf.* Lemma 1.4.8), it remains to construct a family of standard opens of $\mathrm{Spec}(A)$ covering Y. However, this can be done by choosing a family of standard opens of $\mathrm{Spec}(A_i)$ covering Y_i for each $i \in I$. □

1.6. Schemes.

1.6.1. An object $X \in \mathsf{Shv}$ is called a *scheme* if it admits an open cover $X_i \to X$ ($i \in I$) such that each X_i is an affine scheme. In making this definition, we implicitly invoked the fact that every affine scheme belongs to Shv (*cf.* Proposition 1.2.7).

Write Sch for the full subcategory of Shv consisting of schemes.

Lemma 1.6.2. *If $f : Y \to X$ is an open immersion in* Shv *and X is a scheme, then Y is a scheme. (It is called an* open subscheme *of X.*[7]*)*

Proof. If X is an affine scheme, this follows from the observation of §1.5.5.

If X is any scheme, we consider an open cover $X_i \to X$ ($i \in I$) by affine schemes and form $Y_i := Y \times_X X_i$. Since open immersions are stable under base change, $Y_i \to X_i$ is an open immersion, so Y_i is a scheme. We have thus found an open cover $Y_i \to Y$ ($i \in I$) of Y by schemes. We conclude by passing to an open cover of each Y_i by affine schemes. □

Example 1.6.3 (Gluing along open immersions)**.** A simple way of constructing new schemes from old ones is by "gluing" along open immersions. More precisely, given open immersions $U \to X_1$ and $U \to X_2$ in Sch, we may form the push-out $X := X_1 \sqcup_U X_2$ in Shv. We claim that X is a scheme.

Indeed, the natural morphism $X_1 \sqcup X_2 \to X$ in Shv is an epimorphism, so it suffices to prove that $X_1 \to X$ and $X_2 \to X$ are open immersions. (Then open covers of X_1 and X_2 by affine schemes will form an open cover of X by affine schemes.) Let us prove that $X_1 \to X$ is an open immersion. By Corollary 1.5.11, it is enough to that it is an open immersion after base change along $X_1 \to X$ and $X_2 \to X$. However, these two base changes are given by the identity on X_1, respectively the open immersion $U \to X_2$.

1.6.4. We have constructed a chain of fully faithful functors:

$$\mathsf{Sch}^{\mathrm{aff}} \to \mathsf{Sch} \to \mathsf{Shv} \to \mathsf{PShv}.$$

We have already seen that the full subcategories $\mathsf{Sch}^{\mathrm{aff}}$ and Shv of PShv are closed under limits. The situation with Sch is slightly less pleasant.

Proposition 1.6.5. *The full subcategory* Sch *of* Shv *is closed under finite limits.*

Proof. Since Sch contains the terminal object $\mathrm{Spec}(\mathbf{Z})$ of Shv, it suffices to show that Sch contains fiber products. In other words, given a Cartesian diagram in Shv:

$$\begin{array}{ccc} X \times_S Y & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & S \end{array}$$

where X, Y, S are schemes, we need to show that $X \times_S Y$ is also a scheme.

[7]Open subschemes of affine schemes may not be affine.

If X, Y, S are all affine, this holds because the full subcategory $\mathsf{Sch}^{\mathrm{aff}}$ of Shv is closed under fiber products, which are computed by tensor products of rings (*cf.* Example 1.1.6).

If X, Y, S are schemes, it suffices to construct an open cover of $X \times_S Y$ by affine schemes. Observe that for any open immersions $U \to X$, $T \to S$, and $V \to Y$ such that $U \to X \to S$ and $V \to Y \to S$ factor through T, we have a monomorphism:

$$U \times_T V \to X \times_S Y.$$

It is an open immersion because it is the fiber product of $U \times_X (X \times_S Y)$ with $(X \times_S Y) \times_Y V$ over $X \times_S Y$, where both morphisms are open immersions.

Now, we take an open cover $S_i \to S$ $(i \in I)$ of S by affine schemes, as well as open covers $X_{ij} \to X \times_S S_i$ $(j \in J_i)$, $Y_{ik} \to Y \times_S S_i$ $(k \in K_i)$ by affine schemes, which exist because $X \times_S S_i$, $Y \times_S S_i$ are schemes (*cf.* Lemma 1.6.2). The resulting family:

$$X_{ij} \times_{S_i} Y_{ik} \to X \times_S Y \quad (i \in I,\ j \in J_i,\ k \in K_i)$$

is then an open cover of $X \times_S Y$ by affine schemes. □

Remark 1.6.6. For a pair of open immersions $U_1 \to X$, $U_2 \to X$, we will often write $U_1 \cap U_2$ for the fiber product $U_1 \times_X U_2$ and call it the *intersection* of the two open subschemes. Note that this is indeed the intersection as subfunctors of X.

Remark 1.6.7. The full subcategory Sch of PShv does not contain infinite products. In fact, the infinite product ot the projective line is not a scheme (*cf.* [Sta18, 078E]). This fact does not seem to affect life very much though.

1.6.8. *Schemes as colimits of affines.* We will now make precise the idea that schemes are "glued" from affine schemes in the category Shv.

Let X be a scheme and $X_i \to X$ $(i \in I)$ be an open cover by affine schemes. Then we obtain an epimorphism $Y \to X$ in Shv with $Y := \bigsqcup_{i \in I} X_i$. Note that $Y \times_X Y$ is canonically identified with $\bigsqcup_{i,j \in I} X_i \times_X X_j$ (*cf.* Lemma 1.4.2), so we obtain a coequalizer diagram in Shv (*cf.* Lemma 1.4.6):

$$\bigsqcup_{i,j \in I} X_i \times_X X_j \rightrightarrows \bigsqcup_{i \in I} X_i \to X \tag{1.26}$$

On the other hand, Proposition 1.6.5 shows that each $X_i \times_X X_j$ is again a scheme. Hence we may find an open cover $X_{ijk} \to X_i \times_X X_j$ $(k \in K_{ij})$ by affine schemes for each $i, j \in I$. Then the following diagram is again a coequalizer in Shv:

$$\bigsqcup_{\substack{i,j \in I \\ k \in K_{ij}}} X_{ijk} \rightrightarrows \bigsqcup_{i \in I} X_i \to X \tag{1.27}$$

This gives a presentation of X as an iterated colimit (in Shv) of affine schemes.

Corollary 1.6.9. *The functor* $\mathsf{Sch}^{\mathrm{aff}} \to \mathsf{Sch}$ *admits a left adjoint.*

Proof. Given a scheme X, the Hom-set[8] $\mathrm{Hom}(X, \mathbb{A}^1_{\mathbf{Z}})$ has a ring structure induced from the ring structure on $\mathbb{A}^1_{\mathbf{Z}}$ (*cf.* Example 1.1.5). Note that if X is an affine scheme Spec(A), then the ring $\mathrm{Hom}(X, \mathbb{A}^1_{\mathbf{Z}})$ is identified with A.

We argue that the functor sending X to $\mathrm{Spec}(\mathrm{Hom}(X, \mathbb{A}^1_{\mathbf{Z}}))$ induces a bijection:

$$\mathrm{Hom}(Y, \mathrm{Spec}(A)) \xrightarrow{\simeq} \mathrm{Hom}(\mathrm{Spec}(\mathrm{Hom}(Y, \mathbb{A}^1_{\mathbf{Z}})), \mathrm{Spec}(A)) \tag{1.28}$$

for every $Y \in \mathsf{Sch}$ and $A \in \mathsf{Ring}$. This will prove that the functor $\mathrm{Spec}(\mathrm{Hom}(\cdot, \mathbb{A}^1_{\mathbf{Z}}))$ provides a left adjoint to the inclusion $\mathsf{Sch}^{\mathrm{aff}} \to \mathsf{Sch}$.

[8]This is an object of Set since it can be realized as a subset of $\prod_{i \in I} \mathrm{Hom}(X_i, \mathbb{A}^1_{\mathbf{Z}})$ for an open cover of X by affine scheme X_i $(i \in I)$.

To prove the bijectivity of (1.28), we may present Y as a coproduct of affine schemes as in (1.27) and reduce to the case where Y is itself affine, where the assertion is clear. □

Remark 1.6.10. Informally, the presentation (1.27) says that every scheme is a quotient of a disjoint union of affine schemes by Zariski (*i.e.* disjoint union of open immersions) equivalence relations.

This is analogous to the fact that every topological manifold is the quotient (in Top) of a disjoint union of Euclidean spaces by equivalence relations defined by open subspaces.

Remark 1.6.11. In the colimit presentation (1.27), it is possible to choose X_{ijk} such that the morphisms $X_{ijk} \to X_i$, $X_{ijk} \to X_j$ are standard opens for all $i, j \in I$, $k \in K_{ij}$. This is due to the following fact (the "affine communication lemma"): *given open immersions* $\mathrm{Spec}(A) \to X$, $\mathrm{Spec}(B) \to X$*, the scheme* $\mathrm{Spec}(A) \times_X \mathrm{Spec}(B)$ *admits an open cover by affine schemes which are standard opens in both* $\mathrm{Spec}(A)$ *and* $\mathrm{Spec}(B)$*.*

To prove this assertion, let us consider an open cover of $\mathrm{Spec}(A) \times_X \mathrm{Spec}(B)$ by $\mathrm{Spec}(A_i)$ ($i \in I$) where each $\mathrm{Spec}(A_i) \to \mathrm{Spec}(A)$ is a standard open. Similarly, $\mathrm{Spec}(A) \times_X \mathrm{Spec}(B)$ admits an open cover by $\mathrm{Spec}(B_j)$ ($j \in J$) where each $\mathrm{Spec}(B_j) \to \mathrm{Spec}(B)$ is a standard open. The we claim that $\mathrm{Spec}(A_i) \times_X \mathrm{Spec}(B_j)$ is a standard open in both $\mathrm{Spec}(A)$ and $\mathrm{Spec}(B)$. Indeed, from the following two Cartesian squares of schemes:

$$\begin{array}{ccccc} \mathrm{Spec}(A_i) \times_X \mathrm{Spec}(B_j) & \longrightarrow & \mathrm{Spec}(A_i) & \xrightarrow{\simeq} & \mathrm{Spec}(A_i) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}(B_j) & \longrightarrow & \mathrm{Spec}(A) \times_X \mathrm{Spec}(B) & \to & \mathrm{Spec}(A) \end{array}$$

we deduce that $\mathrm{Spec}(A_i) \times_X \mathrm{Spec}(B_j) \to \mathrm{Spec}(B_j)$ is a standard open. Hence the composition $\mathrm{Spec}(A_i) \times_X \mathrm{Spec}(B_j) \to \mathrm{Spec}(B_j) \to \mathrm{Spec}(B)$ is a standard open. The same holds for the composition $\mathrm{Spec}(A_i) \times_X \mathrm{Spec}(B_j) \to \mathrm{Spec}(A_i) \to \mathrm{Spec}(A)$. Finally, it is clear that the collection $\mathrm{Spec}(A_i) \times_X \mathrm{Spec}(B_j) \to \mathrm{Spec}(A) \times_X \mathrm{Spec}(B)$ ($i \in I$, $j \in J$) is an open cover.

1.6.12. The presentation (1.27) of a scheme X has the slight disadvantage that it is not "canonical", *i.e.* it depends on the choice of an open cover by affine schemes as well as the choice of such for each double overlap. We shall now present X as a colimit of affine schemes in a canonical manner.

By formal nonsense, every object $X \in \mathsf{Shv}$ is a colimit of representable sheaves. Namely, if we consider the category $(\mathsf{Sch}^{\mathrm{aff}})_{/X}$ of affine schemes over X, then X is identified with the colimit taken over it:[9]

$$\operatorname*{colim}_{\mathrm{Spec}(R) \to X} \mathrm{Spec}(R) \xrightarrow{\simeq} X.$$

If X is a scheme, then a much more economical presentation is available.

1.6.13. *The standard Zariski site.* Let X be a scheme. Denote by X_{SZar} the subcategory of $(\mathsf{Sch}^{\mathrm{aff}})_{/X}$ whose objects are open immersions $\mathrm{Spec}(R) \to X$ and whose morphisms are standard opens $\mathrm{Spec}(R') \to \mathrm{Spec}(R)$ over X.

Note that X_{SZar} has the structure of a site, with covers given by the standard open covers. Every Zariski sheaf over $(\mathsf{Sch}^{\mathrm{aff}})_{/X}$ thus restricts to a sheaf over X_{SZar}.

Corollary 1.6.14. *Let* X *be a scheme. The canonical map below is an isomorphism in* Shv*:*

$$\operatorname*{colim}_{\substack{\mathrm{Spec}(R) \to X \\ \text{in } X_{\mathrm{SZar}}}} \mathrm{Spec}(R) \xrightarrow{\simeq} X. \tag{1.29}$$

[9]Caution: this colimit is indexed by a "large category", *i.e.* its objects do not form an object of Set. Consequently, limits/colimits in Set indexed by this category may not exist.

Proof. Consider the coequalizer presentation (1.27) of X where we take I to be the set of all open immersions $X_i \to X$ where X_i is affine, and K_{ij} to be the set of all open immersions $X_{ijk} \to X_{ij}$ where $X_{ijk} \to X_i$, $X_{ijk} \to X_j$ are both standard opens (*cf.* Remark 1.6.11). Then the coequalizer of the morphisms:

$$\bigsqcup_{\substack{i,j\in I \\ k\in K_{ij}}} X_{ijk} \rightrightarrows \bigsqcup_{i\in I} X_i$$

is identified with the colimit in (1.29). □

1.7. The "qcqs" condition.

1.7.1. There is a condition on schemes that allows one to replace the infinite colimit in (1.27) by a finite one. It has a code name "qcqs" and stands for "quasi-compact and quasi-separated". We shall explain in this subsection what these terms mean.

We shall work in the category Sch of schemes and make heavy use of the fact that this category admits finite limits (*cf.* Proposition 1.6.5). A property P of morphisms of schemes is *stable under base change* if given a Cartesian square in Sch:

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ \downarrow & & \downarrow \\ Y & \xrightarrow{f} & X \end{array} \tag{1.30}$$

then f satisfies P implies that f' satisfies P.

A property P of morphisms of schemes is *local on the target* if a morphism $f : Y \to X$ in Sch and a family of morphisms $X_i \to X$ ($i \in I$) in Sch such that $\bigsqcup_{i\in I} X_i \to X$ is an epimorphism of Zariski sheaves and each base change $f_i : Y \times_X X_i \to X_i$ satisfies P, then so does f.

1.7.2. A scheme X is called *quasi-compact* if every open cover of X (*cf.* §1.5.5) has a finite subcover. Note that given a finite collection X_i ($i \in I$) of quasi-compact schemes, their coproduct $\bigsqcup_{i\in I} X_i$ in Shv remains a quasi-compact scheme.

Lemma 1.5.6 implies that affine schemes are quasi-compact.

Lemma 1.7.3. *Let* X *be a scheme. The following conditions are equivalent:*

(1) X *is quasi-compact;*
(2) X *admits a finite open cover by affine schemes.*

Proof. (1) ⇒ (2). Take an arbitrary open cover of X by affine schemes. Since X is quasi-compact, we find a finite subcover.

(2) ⇒ (1). Let $X_i \to X$ ($i \in I$) be a finite open cover by affine schemes. Let $U_j \to X$ ($j \in J$) be an arbitrary open cover. For each $i \in I$, the collection $X_i \times_X U_j \to X_i$ ($j \in J$) is an open cover of X_i. Since X_i is affine, hence quasi-compact (*cf.* Lemma 1.5.6), we find an open cover $X_i \times_X U_j \to X_i$ ($j \in J_i$) for a finite subset $J_i \subset J$. Since I is finite, $\bigcup_i J_i$ is also finite, so $U_j \to X$ ($j \in \bigcup_i J_i$) is a finite subcover. □

1.7.4. We now define a relative version of quasi-compactness.

A morphism $f : Y \to X$ of schemes is called *quasi-compact* if for every R-point of X ($R \in \mathsf{Ring}$), the fiber product $Y \times_X \mathrm{Spec}(R)$ is quasi-compact. By definition, the property P = "quasi-compact" is stable under base change. The fact that it is closed under composition follows from Lemma 1.7.3.

Note that the following assertions about a scheme X are equivalent:

(1) X is quasi-compact;
(2) some morphism from X to an affine scheme is quasi-compact;

(3) any morphism from X to an affine scheme is quasi-compact.

Clearly, (3) ⇒ (2) ⇒ (1). To prove (1) ⇒ (3), we need to show that if X is quasi-compact, then so is any fiber product of the form $X \times_{\mathrm{Spec(R)}} \mathrm{Spec}(R')$, but this follows again from Lemma 1.7.3.

Lemma 1.7.5. *Quasi-compactness of morphisms in* Sch *is local on the target.*

Proof. Using Lemma 1.5.10, we reduce to the assertion for standard open covers: Given a morphism $f : Y \to \mathrm{Spec}(A)$ of schemes and a standard open cover $\mathrm{Spec}(A_i) \to \mathrm{Spec}(A)$ ($i \in I$) such that each base change $Y_i := Y \times_{\mathrm{Spec(A)}} \mathrm{Spec}(A_i)$ is quasi-compact, then Y is quasi-compact. This follows from Lemma 1.7.3, because each Y_i admits a finite open cover by affine schemes and I is finite. □

1.7.6. A morphism $f : Y \to X$ of schemes is called *quasi-separated* if the diagonal $\Delta_f : Y \to Y \times_X Y$ is quasi-compact. Since the property of being quasi-compact is stable under base change, so is the property of being quasi-separated.

Let us show that being quasi-separated is closed under composition. This follows from inspecting the diagram below associated to morphisms $g : Z \to Y$, $f : Y \to X$ of schemes:

$$\begin{array}{ccccc} Z & \xrightarrow{\Delta_g} & Z \times_Y Z & \longrightarrow & Y \\ & \searrow^{\Delta_{f\cdot g}} & \downarrow & & \downarrow \Delta_f \\ & & Z \times_X Z & \longrightarrow & Y \times_X Y \end{array}$$

Finally, we say that a scheme X is *quasi-separated* if the morphism $X \to \mathrm{Spec}(\mathbf{Z})$ is.

Lemma 1.7.7. *Quasi-separatedness of morphisms in* Sch *is local on the target.*

Proof. Let $f : Y \to X$ be a morphism of schemes and $X_i \to X$ ($i \in I$) be a family of morphisms of schemes with base change $f_i : Y_i := Y \times_X X_i \to X_i$ for each $i \in I$. Then the following diagram formed by their diagonals is Cartesian:

$$\begin{array}{ccc} Y_i & \xrightarrow{\Delta_{f_i}} & Y_i \times_{X_i} Y_i \\ \downarrow & & \downarrow \\ Y & \xrightarrow{\Delta_f} & Y \times_X Y \end{array}$$

Furthermore, if $\bigsqcup_{i\in I} X_i \to X$ is an epimorphism, then so is its base change $\bigsqcup_{i\in I} Y_i \times_{X_i} Y_i \to Y \times_X Y$ (*cf.* Lemma 1.4.8). The assertion now follows from Lemma 1.7.5. □

Lemma 1.7.8. *Let* X *be a scheme. The following conditions are equivalent:*

(1) X *is quasi-separated;*
(2) for any open affine subschemes U_1, U_2 *of* X*, their intersection* $U_1 \cap U_2$ *is quasi-compact.*

Proof. (1) ⇒ (2). Let U_1, U_2 be open subschemes of X. The intersection $U_1 \cap U_2 := U_1 \times_X U_2$ is the fiber product:

$$\begin{array}{ccc} U_1 \times_X U_2 & \longrightarrow & X \\ \downarrow & & \downarrow \Delta \\ U_1 \times U_2 & \longrightarrow & X \times X \end{array}$$

If U_1, U_2 are affine, then the same holds for $U_1 \times U_2$, hence for $U_1 \times_X U_2$ as Δ is assumed quasi-compact.

(2) ⇒ (1). Take any open cover $X_i \to X$ ($i \in I$) by affine schemes. The hypothesis shows that $X_i \cap X_j$ is quasi-compact for each pair $i, j \in I$. Then $X_i \times X_j \to X \times X$ ($i, j \in I$) is an open cover, such that the base change $X_i \cap X_j \cong (X_i \times X_j) \times_{X \times X} X$ is quasi-compact. Since $X_i \times X_j$ is affine, this implies that $X_i \cap X_j \to X_i \times X_j$ is a quasi-compact morphism, so we may conclude by Lemma 1.7.5. □

1.7.9. Let X be a quasi-compact quasi-separated scheme. Then we may exhibit X as a finite colimit of affine schemes in the category Shv.

Indeed, since X is quasi-compact, we may take a finite open cover $X_i \to X$ ($i \in I$) by affine schemes (*cf.* Lemma 1.7.3). The intersection $X_{ij} := X_i \times_X X_j$ is quasi-compact since X is quasi-separated (*cf.* Lemma 1.7.8), so we may take a finite open cover $X_{ijk} \to X_{ij}$ ($k \in K_{ij}$) of each X_{ij}. This allows us to present X as a coequalizer in Shv:

$$\bigsqcup_{\substack{i,j \in I \\ k \in K_{ij}}} X_{ijk} \rightrightarrows \bigsqcup_{i \in I} X_i \to X \tag{1.31}$$

where ther index sets I and K_{ij} ($i, j \in I$) are all *finite*.

If X is in addition separated, then X_{ij} is already affine (*cf.* the proof of Lemma 1.7.8), so X is the coequalizer in Shv of affine schemes:

$$\bigsqcup_{i,j \in I} X_{ij} \rightrightarrows \bigsqcup_{i \in I} X_i \to X \tag{1.32}$$

Lemma 1.7.10. *Given a diagram in* Sch*:*

$$\begin{array}{ccc} Y & \xrightarrow{f'} & X' \\ & \searrow^{f} & \downarrow g \\ & & X \end{array}$$

the following statements hold:

(1) if f is quasi-separated, then so is f';
(2) if f is quasi-compact and g is quasi-separated, then f' is quasi-compact.

Proof. For (1), we note that Δ_f factors as $\Delta_{f'} : Y \to Y \times_{X'} Y$ followed by the monomorphism $Y \times_{X'} Y \to Y \times_X Y$. Thus, the base change of $\Delta_{f'}$ along any morphism $\mathrm{Spec}(R) \to Y \times_{X'} Y$ ($R \in$ Ring) is calculated by the base change of Δ_f.

For (2), we inspect the commutative diagram with two Cartesian squares below:

$$\begin{array}{ccccc} & & Y & \xrightarrow{f} & X \\ & & \uparrow & & \uparrow \\ Y & \longrightarrow & Y \times_X X' & \longrightarrow & X' \\ \downarrow & & \downarrow & & \\ X' & \xrightarrow{\Delta_g} & X' \times_X X' & & \end{array}$$

where the middle composition is f'. Thus f' is the composition of base changes of Δ_g and f, both being quasi-compact. □

Corollary 1.7.11. *In the category* $\mathsf{Sch}^{\mathrm{qcqs}}$ *of quasi-compact quasi-separated schemes, every morphism is quasi-compact and quasi-separated.*

Proof. This follows immediately from Lemma 1.7.10. □

1.8. Local properties.

1.8.1. In this subsection, we systematically generalize a class of properties of rings and of their morphisms to schemes. These properties are "Zariski local" (and for morphisms, "Zariski local" both on the source and on the target.)

1.8.2. A property P of rings is called *Zariski local* if the following statements hold:

(1) if $A \in \mathsf{Ring}$ satisfies P, then A_f satisfies P for any $f \in A$;
(2) if $\mathrm{Spec}(A_i) \to \mathrm{Spec}(A)$ $(i \in I)$ is a standard open cover where each A_i satisfies P, then A satisfies P.

Let P be a Zariski local property of rings. Then a scheme X is said to *(locally) satisfy* P if for every open immersion $\mathrm{Spec}(A) \to X$, the ring A satisfies P. By (1) & (2), this is equivalent to the existence of an open cover $\mathrm{Spec}(A_i) \to X$ $(i \in I)$ such that each ring A_i satisfies P (*cf.* Remark 1.6.11).

In proving the following statements, we shall repeatedly use fact that given a morphism $M \to N$ of A-modules and a standard open cover $\mathrm{Spec}(A_i) \to \mathrm{Spec}(A)$ $(i \in I)$ such that each base change $M \otimes_A A_i \to N \otimes_A A_i$ is injective (respectively, surjective, bijective), then the same holds for $M \to N$ (*cf.* Lemma 1.2.9).

Lemma 1.8.3. *The property* P = *"reduced" is Zariski local. (Schemes locally satisfying* P *are called* reduced.*)*

Proof. (1) Let A be a reduced ring and $f \in A$. Suppose that $(a/f)^n = 0 \in A_f$, where $a \in A$ and $n \in \mathbf{Z}_{\geq 1}$. Then a becomes nilpotent after multiplying by a power of f, so $a = 0$.

(2) Suppose that $\mathrm{Spec}(A_i) \to A$ $(i \in I)$ is a standard open cover where each A_i is reduced. Let $\sqrt{0} \subset A$ be its nilradical. Since A_i is reduced, $\sqrt{0} \otimes_A A_i = 0$ for each $i \in I$, so $\sqrt{0} = 0$. ☐

Lemma 1.8.4. *The property* P = *"Noetherian" is Zariski local. (Schemes locally satisfying* P *are called* locally Noetherian.*)*

Proof. Note that the property of a module to be finite is Zariski local. Namely,

(1) if A is a ring and M is a finite A-module, then M_f is a finite A_f-module for every $f \in A$;
(2) if $\mathrm{Spec}(A_i) \to \mathrm{Spec}(A)$ $(i \in I)$ is a standard open cover and $M \in \mathsf{Mod}_A$ is such that each $M \otimes_A A_i$ is a finite A_i-module, then M is a finite A-module.

(1) is clear. To prove (2), note that for each $i \in I$, we may find a morphism $A^{\oplus J_i} \to M$, where J_i is a finite set, that becomes surjective after tensoring with A_i. The sum $\bigoplus_{i \in I} A^{\oplus J_i} \to M$ is thus surjective because it becomes so after tensoring with each A_i.

To prove that Zariski locality of P = "Noetherian", we apply the above observation to ideals of A, using the fact that every ideal of A_f is of the form $\mathfrak{a}_f$ for an ideal $\mathfrak{a}$ of A. ☐

1.8.5. A property P of ring maps is called *Zariski local* if the following statements hold:

(1) if $A \to B$ in Ring satisfies P, then $A_f \to B_f$ satisfies P for any $f \in A$;
(2) given $A, B \in \mathsf{Ring}$, $f \in A$, $g \in B$, and a morphism $A_f \to B$ satisfies P, then so does the composition:
$$A \to A_f \to B \to B_g.$$
(3) given $A \to B$ in Ring and a standard open cover $\mathrm{Spec}(B_i) \to \mathrm{Spec}(B)$ $(i \in I)$ where each $A \to B_i$ satisfies P, then $A \to B$ satisfies P;

Note that if a property P is stable under base change, then it satisfies (1). If a property P is stable under composition and standard opens satisfy P, then P satisfies (2).

1.8.6. Let P be a Zariski local property of ring maps. Let $f : Y \to X$ be a morphism of schemes.

The morphism f is said to *(locally) satisfy* P if for every open immersions $\mathrm{Spec}(B) \to Y$, $\mathrm{Spec}(A) \to X$ such that the restriction of f to $\mathrm{Spec}(B)$ factors through $\mathrm{Spec}(A)$, the induced ring map $A \to B$ satisfies P. If P is stable under base change, then the same condition holds for any morphism $\mathrm{Spec}(B) \to Y$ and open immersion $\mathrm{Spec}(A) \to Y \times_X \mathrm{Spec}(B)$.

By (1)–(3) above, this is equivalent to the existence of open covers $\mathrm{Spec}(B_i) \to Y$, $\mathrm{Spec}(A_i) \to X$ ($i \in I$) such that the restriction of f to each $\mathrm{Spec}(B_i)$ factors through $\mathrm{Spec}(A_i)$, and the induced ring map $A_i \to B_i$ satisfies P.

Remark 1.8.7. Let P be a Zariski local property of ring maps which is also stable under base change, *i.e.* given a ring map $A \to B$ satisfying P and another ring map $A \to A'$, the induced map $A' \to A' \otimes_A B$ also satisfies P. Then the property of "(locally) satisfying P" for morphisms of schemes is also stable under base change and local on the target (*cf.* §1.7.1).

Indeed, stability under base change is clear. To prove locality on the target, we may use base change and Lemma 1.5.10 to reduce to the case of standard open covers, where it follows from the definition of Zariski locality.

Lemma 1.8.8. *The property* P = *"open immersion" is Zariski local.*

Proof. Since the property of being an open immersion is stable under base change and composition, it satisfies (1) & (2). Condition (3) is a special case of Corollary 1.5.11. ☐

Lemma 1.8.9. *The property* P = *"flat" is Zariski local. (Morphisms of schemes locally satisfying* P *are called* flat*.)*

Proof. Flatness is stable under base change and composition and open immersions are flat, so P = "flat" satisfies (1) & (2). Property (3) follows from the fact that injectivity of maps of A-modules can be checked over a standard open cover of $\mathrm{Spec}(A)$. ☐

1.8.10. Recall that a morphism $f : A \to B$ in Ring is said to be *of finite type* if B is a quotient of the A-algebra $A[x_1, \cdots, x_n]$ for some $n \in \mathbf{Z}_{\geq 0}$. In this case, we also call B a *finitely generated* A-algebra. Note that this condition is equivalent to the existence of a closed immersion $\mathrm{Spec}(B) \to \mathbb{A}^n_A$ ($n \in \mathbf{Z}_{\geq 0}$).

A morphism $f : A \to B$ in Ring is *of finite presentation* if B is a quotient of the A-algebra $A[x_1, \cdots, x_n]$ for some $n \in \mathbf{Z}_{\geq 0}$ by a finitely generated ideal $I = (f_1, \cdots, f_m)$ ($m \in \mathbf{Z}_{\geq 0}$). In this case, we also call B a *finitely presented* A-algebra.

Note that the condition for B to be finitely presented means precisely that $\mathrm{Spec}(B)$ is a fiber product in $\mathsf{Sch}_{/\mathrm{Spec}(A)}$ of the following form, for some $n, m \in \mathbf{Z}_{\geq 0}$:

$$\begin{array}{ccc} \mathrm{Spec}(B) & \longrightarrow & \mathbb{A}^n_A \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & \mathbb{A}^m_A \end{array} \tag{1.33}$$

Indeed, being finitely presented is *a priori* equivalent to the existence of (1.33) where the bottom morphism corresponds to the origin $0 : \mathrm{Spec}(A) \to \mathbb{A}^m_A$, but any Cartesian square (1.33) is isomorphic to one of this form, by applying a translation on $\mathbb{A}^m_A$.

Lemma 1.8.11. *The following properties are Zariski local:*

(1) P = *"of finite type";*
(2) P = *"of finite presentation".*

(Morphisms of schemes locally satisfying P *are called* locally of finite type*, respectively* locally of finite presentation*.)*

Proof. Both properties are stable under base change and composition, and standard opens are of finite presentation because $\mathrm{A}_f \cong \mathrm{A}[t]/(ft-1)$.

It remains to prove that given $\mathrm{A} \to \mathrm{B}$ in Ring and a standard open cover $\mathrm{Spec}(\mathrm{B}_i) \to \mathrm{Spec}(\mathrm{B})$ $(i \in \mathrm{I})$ where each $\mathrm{A} \to \mathrm{B}_i$ is of finite type (respectively, of finite presentation), then $\mathrm{A} \to \mathrm{B}$ is of finite type (respectively, of finite presentation). This follows from a direct argument, see [Sta18, 00EP] for details. □

Remark 1.8.12. If X is locally Noetherian, then any morphism $f : \mathrm{Y} \to \mathrm{X}$ is locally of finite type if and only if it is locally of finite presentation. This follows from Hilbert's basis theorem, which asserts that if A is a Noetherian ring, then so is $\mathrm{A}[x]$ (*cf.* [Sta18, 00FN]).

Lemma 1.8.13. *Given a diagram in* Sch*:*

$$\begin{array}{ccc} \mathrm{Y} & \xrightarrow{f'} & \mathrm{X}' \\ & \underset{f}{\searrow} & \downarrow{\scriptstyle g} \\ & & \mathrm{X} \end{array}$$

the following statements hold:

1. *if f is locally of finite type, then so is f';*
2. *if f is locally of finite presentation and g is locally of finite type, then f' is locally of finite presentation.*

Proof. Since these properties are local, this reduces to the case where all three schemes are affine, say $\mathrm{X} = \mathrm{Spec}(\mathrm{A})$, $\mathrm{X}' = \mathrm{Spec}(\mathrm{A}')$, and $\mathrm{Y} = \mathrm{Spec}(\mathrm{B})$.

Statement (1) is clear. For statement (2): if f is the base change of some morphism $\mathbb{A}^n_{\mathrm{A}} \to \mathbb{A}^m_{\mathrm{A}}$ $(m, n \in \mathbf{Z}_{\geq 0})$ along $0 : \mathrm{Spec}(\mathrm{A}) \to \mathbb{A}^m_{\mathrm{A}}$ and there is a closed immersion $\iota : \mathrm{Spec}(\mathrm{A}') \to \mathbb{A}^r_{\mathrm{A}}$ $(r \in \mathbf{Z}_{\geq 0})$, then f' is the base change of the induced morphism $\mathbb{A}^{n+r}_{\mathrm{A}} \to \mathbb{A}^{m+r}_{\mathrm{A}}$ (identity on the last r factors) along the morphism $\mathrm{Spec}(\mathrm{A}') \to \mathbb{A}^{m+r}_{\mathrm{A}}$, which is 0 on the first m factors and ι on the last r factors. □

Remark 1.8.14. Let X be a scheme.

Lemma 1.8.13 implies that in the category of schemes locally of finite type (respectively, locally of finite presentation) over X, every morphism is locally of finite type (respectively, locally of finite presentation).

In particular, this implies that the category of schemes locally of finite type (respectively, locally of finite presentation) over X, viewed as a full subcategory of the category of schemes over X, is closed under finite limits.

1.8.15. The following result characterizes morphisms locally of finite presentation in terms of the underlying morphisms in PShv.

Proposition 1.8.16. *Let $f : \mathrm{Y} \to \mathrm{X}$ be a morphism in* Sch*. The following are equivalent:*

1. *f is locally of finite presentation;*
2. *Y, viewed as a presheaf on $(\mathsf{Sch}^{\mathrm{aff}})_{/\mathrm{X}}$, commutes with filtered colimits,* i.e. *given any cofiltered diagram* $\mathrm{Spec}(\mathrm{R}_i)$ *$(i \in \mathcal{I})$ of affine schemes over X, the natural map:*
$$\operatorname*{colim}_{i \in \mathcal{I}} \mathrm{Hom}_{/\mathrm{X}}(\mathrm{Spec}(\mathrm{R}_i), \mathrm{Y}) \to \mathrm{Hom}_{/\mathrm{X}}(\lim_{i \in \mathcal{I}} \mathrm{Spec}(\mathrm{R}_i), \mathrm{Y}) \tag{1.34}$$
is bijective.

Proof. Let us first prove the assertion when f is a morphism of affine schemes.

For (1) $\Rightarrow$ (2), we first observe that (1.34) is bijective for $X = \mathrm{Spec}(A)$ ($A \in \mathsf{Ring}$) and $Y := \mathbb{A}^1_A \to \mathrm{Spec}(A)$. Then we note that both sides of (1.34) commute with finite limits in Y. Using the Cartesian square (1.33), we then see that (1.34) is bijective for any morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ locally of finite presentation.

For (2) $\Rightarrow$ (1), we observe that given any morphism $A \to B$ in Ring, the category $\mathcal{I}$ of factorizations $A \to R_i \to B$, where $A \to R_i$ is of finite presentation, is filtered (*cf.* Remark 1.8.14), and B is identified with $\mathrm{colim}_{i\in\mathcal{I}} R_i$. Thus, if (1.34) is bijective, then the isomorphism $\lim_{i\in\mathcal{I}} \mathrm{Spec}(R_i) \xrightarrow{\simeq} \mathrm{Spec}(B)$ factors through some $\mathrm{Spec}(R_i) \to \mathrm{Spec}(B)$. This implies that its section $\mathrm{Spec}(B) \to \mathrm{Spec}(R_i)$ is of finite presentation (*cf.* Lemma 1.8.13), so the composite $\mathrm{Spec}(B) \to \mathrm{Spec}(R_i) \to \mathrm{Spec}(A)$ is also of finite presentation.

Now, we treat the case of an arbitrary morphism $f : Y \to X$ in Sch. The implication (2) $\Rightarrow$ (1) is immediate, since the property of being locally of finite presentation is local on the source and the target. Let us assume (1) and prove (2).

To prove that (1.34) is injective, we consider two morphisms g_i, g_i' from $\mathrm{Spec}(R_i)$ to Y over X which are equalized over $\mathrm{Spec}(R)$. We want to show $g_i = g_i'$. This may be checked over any standard open cover of $\mathrm{Spec}(R_i)$, so by localizing, we reduce to the case where $f : Y \to X$ is a morphism of affine schemes.

To prove that (1.34) is surjective, we consider any morphism $g : \mathrm{Spec}(R) \to Y$ over X. We want to show that g factors through some $g_i : \mathrm{Spec}(R_i) \to Y$ over X. Consider a standard open cover of $\mathrm{Spec}(R_j) \to \mathrm{Spec}(R)$ ($j \in J$) where the restriction of g factors as:

$$\begin{array}{ccccc} & & \mathrm{Spec}(B) & \subset & Y \\ & \overset{g_j}{\nearrow} & \downarrow & & \downarrow f \\ \mathrm{Spec}(R_j) & \longrightarrow & \mathrm{Spec}(A) & \subset & X \end{array}$$

for open affine subschemes $\mathrm{Spec}(B) \subset Y$, $\mathrm{Spec}(A) \subset X$. Say that R_j is the localization of R at some $a_j \in R$. Since J is finite, there is an object $i \in \mathcal{I}$ so that each a_j lifts along $R_i \to R$. Then $R_j \cong R_{a_j}$ is the colimit of $(R_{i'})_{a_j}$ over $i' \in \mathcal{I}_{i/}$. (We used the observation that for any $i \in \mathcal{I}$, the functor $\mathcal{I}_{i/} \to \mathcal{I}$ is cofinal, so any colimit over $\mathcal{I}$ may be computed over $\mathcal{I}_{i/}$, *cf.* [Sta18, 0BUC].) By the affine case treated above, there is some $i' \in \mathcal{I}_{i/}$ such that each g_j factors through $\mathrm{Spec}((R_{i'})_{a_j})$. Thus, there is some $i'' \in \mathcal{I}_{i'/}$ such that the factorizations glue to a morphism $\mathrm{Spec}(R_{i''}) \to Y$ through which g factors. □

1.8.17. Let $f : Y \to X$ be a morphism in Sch. We say that f is:

(1) *of finite type* if it is locally of finite type and quasi-compact;
(2) *of finite presentation* if it is locally of finite presentation, quasi-compact, and quasi-separated.

In particular, if X is quasi-compact and f is of finite type, then Y is quasi-compact. If X is quasi-compact and quasi-separated and X is of finite presentation, then Y is also quasi-compact and quasi-separated.

2. Quasi-coherent sheaves

Recall that for every ring R, there is an abelian category $\mathsf{Mod}_{\mathrm{R}}$ of R-modules. In this section, we shall assign to each scheme X an abelian category $\mathsf{QCoh}(\mathrm{X})$ of "quasi-coherent sheaves" which generalizes $\mathsf{Mod}_{\mathrm{R}}$ for X = Spec(R). The assignment $\mathrm{X} \mapsto \mathsf{QCoh}(\mathrm{X})$ will be contravariantly functorial, *i.e.* to every morphism $f : \mathrm{Y} \to \mathrm{X}$ of schemes, there is a functor called "pullback":

$$f^* : \mathsf{QCoh}(\mathrm{X}) \to \mathsf{QCoh}(\mathrm{Y}).$$

When f is quasi-compact and quasi-separated, f^* has a well-behaved right adjoint f_*, called "pushforward". We will study the interaction between pullback and pushforward.

We will also define vector bundles over a scheme X as objects of $\mathsf{QCoh}(\mathrm{X})$ satisfying certain properties; we think of vector bundles as vector spaces "parametrized by X". When X is locally Noetherian, $\mathsf{QCoh}(\mathrm{X})$ also contains the full abelian subcategory $\mathsf{Coh}(\mathrm{X})$ of "coherent sheaves", which can be thought of as vector bundles with singularities.

2.1. Definitions.

2.1.1. *Geometry "relative to* X*".* In practice, we sometimes fix a "base" presheaf X and restrict our attention to presheaves lying over X. (For example, the subject of complex algebraic geometry has Spec($\mathbf{C}$) as the base.)

When doing so, we will invoke the canonical identification between $\mathsf{PShv}_{/\mathrm{X}}$ and the category of presheaves on $(\mathsf{Sch}^{\mathrm{aff}})_{/\mathrm{X}}$:

$$\mathsf{PShv}_{/\mathrm{X}} \xrightarrow{\simeq} \mathsf{Fun}(((\mathsf{Sch}^{\mathrm{aff}})_{/\mathrm{X}})^{\mathrm{op}}, \mathsf{Set}),$$

where a presheaf $\mathrm{Y} \in \mathsf{PShv}_{/\mathrm{X}}$ is sent to the functor $\mathrm{Y}_{/\mathrm{X}}$ carrying $\mathrm{Spec(R)} \to \mathrm{X}$ to the set of morphisms $\mathrm{Spec(R)} \to \mathrm{Y}$ over X, and the converse sends a functor $\mathrm{Y}_{/\mathrm{X}}$ to the presheaf Y whose R-points consist of an R-point of X and an element of $\mathrm{Y}_{/\mathrm{X}}(\mathrm{Spec(R)} \to \mathrm{X})$.

If X = Spec(A) for some $\mathrm{A} \in \mathsf{Ring}$, then $(\mathsf{Sch}^{\mathrm{aff}})_{/\mathrm{X}}$ is opposite to the category of A-algebras, so $\mathsf{PShv}_{/\mathrm{X}}$ is identified with $\mathsf{Fun}(\mathsf{Ring}_{\mathrm{A}/}, \mathsf{Set})$.

2.1.2. Let X be a scheme. There is a presheaf of rings $\mathcal{O}_{\mathrm{X}}$ on the category $(\mathsf{Sch}^{\mathrm{aff}})_{/\mathrm{X}}$ of affine schemes over X. It sends an affine scheme Spec(R) over X to the ring R, and a morphism $\mathrm{Spec(R')} \to \mathrm{Spec(R)}$ of affine schemes over X to the induced ring map $\mathrm{R} \to \mathrm{R'}$.

Note that $\mathcal{O}_{\mathrm{X}}$ is representable by a scheme over X, namely $\mathbb{A}^1_{\mathrm{X}} := \mathbb{A}^1_{\mathbf{Z}} \times \mathrm{X}$. In particular, $\mathcal{O}_{\mathrm{X}}$ is a sheaf with respect to standard open covers (*cf.* Proposition 1.2.7). We call $\mathcal{O}_{\mathrm{X}}$ the *structure sheaf* of the scheme X.

2.1.3. Given a scheme X, an $\mathcal{O}_{\mathrm{X}}$*-module* is a presheaf $\mathcal{M}$ on $(\mathsf{Sch}^{\mathrm{aff}})_{/\mathrm{X}}$ equipped with a module structure over $\mathcal{O}_{\mathrm{X}}$, *i.e.* its value $\mathcal{M}(\mathrm{R})$ at every affine scheme Spec(R) over X has an R-module structure, and for every morphism $\mathrm{Spec(R')} \to \mathrm{Spec(R)}$ of affine schemes over X, the induced map $\mathcal{M}(\mathrm{R}) \to \mathcal{M}(\mathrm{R'})$ is R-linear. Write $\mathsf{Mod}_{\mathcal{O}_{\mathrm{X}}}$ for the category of $\mathcal{O}_{\mathrm{X}}$-modules.

An $\mathcal{O}_{\mathrm{X}}$-module is called *quasi-coherent* if for every morphism $\mathrm{Spec(R')} \to \mathrm{Spec(R)}$ of affine schemes over X, the R-linear map $\mathcal{M}(\mathrm{R}) \to \mathcal{M}(\mathrm{R'})$ induces a bijection:

$$\mathcal{M}(\mathrm{R}) \otimes_{\mathrm{R}} \mathrm{R'} \xrightarrow{\simeq} \mathcal{M}(\mathrm{R'}). \tag{2.1}$$

Lemma 2.1.4. *Let* X *be a scheme. Every quasi-coherent* $\mathcal{O}_{\mathrm{X}}$*-module is a sheaf with respect to standard open covers.*

Proof. This is a restatement of Lemma 1.2.9. □

2.1.5. Because of Lemma 2.1.4, we call quasi-coherent $\mathcal{O}_X$-modules *quasi-coherent sheaves.* They form a category $\mathsf{QCoh}(X)$, where a morphism $\mathcal{M} \to \mathcal{N}$ in $\mathsf{QCoh}(X)$ is a morphism of presheaves on $(\mathsf{Sch}^{\mathrm{aff}})_{/X}$ such that for every affine scheme $\mathrm{Spec}(R)$ over X, the induced morphism $\mathcal{M}(R) \to \mathcal{N}(R)$ is R-linear.

Given an R-point x of X, we write $\mathcal{M}|_x$ for the R-module $\mathcal{M}(R)$ and call it the *fiber* of $\mathcal{M}$ at the R-point x.

2.1.6. Tautologically, $\mathsf{QCoh}(X)$ may be presented as a 2-limit of the categories of modules indexed by affine schemes over X:

$$\mathsf{QCoh}(X) \xrightarrow{\simeq} \lim_{\mathrm{Spec}(R)\to X} \mathsf{Mod}_R, \tag{2.2}$$

where the functor $\mathsf{Mod}_R \to \mathsf{Mod}_{R'}$ for each morphism of affine schemes $\mathrm{Spec}(R') \to \mathrm{Spec}(R)$ over X is given by $(\cdot) \otimes_R R'$.

Concretely, the equivalence (2.2) means that a quasi-coherent sheaf on X is a compatible system $(\mathcal{M}|_x)$ of R-modules for each R-point x of X. Here, being "compatible" means that for each morphism of affine schemes $\mathrm{Spec}(R') \to \mathrm{Spec}(R)$ over X, there is an isomorphism of R'-modules $\varphi_{x,x'} : \mathcal{M}|_x \otimes_R R' \cong \mathcal{M}|_{x'}$ for $x' := x|_{R'}$, making the following diagram commute for any morphisms of affine schemes $\mathrm{Spec}(R'') \to \mathrm{Spec}(R') \to \mathrm{Spec}(R)$ over X:

$$\begin{array}{ccc} (\mathcal{M}|_x \otimes_R R') \otimes_{R'} R'' & \xrightarrow{\varphi_{x,x'}} & \mathcal{M}|_{x'} \otimes_{R'} R'' \\ \downarrow{\simeq} & & \downarrow{\varphi_{x',x''}} \\ \mathcal{M}|_x \otimes_R (R' \otimes_{R'} R'') & \xrightarrow{\varphi_{x,x''}} & \mathcal{M}|_{x''} \end{array}$$

Remark 2.1.7. It follows from (2.2) that $\mathsf{QCoh}(X)$ admits colimits, and the canonical functor $\mathsf{QCoh}(X) \to \mathsf{Mod}_R$, for any affine scheme $\mathrm{Spec}(R)$ over X, preserves colimits.

Indeed, this is a general phenomenon: given a diagram of categories $\mathcal{C}_i$ ($i \in \mathcal{I}$) where each $\mathcal{C}_i$ admits colimits and the functor $\mathcal{C}_i \to \mathcal{C}_j$, for each morphism $i \to j$ in $\mathcal{I}$, preserves colimits, then the 2-limit $\mathcal{C} := \lim_{i\in\mathcal{I}} \mathcal{C}_i$ admits colimits and the natural functor $\mathcal{C} \to \mathcal{C}_i$, for each $i \in \mathcal{I}$, preserves them.

Remark 2.1.8. The category $\mathsf{QCoh}(X)$ has the structure of a symmetric monoidal category (*cf.* [Sta18, 0FFJ]). The monoidal product is given by tensor product of R-modules for any R-point x of X, *i.e.* for any $\mathcal{M}, \mathcal{N} \in \mathsf{QCoh}(X)$, we set:

$$(\mathcal{M} \otimes \mathcal{N})|_x := \mathcal{M}|_x \otimes_R \mathcal{N}|_x.$$

(In making this definition, we invoked the canonical isomorphism of R'-modules:

$$(\mathcal{M}|_x \otimes_R \mathcal{N}|_x) \otimes_R R' \xrightarrow{\simeq} \mathcal{M}|_{x'} \otimes_{R'} \mathcal{M}|_{x'}$$

for any morphism $\mathrm{Spec}(R') \to \mathrm{Spec}(R)$ and $x' := x|_R$.) The monoidal unit of $\mathsf{QCoh}(X)$ is the structure sheaf $\mathcal{O}_X$.

Lemma 2.1.9. *Let* $X = \mathrm{Spec}(A)$ *be an affine scheme. There is a canonical equivalence:*

$$\mathsf{QCoh}(X) \xrightarrow{\simeq} \mathsf{Mod}_A. \tag{2.3}$$

Proof. This follows from the equivalence (2.2), because the index category has an initial object given by the identity on $\mathrm{Spec}(A)$. ☐

2.1.10. Let $f : Y \to X$ be a morphism of schemes. Then there is a functor:

$$f^* : \mathsf{QCoh}(X) \to \mathsf{QCoh}(Y), \tag{2.4}$$

Indeed, for each $\mathcal{M} \in \mathsf{QCoh}(X)$, $f^*\mathcal{M}$ is defined by $(f^*\mathcal{M})|_y := \mathcal{M}|_{f(y)}$ for any R-point y of Y. For each morphism $\mathcal{M} \to \mathcal{N}$ in $\mathsf{QCoh}(X)$, $f^*\mathcal{M} \to f^*\mathcal{N}$ is given by the R-linear map $\mathcal{M}|_{f(y)} \to \mathcal{N}|_{f(y)}$ for any R-point y of Y.

We call (2.4) the *pullback* functor of quasi-coherent sheaves. Note that it preserves colimits (*cf.* Remark 2.1.7) and is symmetric monoidal (*cf.* Remark 2.1.8, [Sta18, 0FFY]).

Remark 2.1.11. When $Y = \mathrm{Spec}(R)$ is affine, then $f^*\mathcal{M}$ is the R-module $\mathcal{M}|_f$ under the equivalence of Lemma 2.1.9. (The difference in thinking of $f^*\mathcal{M}$ as the pullback or the fiber at an R-point is purely psychological.)

2.1.12. Any construction on modules which behaves functorially under tensor product of rings can be carried out on $\mathsf{QCoh}(X)$.

For example, given $\mathcal{M} \in \mathsf{QCoh}(X)$ and $n \in \mathbf{Z}_{\geq 1}$, we define $\bigwedge^n \mathcal{M} \in \mathsf{QCoh}(X)$ by $(\bigwedge^n \mathcal{M})|_x := \bigwedge^n(\mathcal{M}|_x)$ for any R-point x of X. This is well-defined because given a ring map $R \to R'$ and $x' := x|_{R'}$, there holds $\bigwedge^n(\mathcal{M}|_x) \otimes_R R' \cong \bigwedge^n(\mathcal{M}|_{x'})$.

The analogous construction can be carried out for the symmetric power $\mathrm{Sym}^n \mathcal{M}$.

2.1.13. *Vector bundles.* Let us now define the notion of vector bundles.

Let X be a scheme. A quasi-coherent $\mathcal{O}_X$-module is called *free* if it is isomorphic to $\bigoplus_{i\in I} \mathcal{O}_X$ for some set I. If I is in addition finite, then $\mathcal{M}$ is called *finite free* and the cardinality $|I|$ is called the *rank* of $\mathcal{M}$.

A quasi-coherent $\mathcal{O}_X$-module is called *locally free* (respectively, *finite locally free*) if there exists an open cover $f_i : X_i \to X$ $(i \in I)$ such that each pullback $(f_i)^*\mathcal{M}$ is free (respectively, finite free). If each pullback $(f_i)^*\mathcal{M}$ is finite free of the same rank r, we also say that $\mathcal{M}$ is finite locally free *of rank* r.

We shall also call finite locally free $\mathcal{O}_X$-modules *vector bundles* over X and those of rank 1 *line bundles* over X. Let us denote the full subcategory of vector bundles by:

$$\mathsf{Vect}(X) \subset \mathsf{QCoh}(X).$$

2.1.14. Almost every good property enjoyed by finite modules holds because of Nakayama's lemma, so let us review it.

Proposition 2.1.15 (Nakayama's lemma)**.** *Let* A *be a ring,* $\mathfrak{a} \subset A$ *be an ideal, and* M *be a finite* A*-module. Suppose that* $M \otimes_A A/\mathfrak{a} \cong 0$*. Then there exists some* $f \in A$ *whose image in* $A/\mathfrak{a}$ *is invertible and* $M_f \cong 0$*.*

Proof. Let $x_1, \cdots, x_n \in M$ be a set of generators. The hypothesis $M \otimes_A A/\mathfrak{a} \cong 0$ means that $\mathfrak{a}M = M$. Thus each x_i is of the form $\sum_{j=1}^n a_{ij}x_j$ for $a_{ij} \in \mathfrak{a}$. Denote by $T : A^{\oplus n} \to A^{\oplus n}$ the endomorphism $e_i \mapsto e_i - \sum_{j=1}^n a_{ij}e_j$. Then the composition of T with the map $A^{\oplus n} \to M$, $e_i \mapsto x_i$ vanishes. Let adj(T) be the *adjugate* of T, *i.e.* the dual of $\bigwedge^{n-1} T$ under the pairing $A^{\oplus n} \otimes \bigwedge^{n-1}(A^{\oplus n}) \to \bigwedge^n(A^{\oplus n}) \cong A$. Then $T \cdot \mathrm{adj}(T) = \det(T) \cdot \mathbf{1}$, so:

$$\det(T) \cdot x_i = 0 \text{ for all } 1 \leq i \leq n.$$

Since $\det(T) \in 1 + \mathfrak{a}$, we see that $f := \det(T)$ suffices. □

Remark 2.1.16. We state Nakayama's lemma in this form because it has a clear geometric interpretation: if the pullback of M to the closed subscheme $\mathrm{Spec}(A/\mathfrak{a})$ of $\mathrm{Spec}(A)$ vanishes, then it already vanishes over some standard open $\mathrm{Spec}(A_f)$ containing $\mathrm{Spec}(A/\mathfrak{a})$ (*i.e.* the closed immersion $\mathrm{Spec}(A/\mathfrak{a}) \to \mathrm{Spec}(A)$ factors through $\mathrm{Spec}(A_f)$).

Note that if A is local and $\mathfrak{a}$ is the maximal ideal, then any such f is a unit, so the conclusion $M_f \cong 0$ is equivalent to $M \cong 0$.

Lemma 2.1.17. *Let* $\mathrm{X} = \mathrm{Spec}(\mathrm{A})$ *be an affine scheme. Then finite locally free* $\mathcal{O}_{\mathrm{X}}$*-modules are precisely the finite projective* A*-modules under the equivalence* (2.3)*.*

Proof. Let $\mathcal{M}$ be an A-module.

Assume that $\mathcal{M}$ is finite locally free. Let us prove that $\mathcal{M}$ is finite projective. The assumption implies that we have a standard open cover $\mathrm{Spec}(\mathrm{A}_i) \to \mathrm{Spec}(\mathrm{A})$ ($i \in \mathrm{I}$) such that each $\mathcal{M}_i := \mathcal{M} \otimes_{\mathrm{A}} \mathrm{A}_i$ is finite free. This implies that $\mathcal{M}$ is a finite A-module, since being finite is a Zariski local property (*cf.* the proof of Lemma 1.8.4). It remains to prove that $\mathcal{M}$ is a projective object of $\mathsf{QCoh}(\mathrm{X})$, *i.e.* $\mathrm{Hom}(\mathcal{M}, \cdot)$ preserves surjections.

We first note that $\mathcal{M}$ is a finitely presented A-module, *i.e.* it is the cokernel of some map $\mathrm{A}^{\oplus r'} \to \mathrm{A}^{\oplus r}$. Indeed, choose a surjection $\mathrm{A}^{\oplus r} \to \mathcal{M}$ and write K for the kernel, we show that K is finite by proving that each $\mathrm{K}_i := \mathrm{K} \otimes_{\mathrm{A}} \mathrm{A}_i$ is a finite A_i-module. Since $\mathcal{M}_i$ is free, K_i is a direct summand of $\mathrm{A}_i^{\oplus r}$, hence finite.

Since $\mathcal{M}$ is a finitely presented A-module, we have a natural isomorphism:

$$\mathrm{Hom}(\mathcal{M}, \mathcal{N}) \otimes_{\mathrm{A}} \mathrm{A}_i \xrightarrow{\simeq} \mathrm{Hom}(\mathcal{M}, \mathcal{N} \otimes_{\mathrm{A}} \mathrm{A}_i). \tag{2.5}$$

The condition of being finite locally free now implies that $\mathcal{M}^{\vee} \otimes_{\mathrm{A}} \mathcal{N} \to \mathrm{Hom}(\mathcal{M}, \mathcal{N})$ is bijective: this reduces to the bijectivity of $\mathcal{M}_i^{\vee} \otimes_{\mathrm{A}_i} \mathcal{N}_i \to \mathrm{Hom}(\mathcal{M}_i, \mathcal{N}_i)$ using (2.5), which holds because $\mathcal{M}_i$ is finite free. In particular, $\mathrm{Hom}(\mathcal{M}, \cdot)$ preserves surjections.

Let us now prove the converse. Assume that $\mathcal{M}$ is finite projective. Note that this implies that $\mathrm{S}^{-1}\mathcal{M}$ is a finite projective $\mathrm{S}^{-1}\mathrm{A}$-module for any multiplicative subset $\mathrm{S} \subset \mathrm{A}$. Indeed, this follows immediately from the equivalence being finite projective and being a direct summand of a finite free module.

For any prime $\mathfrak{p}$, we shall prove that $\mathcal{M}_{\mathfrak{p}}$ is finite free. Indeed, we may lift a basis of $\mathrm{M}_{\mathfrak{p}} \otimes_{\mathrm{A}_{\mathfrak{p}}} \kappa(\mathfrak{p})$, for $\kappa(\mathfrak{p}) := \mathrm{A}_{\mathfrak{p}}/\mathfrak{p}\mathrm{A}_{\mathfrak{p}}$, to generators of the $\mathrm{A}_{\mathfrak{p}}$-module $\mathrm{M}_{\mathfrak{p}}$ by Nakayama's lemma (*cf.* Proposition 2.1.15). This gives a surjection $\mathrm{A}_{\mathfrak{p}}^{\oplus r} \to \mathrm{M}_{\mathfrak{p}}$ for some $r \in \mathbf{Z}_{\geq 0}$. Because $\mathrm{M}_{\mathfrak{p}}$ is projective, the kernel of this surjection is a finite $\mathrm{A}_{\mathfrak{p}}$-module whose base change to $\kappa(\mathfrak{p})$ vanishes, so by Nakayama's lemma again it vanishes, giving $\mathrm{A}_{\mathfrak{p}}^{\oplus r} \xrightarrow{\simeq} \mathrm{M}_{\mathfrak{p}}$.

Next, we claim that there exists $f \notin \mathfrak{p}$ such that $\mathcal{M}_f$ is a finite free A_f-module. Indeed, $\mathcal{M}_{\mathfrak{p}}$ is a filtered colimit of $\mathcal{M}_f$ over $f \notin \mathfrak{p}$. This implies that for some $f \notin \mathfrak{p}$, we have a morphism $\mathrm{A}_f^{\oplus r} \to \mathcal{M}_f$ which becomes an isomorphism after localizing at $\mathfrak{p}$. This implies that its cokernel, being a finite A_f-module which vanishes after localizing at $\mathfrak{p}$, vanishes after possibly modifying the element $f \notin \mathfrak{p}$. Thus we obtain a surjective map $\mathrm{A}_f^{\oplus r} \to \mathcal{M}_f$ for some $f \notin \mathfrak{p}$. However, because $\mathcal{M}_f$ is projective, the kernel is again a finite A_f-module which vanishes after localizing at $\mathfrak{p}$.[10] By modifying the element $f \notin \mathfrak{p}$ once more, we obtain an isomorphism $\mathrm{A}_f^{\oplus r} \cong \mathcal{M}_f$.

Thus, for each prime $\mathfrak{p}$ of A, we have found an element $f \notin \mathfrak{p}$ such that $\mathcal{M}_f$ is finite free. The collection of open immersions $\mathrm{Spec}(\mathrm{A}_f) \to \mathrm{Spec}(\mathrm{A})$ over such f forms an open cover, so $\mathcal{M}$ is finite locally free. □

2.1.18. *Coherent sheaves.* If X is a locally Noetherian scheme, we define $\mathsf{Coh}(\mathrm{X})$ to be the full subcategory of $\mathsf{QCoh}(\mathrm{X})$ consisting of objects $\mathcal{M}$ such that $\mathcal{M}|_x$ is a finite R-module for every R-point x of X. Objects of $\mathsf{Coh}(\mathrm{X})$ are called *coherent sheaves*.

If $\mathrm{X} = \mathrm{Spec}(\mathrm{A})$ is affine, then $\mathsf{Coh}(\mathrm{X})$ is equivalent to category of finite A-modules under the equivalence (2.3).

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[10]Note a subtlety in this proof: we are using the projectivity of $\mathcal{M}$ rather than the projectivity of $\mathcal{M}_{\mathfrak{p}}$. In general, a finite A-module $\mathcal{M}$ whose localization at every prime $\mathfrak{p}$ is free may *not* be locally free (*cf.* MathOverflow, Question 13817).

Thus, for a locally Noetherian scheme X, we have three categories:

$$\mathsf{Vect}(\mathrm{X}) \subset \mathsf{Coh}(\mathrm{X}) \subset \mathsf{QCoh}(\mathrm{X}).$$

Remark 2.1.19. If X is not locally Noetherian, the above definition of $\mathsf{Coh}(\mathrm{X})$ needs to be modified in order to yield an abelian category (*cf.* [Sta18, 01BU]). We will always restrict to locally Noetherian schemes when we speak of coherent sheaves.

2.2. Zariski descent.

2.2.1. We shall prove that the assignment $\mathrm{X} \mapsto \mathsf{QCoh}(\mathrm{X})$ satisfies Zariski descent. This will be deduced from two facts: the Zariski descent of the category of sheaves (*cf.* Proposition 1.4.13) and the fact that quasi-coherence is a "local property".

Let us consider a family of morphisms $\mathrm{X}_i \to \mathrm{X}$ ($i \in \mathrm{I}$) of schemes. Denote by $\mathsf{QCoh}(\{\mathrm{X}_i\})$ the category of descent data for quasi-coherent sheaves with respect to $\mathrm{X}_i \to \mathrm{X}$ ($i \in \mathrm{I}$). Namely, an object of $\mathsf{QCoh}(\{\mathrm{X}_i\})$ is a collection $\mathcal{M}_i \in \mathsf{QCoh}(\mathrm{X}_i)$ ($i \in \mathrm{I}$) equipped with isomorphisms of their pullbacks to $\mathrm{X}_{ij} := \mathrm{X}_i \times_{\mathrm{X}} \mathrm{X}_j$:

$$\varphi_{ij} : \mathcal{M}_i|_{\mathrm{X}_{ij}} \xrightarrow{\simeq} \mathcal{M}_j|_{\mathrm{X}_{ij}} \in \mathsf{QCoh}(\mathrm{X}_{ij})$$

for each $i, j \in \mathrm{I}$, making the following diagram commute (*cf.* §1.4.10):

$$\begin{array}{ccc} \mathcal{M}_i|_{\mathrm{X}_{ijk}} & \xrightarrow{p_{12}^*\varphi_{ij}} & \mathcal{M}_j|_{\mathrm{X}_{ijk}} \\ & \searrow^{p_{13}^*\varphi_{ik}} & \downarrow{\scriptstyle p_{23}^*\varphi_{jk}} \\ & & \mathcal{M}_k|_{\mathrm{X}_{ijk}} \end{array} \tag{2.6}$$

A morphism $(\{\mathcal{M}_i\}, \{\varphi_{ij}\}) \to (\{\mathcal{N}_i\}, \{\psi_{ij}\})$ is a collection of morphisms $\mathcal{M}_i \to \mathcal{N}_i$ in $\mathsf{QCoh}(\mathrm{X}_i)$ ($i \in \mathrm{I}$) which intertwine φ_{ij} with ψ_{ij} for each $i, j \in \mathrm{I}$.

Proposition 2.2.2 (Zariski descent of QCoh)**.** *Let $\mathrm{X}_i \to \mathrm{X}$ ($i \in \mathrm{I}$) be a family of morphisms in Sch such that $\bigsqcup_{i\in\mathrm{I}} \mathrm{X}_i \to \mathrm{X}$ is an epimorphism in Shv. Then the pullback functor defines an equivalence of categories:*

$$\mathsf{QCoh}(\mathrm{X}) \xrightarrow{\simeq} \mathsf{QCoh}(\{\mathrm{X}_i\}).$$

2.2.3. Before proving Proposition 2.2.2, let us first observe that Proposition 1.4.13 implies descent for *sheaves* of $\mathcal{O}_{\mathrm{X}}$-modules, *i.e.* $\mathcal{O}_{\mathrm{X}}$-modules whose underlying presheaves on $(\mathsf{Sch}^{\mathrm{aff}})_{/\mathrm{X}}$ are Zariski sheaves.

Indeed, we may view $\mathcal{O}_{\mathrm{X}}$ as a ring object in $\mathsf{Shv}_{/\mathrm{X}}$. Then the category of sheaves of $\mathcal{O}_{\mathrm{X}}$-modules is equivalent to the category of $\mathcal{O}_{\mathrm{X}}$-module objects in $\mathsf{Shv}_{/\mathrm{X}}$, *i.e.* objects $\mathcal{M} \in \mathsf{Shv}_{/\mathrm{X}}$ equipped with structural morphisms in $\mathsf{Shv}_{/\mathrm{X}}$:

$$0 : \mathrm{X} \to \mathcal{M}$$
$$\mathrm{add} : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$$
$$\mathrm{act} : \mathcal{O}_{\mathrm{X}} \times \mathcal{M} \to \mathcal{M}$$

satisfying the axioms of a module.

Since the assignment $\mathrm{X} \mapsto \mathsf{Shv}_{/\mathrm{X}}$ satisfies descent (*cf.* Proposition 1.4.13), so does the assignment of the category of sheaves of $\mathcal{O}_{\mathrm{X}}$-modules to each $\mathrm{X} \in \mathsf{Sch}$, by applying descent to the underlying object of $\mathsf{Shv}_{/\mathrm{X}}$ together with the above structural morphisms.

Proof of Proposition 2.2.2. Using descent of sheaves of $\mathcal{O}_{\mathrm{X}}$-modules (*cf.* §2.2.3), it remains to prove that the condition of quasi-coherence also descends. Namely, given a sheaf of $\mathcal{O}_{\mathrm{X}}$-modules $\mathcal{M}$ such that $\mathcal{M}|_{\mathrm{X}_i}$ is quasi-coherent for each $i \in \mathrm{I}$, we want to prove that $\mathcal{M}$

is quasi-coherent. Since $\bigsqcup_{i\in \mathrm{I}} \mathrm{X}_i \to \mathrm{X}$ is an epimorphism, this reduces to the case where $\mathrm{X} = \mathrm{Spec}(\mathrm{R})$ and $\mathrm{X}_i = \mathrm{Spec}(\mathrm{R}_i) \to \mathrm{Spec}(\mathrm{R})$ ($i \in \mathrm{I}$) is a standard open cover, and it is enough to prove that the induced map:

$$\mathcal{M}(\mathrm{R}) \otimes_{\mathrm{R}} \mathrm{R}' \to \mathcal{M}(\mathrm{R}')$$

is an isomorphism for any ring map $\mathrm{R} \to \mathrm{R}'$.

As usual, we write $\mathrm{R}_{ij} := \mathrm{R}_i \otimes_{\mathrm{R}} \mathrm{R}_j$, $\mathrm{R}'_i := \mathrm{R}_i \otimes_{\mathrm{R}} \mathrm{R}'$, and $\mathrm{R}'_{ij} := \mathrm{R}_{ij} \otimes_{\mathrm{R}} \mathrm{R}'$ ($i, j \in \mathrm{I}$). Since $\mathcal{M}$ is a sheaf, we have a morphism of equalizers:

$$\begin{array}{ccccc} \mathcal{M}(\mathrm{R}) & \longrightarrow & \bigoplus_{i\in\mathrm{I}} \mathcal{M}(\mathrm{R}_i) & \rightrightarrows & \bigoplus_{i,j\in\mathrm{I}} \mathcal{M}(\mathrm{R}_{ij}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{M}(\mathrm{R}') & \longrightarrow & \bigoplus_{i\in\mathrm{I}} \mathcal{M}(\mathrm{R}'_i) & \rightrightarrows & \bigoplus_{i,j\in\mathrm{I}} \mathcal{M}(\mathrm{R}'_{ij}) \end{array} \tag{2.7}$$

Because $\mathrm{M}|_{\mathrm{Spec}(\mathrm{R}_i)}$ is quasi-coherent, we have natural isomorphisms:

$$\mathcal{M}(\mathrm{R}_i) \otimes_{\mathrm{R}} \mathrm{R}' \xrightarrow{\simeq} \mathcal{M}(\mathrm{R}'_i), \quad \mathcal{M}(\mathrm{R}_{ij}) \otimes_{\mathrm{R}} \mathrm{R}' \xrightarrow{\simeq} \mathcal{M}(\mathrm{R}'_{ij}).$$

Thus, it suffices to prove that the top row of (2.7) remains an equalizer after applying $(\cdot) \otimes_{\mathrm{R}} \mathrm{R}'$. This amounts to the injectivity of the induced map of R'-modules:

$$\mathcal{M}(\mathrm{R}) \otimes_{\mathrm{R}} \mathrm{R}' \to \bigoplus_{i\in\mathrm{I}} \mathcal{M}(\mathrm{R}_i) \otimes_{\mathrm{R}} \mathrm{R}'. \tag{2.8}$$

To prove that (2.8) is injective, it is enough to do so after applying $(\cdot) \otimes_{\mathrm{R}} \mathrm{R}'_j$ for each $j \in \mathrm{I}$ (*cf.* Lemma 1.2.9). Let us calculate $\mathcal{M}(\mathrm{R}) \otimes_{\mathrm{R}} \mathrm{R}'_j$ and $\mathcal{M}(\mathrm{R}_i) \otimes_{\mathrm{R}} \mathrm{R}'_j$.

Since $\mathrm{R} \to \mathrm{R}_j$ is flat, $\mathcal{M}(\mathrm{R}) \otimes_{\mathrm{R}} \mathrm{R}_j$ is the equalizer of the two parallel arrows:

$$\bigoplus_{i\in\mathrm{I}} \mathcal{M}(\mathrm{R}_i) \otimes_{\mathrm{R}} \mathrm{R}_j \rightrightarrows \bigoplus_{i_1,i_2\in\mathrm{I}} \mathcal{M}(\mathrm{R}_{i_1 i_2}) \otimes_{\mathrm{R}} \mathrm{R}_j. \tag{2.9}$$

Since $\mathcal{M}|_{\mathrm{Spec}(\mathrm{R}_i)}$ is quasi-coherent, the R_j-module $\mathcal{M}(\mathrm{R}_i) \otimes_{\mathrm{R}} \mathrm{R}_j$ is identified with $\mathcal{M}(\mathrm{R}_{ij})$, and likewise $\mathcal{M}(\mathrm{R}_{i_1 i_2}) \otimes_{\mathrm{R}} \mathrm{R}_j \cong \mathcal{M}(\mathrm{R}_{i_1 i_2 j})$ for $\mathrm{R}_{i_1 i_2 j} := \mathrm{R}_{i_1 i_2} \otimes_{\mathrm{R}} \mathrm{R}_j$. Since $\mathrm{Spec}(\mathrm{R}_{ij}) \to \mathrm{Spec}(\mathrm{R}_j)$ ($i \in \mathrm{I}$) is a standard open cover, the equalizer of (2.9) is identified with $\mathcal{M}(\mathrm{R}_j)$ (*cf.* Lemma 1.2.9). Thus, we find:

$$\begin{aligned} \mathcal{M}(\mathrm{R}) \otimes_{\mathrm{R}} \mathrm{R}'_j &\xrightarrow{\simeq} \mathcal{M}(\mathrm{R}) \otimes_{\mathrm{R}} \mathrm{R}_j \otimes_{\mathrm{R}_j} \mathrm{R}'_j \\ &\xrightarrow{\simeq} \mathcal{M}(\mathrm{R}_j) \otimes_{\mathrm{R}_j} \mathrm{R}'_j \xrightarrow{\simeq} \mathcal{M}(\mathrm{R}'_j). \end{aligned}$$

On the other hand each $\mathcal{M}(\mathrm{R}_i) \otimes_{\mathrm{R}} \mathrm{R}'_j$ is identified with $\mathcal{M}(\mathrm{R}_i) \otimes_{\mathrm{R}_i} \mathrm{R}'_{ij}$, which is identified with $\mathcal{M}(\mathrm{R}'_{ij})$ by quasi-coherence of $\mathcal{M}|_{\mathrm{Spec}(\mathrm{R}_i)}$.

Altogether, each map $\mathcal{M}(\mathrm{R}) \otimes_{\mathrm{R}} \mathrm{R}'_j \to \mathcal{M}(\mathrm{R}_i) \otimes_{\mathrm{R}} \mathrm{R}'_j$ induced from (2.8) by tensoring with R'_j is given by the restriction $\mathcal{M}(\mathrm{R}'_j) \to \mathcal{M}(\mathrm{R}'_{ij})$. Since $\mathrm{Spec}(\mathrm{R}'_{ij}) \to \mathrm{Spec}(\mathrm{R}'_j)$ ($i \in \mathrm{I}$) is a standard open cover, $\mathcal{M}(\mathrm{R}'_j) \to \bigoplus_{i\in\mathrm{I}} \mathcal{M}(\mathrm{R}'_{ij})$ is injective, as desired. □

2.2.4. Let us now give a presentation of $\mathsf{QCoh}(\mathrm{X})$ in the same spirit as Corollary 1.6.14. Recall the category $\mathrm{X}_{\mathrm{SZar}}$ whose objects are open immersions $\mathrm{Spec}(\mathrm{R}) \to \mathrm{X}$ ($\mathrm{R} \in \mathsf{Ring}$) and whose morphisms are standard opens $\mathrm{Spec}(\mathrm{R}') \to \mathrm{Spec}(\mathrm{R})$ over X.

The pullback functor gives rise to a functor:

$$\mathsf{QCoh}(\mathrm{X}) \to \lim_{\substack{\mathrm{Spec}(\mathrm{R})\to\mathrm{X} \\ \text{in } \mathrm{X}_{\mathrm{SZar}}}} \mathsf{Mod}_{\mathrm{R}}. \tag{2.10}$$

Corollary 2.2.5. *For any scheme* X, *the functor* (2.10) *is an equivalence.*

Proof. The collection of all open immersions $\mathrm{Spec}(\mathrm{R}) \to \mathrm{X}$ ($\mathrm{R} \in \mathsf{Ring}$) satisfies the hypothesis of Proposition 2.2.2. Thus, $\mathsf{QCoh}(\mathrm{X})$ is equivalent to the category consisting of collections $\mathcal{M}_{\mathrm{R}} \in \mathsf{Mod}_{\mathrm{R}}$ for every open immersion $\mathrm{Spec}(\mathrm{R}) \to \mathrm{X}$, together with isomorphisms:

$$\varphi_{\mathrm{R},\mathrm{R}'} : \mathcal{M}_{\mathrm{R}}|_{\mathrm{Spec}(\mathrm{R})\times_{\mathrm{X}}\mathrm{Spec}(\mathrm{R}')} \xrightarrow{\simeq} \mathcal{M}_{\mathrm{R}'}|_{\mathrm{Spec}(\mathrm{R})\times_{\mathrm{X}}\mathrm{Spec}(\mathrm{R}')} \tag{2.11}$$

for every pair of open immersions $\mathrm{Spec}(\mathrm{R}), \mathrm{Spec}(\mathrm{R}') \to \mathrm{X}$, satisfiying the cocycle condition for every triple of open immersions of affine schemes into X.

In this description, the functor (2.10) sends $(\{\mathcal{M}_{\mathrm{R}}\}, \{\varphi_{\mathrm{R},\mathrm{R}'}\})$ to the collection of R-modules $\mathcal{M}_{\mathrm{R}}$ equipped with the isomorphism $\varphi_{\mathrm{R},\mathrm{R}_f} : \mathcal{M}_{\mathrm{R}} \otimes_{\mathrm{R}} \mathrm{R}_f \xrightarrow{\simeq} \mathcal{M}_{\mathrm{R}_f}$ for every standard open $\mathrm{Spec}(\mathrm{R}_f) \to \mathrm{Spec}(\mathrm{R})$ over X.

The assertion that (2.10) is an equivalence amounts to the assertion that the family of isomorphisms (2.11) is uniquely determined by its subfamily where R' is a localization of R at some $f \in \mathrm{R}$. To prove this, we cover each $\mathrm{Spec}(\mathrm{R}) \times_{\mathrm{X}} \mathrm{Spec}(\mathrm{R}')$ by open affine subschemes $\mathrm{Spec}(\mathrm{R}_i)$ ($i \in \mathrm{I}$) which are standard opens in both $\mathrm{Spec}(\mathrm{R})$ and $\mathrm{Spec}(\mathrm{R}')$ (*cf.* Remark 1.6.11). The cocycle condition for the triple $\mathrm{Spec}(\mathrm{R}), \mathrm{Spec}(\mathrm{R}_i), \mathrm{Spec}(\mathrm{R}') \to \mathrm{X}$ ensures that

$$\varphi_{\mathrm{R},\mathrm{R}'}|_{\mathrm{Spec}(\mathrm{R}_i)} = (\varphi_{\mathrm{R}',\mathrm{R}_i})^{-1} \cdot \varphi_{\mathrm{R},\mathrm{R}_i}.$$

Finally, $\varphi_{\mathrm{R},\mathrm{R}'}$ is uniquely determined by its restrictions $\varphi_{\mathrm{R},\mathrm{R}'}|_{\mathrm{Spec}(\mathrm{R}_i)}$ over $i \in \mathrm{I}$, by Proposition 2.2.2 applied to the family of morphisms $\mathrm{Spec}(\mathrm{R}_i) \to \mathrm{Spec}(\mathrm{R}) \times_{\mathrm{X}} \mathrm{Spec}(\mathrm{R}')$ ($i \in \mathrm{I}$). □

Corollary 2.2.6. *Let* X *be a scheme. Then* $\mathsf{QCoh}(\mathrm{X})$ *is an abelian category.*

Proof. We use Corollary 2.2.5 and the fact that a limit of abelian categories $\mathcal{A} := \lim_{i \in \mathrm{I}} \mathcal{A}_i$, where the functors $\mathcal{A}_i \to \mathcal{A}_j$, for each morphism $i \to j$ in I, preserve finite limits and colimits, remains abelian. (For preservation of finite limits, we use the fact that localization is exact.) □

Remark 2.2.7. The proof of Corollary 2.2.6 shows that the finite limits and (arbitrary) colimits in $\mathsf{QCoh}(\mathrm{X})$ are computed by those in $\mathsf{Mod}_{\mathrm{R}}$ over $\mathrm{Spec}(\mathrm{R}) \in \mathrm{X}_{\mathrm{SZar}}$.

In particular, we see more generally that given an open immersion $f : \mathrm{U} \to \mathrm{X}$, the pullback functor $f^* : \mathsf{QCoh}(\mathrm{X}) \to \mathsf{QCoh}(\mathrm{U})$ is exact.

2.2.8. We shall adopt terminology from abelian categories to describe $\mathsf{QCoh}(\mathrm{X})$: given a monomorphism $\mathcal{M}_1 \to \mathcal{M}$, we call $\mathcal{M}_1$ a *subsheaf* of $\mathcal{M}$; given an epimorphism $\mathcal{M} \to \mathcal{M}_2$, we call $\mathcal{M}_2$ a *quotient sheaf* of $\mathcal{M}$.

Note that the properties of being monomorphism and epimorphisms can be verified over an affine open cover of X, where they amount to injectivity and surjectivity of module maps. In view of this fact, we also call monomorphisms in $\mathsf{QCoh}(\mathrm{X})$ *injections* and epimorphisms in $\mathsf{QCoh}(\mathrm{X})$ *surjections.*

Remark 2.2.9. As a consequence of Corollary 2.2.5, any construction on modules compatible with localization at an element also carries over to quasi-coherent sheaves (generalizing the discussion of §2.1.12).

For example, given $\mathrm{X} \in \mathsf{Sch}$ and a morphism $f : \mathcal{M}_1 \to \mathcal{M}_2$ in $\mathsf{QCoh}(\mathrm{X})$, the *image* of f is the subobject $\mathcal{N} \subset \mathcal{M}_2$ such that $\mathcal{N}|_x$ is the image of $\mathcal{M}_1|_x \to \mathcal{M}_2|_x$ for every open immersion $x : \mathrm{Spec}(\mathrm{R}) \to \mathrm{X}$. Note that this description is not valid for arbitrary R-points of X.

2.3. Pushforward.

2.3.1. Given a morphism $f : \mathrm{Y} \to \mathrm{X}$ of schemes, we obtain a pullback functor:

$$f^* : \mathsf{QCoh}(\mathrm{X}) \to \mathsf{QCoh}(\mathrm{Y}). \tag{2.12}$$

Since both $\mathsf{QCoh}(\mathrm{X})$ and $\mathsf{QCoh}(\mathrm{Y})$ have arbitrary colimits and f^* preserves them, the adjoint functor theorem[11] implies that (2.12) admits a right adjoint, called *pushforward*:

$$f_* : \mathsf{QCoh}(\mathrm{Y}) \to \mathsf{QCoh}(\mathrm{X}). \tag{2.13}$$

2.3.2. *Global sections.* Let us first understand f_* when the target X is affine. Suppose that $\mathrm{X} = \mathrm{Spec}(\mathrm{A})$ for some ring A, then $\mathsf{QCoh}(\mathrm{X})$ is canonically equivalent to $\mathsf{Mod}_{\mathrm{A}}$ (*cf.* Lemma 2.1.9), so $f_*\mathcal{N}$ can be viewed as an A-module for any $\mathcal{N} \in \mathsf{QCoh}(\mathrm{Y})$. Its underlying set is computed by:

$$\begin{aligned}\mathrm{Hom}(\mathrm{A}, f_*\mathcal{N}) &\xrightarrow{\simeq} \mathrm{Hom}_{\mathsf{QCoh}(\mathrm{Y})}(f^*\mathrm{A}, \mathcal{N}) \\ &\xrightarrow{\simeq} \mathrm{Hom}_{\mathsf{QCoh}(\mathrm{Y})}(\mathcal{O}_{\mathrm{Y}}, \mathcal{N})\end{aligned}$$

Thus, an element of $f_*\mathcal{N}$ corresponds to a morphism $\mathcal{O}_{\mathrm{Y}} \to \mathcal{N}$ in $\mathsf{QCoh}(\mathrm{Y})$. Thus we call $f_*\mathcal{N}$ the A-module of *global sections* of $\mathcal{N}$. It has the following alternative notations:

$$\Gamma(\mathrm{Y}, \mathcal{N}) \coloneqq \mathrm{H}^0(\mathrm{Y}, \mathcal{N}) \coloneqq f_*\mathcal{N}.$$

Remark 2.3.3. As a special case, if $f : \mathrm{Spec}(\mathrm{B}) \to \mathrm{Spec}(\mathrm{A})$ is a morphism of affine schemes, then f_* corresponds to the functor $\mathsf{Mod}_{\mathrm{B}} \to \mathsf{Mod}_{\mathrm{A}}$ of restriction of scalars. Indeed, this is the right adjoint of the functor $(\cdot) \otimes_{\mathrm{A}} \mathrm{B} : \mathsf{Mod}_{\mathrm{A}} \to \mathsf{Mod}_{\mathrm{B}}$, which corresponds to f^*.

Remark 2.3.4. In general, the formation of f_* is not "local on the target", which causes headache. For example, consider the Cartesian square of schemes:

$$\begin{array}{ccc}\bigsqcup_{i\in\mathrm{I}} \mathbb{A}^1_{\mathbf{Z}} \smallsetminus 0 & \longrightarrow & \bigsqcup_{i\in\mathrm{I}} \mathbb{A}^1_{\mathbf{Z}} \\ \big\downarrow f' & & \big\downarrow f \\ \mathbb{A}^1_{\mathbf{Z}} \smallsetminus 0 & \longrightarrow & \mathbb{A}^1_{\mathbf{Z}}\end{array}$$

where I is an *infinite* set, f identifies each copy of $\mathbb{A}^1_{\mathbf{Z}}$ with the target, and the lower horizontal arrow corresponds to the localization of $\mathbf{Z}[x]$ at x. Then f_* carries the structure sheaf to the $\mathbf{Z}[x]$-module $\prod_{i\in\mathrm{I}} \mathbf{Z}[x]$, whereas $(f')_*$ carries the structure sheaf to $\prod_{i\in\mathrm{I}} \mathbf{Z}[x, x^{-1}]$. The natural map of $\mathbf{Z}[x, x^{-1}]$-modules below is *not* bijective:

$$\Big(\prod_{i\in\mathrm{I}} \mathbf{Z}[x]\Big) \otimes_{\mathbf{Z}[x]} \mathbf{Z}[x, x^{-1}] \to \prod_{i\in\mathrm{I}} \mathbf{Z}[x, x^{-1}].$$

2.3.5. *The base change morphism.* We shall prove that a finiteness condition on f—being quasi-compact and quasi-separated—prevents the pathology of Remark 2.3.4. Before stating the result, let us first explain the general paradigm of "base change" morphisms.

Given a Cartesian diagram in Sch:

$$\begin{array}{ccc}\mathrm{Y}' & \xrightarrow{g'} & \mathrm{Y} \\ \big\downarrow f' & & \big\downarrow f \\ \mathrm{X}' & \xrightarrow{g} & \mathrm{X}\end{array} \tag{2.14}$$

[11] In order to apply the adjoint functor theorem, we need to know that $\mathsf{QCoh}(\mathrm{X})$ and $\mathsf{QCoh}(\mathrm{Y})$ are *presentable*, which involves a set-theoretic condition ("accessibility") in addition to containing all colimits. This set-theoretic condition is indeed satisfied (*cf.* [Sta18, 077K]). However, its proof is not trivial, so it is somewhat disingenuous to invoke this result. In fact, we will only use f_* when f is quasi-compact and quasi-separated, in which case we will construct it explicitly (*cf.* Proposition 2.3.9).

there is a canonical natural transformation of functors from $\mathsf{QCoh}(\mathrm{Y})$ to $\mathsf{QCoh}(\mathrm{X}')$, obtained from the isomorphism $f_*(g')_* \xrightarrow{\simeq} g_*(f')_*$ by adjunction:

$$g^* f_* \to (f')_*(g')^*. \tag{2.15}$$

The natural transformation (2.15) is called the *base change* morphism.

Remark 2.3.6. The base change morphism (2.15) is compatible with concatenation of Cartesian squares. Namely, given Cartesian squares:

$$\begin{array}{ccccc} \mathrm{Y}'' & \xrightarrow{h'} & \mathrm{Y}' & \xrightarrow{g'} & \mathrm{Y} \\ \downarrow{\scriptstyle f''} & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathrm{X}'' & \xrightarrow{h} & \mathrm{X}' & \xrightarrow{g} & \mathrm{X} \end{array} \tag{2.16}$$

Then pulling back (2.15) by h yields a natural transformation:

$$\begin{aligned} (gh)^* f_* \cong h^* g^* f_* &\to h^*(f')_*(g')^* \\ &\to (f'')_*(h')^*(g')^* \cong (f'')_*(g'h')^*, \end{aligned} \tag{2.17}$$

where we use the base change morphism for the left square in (2.16). The natural transformation (2.17) equals the base change morphism associated to the outer rectangle of (2.16).

Lemma 2.3.7. *Suppose that g is a flat morphism of affine schemes (*cf. *Lemma 1.8.9) and f is quasi-compact and quasi-separated. Then* (2.15) *is an isomorphism.*

Proof. We write g as $\mathrm{Spec}(\mathrm{A}') \to \mathrm{Spec}(\mathrm{A})$ for a flat ring map $\mathrm{A} \to \mathrm{A}'$. Since the target of f is affine, Y is itself quasi-compact and quasi-separated. Thus we may present Y as the coequalizer of affine schemes:

$$\textstyle\bigsqcup_{\substack{i,j\in \mathrm{I} \\ k\in \mathrm{K}_{ij}}} \mathrm{Spec}(\mathrm{B}_{ijk}) \rightrightarrows \bigsqcup_{i\in\mathrm{I}} \mathrm{Spec}(\mathrm{B}_i) \to \mathrm{Y}$$

where the index sets I and K_{ij} ($i, j \in \mathrm{I}$) are finite (*cf.* §1.7.9).

Given $\mathcal{N} \in \mathsf{QCoh}(\mathrm{Y})$, we may apply descent of QCoh (*cf.* Proposition 2.2.2) to the cover $\mathrm{Spec}(\mathrm{B}_i) \to \mathrm{Y}$ ($i \in \mathrm{I}$) to express the A-module $f_*\mathcal{N}$ as the equalizer of the morphisms:

$$\textstyle\bigoplus_{i\in\mathrm{I}} \mathcal{N}|_{\mathrm{Spec}(\mathrm{B}_i)} \rightrightarrows \bigoplus_{\substack{i,j\in \mathrm{I} \\ k\in \mathrm{K}_{ij}}} \mathcal{N}|_{\mathrm{Spec}(\mathrm{B}_{ijk})} \tag{2.18}$$

The functor g^* corresponds to $(\cdot)\otimes_{\mathrm{A}} \mathrm{A}'$. Since $\mathrm{A} \to \mathrm{A}'$ is flat, $(f_*\mathcal{N})\otimes_{\mathrm{A}} \mathrm{A}'$ is identified with the equalizer of the tensor product of (2.18) with A', which is identified with $(f')_*(g')^*\mathcal{N}$ by covering Y' with $\mathrm{Spec}(\mathrm{B}_i \otimes_{\mathrm{A}} \mathrm{A}')$ and their overlaps by $\mathrm{Spec}(\mathrm{B}_{ijk} \otimes_{\mathrm{A}} \mathrm{A}')$. □

2.3.8. We shall use Lemma 2.3.7 to calculate f_* for any quasi-compact quasi-separated morphism of schemes $f : \mathrm{Y} \to \mathrm{X}$.

Recall that $\mathsf{QCoh}(\mathrm{X})$ is identified with the limit of the categories $\mathsf{Mod}_{\mathrm{R}}$ over $\mathrm{Spec}(\mathrm{R}) \to \mathrm{X}$ in $\mathrm{X}_{\mathrm{SZar}}$ (*cf.* Corollary 2.2.5). Identifying $f_*\mathcal{N}$, for $\mathcal{N} \in \mathsf{QCoh}(\mathrm{Y})$, amounts to giving an R-module $(f_*\mathcal{N})|_x$ for each open immersion $x : \mathrm{Spec}(\mathrm{R}) \to \mathrm{X}$ together with compatibility isomorphisms:

$$(f_*\mathcal{N})|_x \otimes_{\mathrm{R}} \mathrm{R}' \xrightarrow{\simeq} (f_*\mathcal{N})|_{x'} \tag{2.19}$$

for each standard open $\mathrm{Spec}(\mathrm{R}') \to \mathrm{Spec}(\mathrm{R})$ with $x' := x|_{\mathrm{R}'}$.

For each $x : \mathrm{Spec}(\mathrm{R}) \to \mathrm{X}$, we write $\mathrm{Y}_x := \mathrm{Y} \times_{\mathrm{X}} \mathrm{Spec}(\mathrm{R})$ and $\mathcal{N}|_{\mathrm{Y}_x} \in \mathsf{QCoh}(\mathrm{Y}_x)$ for the pullback of $\mathcal{N}$ to Y_x.

Proposition 2.3.9. *Let $f : \mathrm{Y} \to \mathrm{X}$ be a quasi-compact quasi-separated morphism of schemes. Let $\mathcal{N} \in \mathsf{QCoh}(\mathrm{Y})$. Then $f_*\mathcal{N}$ corresponds to the family of* R*-modules:*

$$(f_*\mathcal{N})|_x := \Gamma(\mathrm{Y}_x, \mathcal{N}|_{\mathrm{Y}_x})$$

for each open immersion $x : \mathrm{Spec}(\mathrm{R}) \to \mathrm{X}$, with compatibility (2.19) *supplied by the base change isomorphism (*cf. *Lemma 2.3.7).*

Proof. This family defines an object of $\mathsf{QCoh}(\mathrm{X})$ by the equivalence of Corollary 2.2.5. It remains to supply a natural bijection for each $\mathcal{M} \in \mathsf{QCoh}(\mathrm{X})$:

$$\mathrm{Hom}_{\mathsf{QCoh}(\mathrm{Y})}(f^*\mathcal{M}, \mathcal{N}) \xrightarrow{\simeq} \lim_{\substack{\mathrm{Spec}(\mathrm{R}) \to \mathrm{X} \\ \text{in } \mathrm{X}_{\mathrm{SZar}}}} \mathrm{Hom}(\mathcal{M}|_x, \Gamma(\mathrm{Y}_x, \mathcal{N}|_{\mathrm{Y}_x})) \tag{2.20}$$

By adjunction for the projection map $f_x : \mathrm{Y}_x \to \mathrm{Spec}(\mathrm{R})$, we have:

$$\mathrm{Hom}(\mathcal{M}|_x, \Gamma(\mathrm{Y}_x, \mathcal{N}|_{\mathrm{Y}_x}) \xrightarrow{\simeq} \mathrm{Hom}_{\mathsf{QCoh}(\mathrm{Y}_x)}((f_x)^*(\mathcal{M}|_x), \mathcal{N}|_{\mathrm{Y}_x}).$$

Note that $(f_x)^*(\mathcal{M}|_x)$ is identified with the pullback of $f^*\mathcal{M}$ to Y_x. Moreoever, for any $\mathcal{N}_1, \mathcal{N}_2 \in \mathsf{QCoh}(\mathrm{Y})$, the limit of $\mathrm{Hom}_{\mathsf{QCoh}(\mathrm{Y}_x)}(\mathcal{N}_1|_{\mathrm{Y}_x}, \mathcal{N}_2|_{\mathrm{Y}_x})$ over $\mathrm{X}_{\mathrm{SZar}}$ is identified with $\mathrm{Hom}_{\mathsf{QCoh}(\mathrm{Y})}(\mathcal{N}_1, \mathcal{N}_2)$ by descent of QCoh (*cf.* the proof of Corollary 2.2.5). This yields the desired isomorphism (2.20). □

Corollary 2.3.10. *Let $f : \mathrm{Y} \to \mathrm{X}$ be a morphism in Sch. If f is quasi-compact and quasi-separated, then $f_* : \mathsf{QCoh}(\mathrm{Y}) \to \mathsf{QCoh}(\mathrm{X})$ commutes with filtered colimits.*

Proof. The assertion reduces to the case where $\mathrm{X} = \mathrm{Spec}(\mathrm{A})$ is an affine scheme (*cf.* Proposition 2.3.9). Thus Y is quasi-compact and quasi-separated, so $f_*\mathcal{M} \cong \Gamma(\mathrm{Y}, \mathcal{M})$ is given by a finite limit in $\mathsf{Mod}_{\mathrm{A}}$ (*cf.* the proof of Lemma 2.3.7). We use the fact that filtered colimits commute with finite limits in $\mathsf{Mod}_{\mathrm{A}}$. □

2.3.11. *Projection formula (baby version).* We shall use the above corollaries to prove an extremely useful result concerning the interaction between f_* and f^*.

Namely, let $f : \mathrm{Y} \to \mathrm{X}$ be a morphism of schemes. Let $\mathcal{M} \in \mathsf{QCoh}(\mathrm{X})$ and $\mathcal{N} \in \mathsf{QCoh}(\mathrm{Y})$. Then there is a morphism in $\mathsf{QCoh}(\mathrm{X})$ natural in $\mathcal{M}$ and $\mathcal{N}$:

$$\mathcal{M} \otimes_{\mathcal{O}_{\mathrm{X}}} f_*\mathcal{N} \to f_*(f^*\mathcal{M} \otimes_{\mathcal{O}_{\mathrm{Y}}} \mathcal{N}). \tag{2.21}$$

Indeed, this is the map obtained by adjunction from the composition:

$$f^*(\mathcal{M} \otimes_{\mathcal{O}_{\mathrm{X}}} f_*\mathcal{N}) \xrightarrow{\simeq} f^*\mathcal{M} \otimes_{\mathcal{O}_{\mathrm{Y}}} f^*f_*\mathcal{N} \to f^*\mathcal{M} \otimes_{\mathcal{O}_{\mathrm{Y}}} \mathcal{N}.$$

Proposition 2.3.12. *Let $f : \mathrm{Y} \to \mathrm{X}$ be a morphism of schemes. Suppose that f is quasi-compact quasi-separated, and $\mathcal{M}$ is locally free. Then* (2.21) *is an isomorphism.*

Proof. Since f is quasi-compact quasi-separated, we may reduce to the case where $\mathrm{X} = \mathrm{Spec}(\mathrm{A})$ is affine (*cf.* Proposition 2.3.9) and $\mathcal{M}$ is free. Note that (2.21) is an isomorphism for $\mathcal{M} = \mathrm{A}$, and its two sides, viewed as functors in $\mathcal{M} \in \mathsf{Mod}_{\mathrm{A}}$, commute with direct sums because they commute with finite sums and filtered colimits (*cf.* Corollary 2.3.10). □

2.4. Relative spectra.

2.4.1. A morphism $f : \mathrm{Y} \to \mathrm{X}$ of schemes is called *affine* if for every R-point x of X ($\mathrm{R} \in \mathsf{Ring}$), the fiber product $\mathrm{Y} \times_{\mathrm{X}} \mathrm{Spec}(\mathrm{R})$ is an affine scheme.

The notion of affine morphisms can be viewed as a relative version of the notion of affine schemes. We shall use the pushforward functor on QCoh to construct a relative version of the adjunction between schemes and affine schemes (*cf.* Corollary 1.6.9).

First, we note that pushforward along an affine morphism is especially pleasant.

Lemma 2.4.2. *Let $f : \mathrm{Y} \to \mathrm{X}$ be an affine morphism in* Sch. *Then:*

(1) *the functor $f_* : \mathsf{QCoh}(\mathrm{Y}) \to \mathsf{QCoh}(\mathrm{X})$ is exact;*
(2) *the natural morphism* (2.21) *is an isomorphism for any $\mathcal{M} \in \mathsf{QCoh}(\mathrm{X})$, $\mathcal{N} \in \mathsf{QCoh}(\mathrm{Y})$:*

$$\mathcal{M} \otimes_{\mathcal{O}_{\mathrm{X}}} f_*\mathcal{N} \to f_*(f^*\mathcal{M} \otimes_{\mathcal{O}_{\mathrm{Y}}} \mathcal{N}).$$

Proof. Since affine morphisms are quasi-compact quasi-separated, both assertions reduce to the case where $\mathrm{X} = \mathrm{Spec}(\mathrm{A})$ is an affine scheme (*cf.* Proposition 2.3.9).

Since f is affine, $\mathrm{Y} = \mathrm{Spec}(\mathrm{B})$ is also affine, so f_* corresponds to restriction of scalar along the ring map $\mathrm{A} \to \mathrm{B}$, which is exact. This proves (1). Statement (2) follows from the fact that given $\mathrm{M} \in \mathsf{Mod}_{\mathrm{A}}$ and $\mathrm{N} \in \mathsf{Mod}_{\mathrm{B}}$, the natural map:

$$\mathrm{M} \otimes_{\mathrm{A}} \mathrm{N} \to (\mathrm{M} \otimes_{\mathrm{A}} \mathrm{B}) \otimes_{\mathrm{B}} \mathrm{N}$$

is an isomorphism. □

2.4.3. Given a symmetric monoidal category $\mathcal{O}$, we write $\mathsf{CAlg}(\mathcal{O})$ for the category of commutative monoids in $\mathcal{O}$. (If we write $\otimes$ for the monoidal product of $\mathcal{O}$ and $\mathbf{1} \in \mathcal{O}$ for its monoidal unit, then objects of $\mathsf{CAlg}(\mathcal{O})$ are objects $\mathrm{A} \in \mathcal{O}$ equipped with maps $\mathrm{A} \otimes \mathrm{A} \to \mathrm{A}$ and $\mathbf{1} \to \mathrm{A}$ satisfying the unitality, associativity, and commutativity axioms.)

Note that Ring is equivalent to $\mathsf{CAlg}(\mathsf{Mod}_{\mathbf{Z}})$, by definition. More generally, the category $\mathsf{Ring}_{\mathrm{R}/}$ of R-algebras ($\mathrm{R} \in \mathsf{Ring}$) is equivalent to $\mathsf{CAlg}(\mathsf{Mod}_{\mathrm{R}})$.

2.4.4. Given a scheme X, an object of $\mathsf{CAlg}(\mathsf{QCoh}(\mathrm{X}))$ is called a *quasi-coherent sheaf of $\mathcal{O}_{\mathrm{X}}$-algebras.* We shall define a *relative spectrum* functor:

$$\mathsf{CAlg}(\mathsf{QCoh}(\mathrm{X}))^{\mathrm{op}} \to \mathsf{Sch}_{/\mathrm{X}}, \quad \mathcal{B} \mapsto \mathrm{Spec}_{\mathrm{X}}(\mathcal{B}), \tag{2.22}$$

where an R-point of the presheaf $\mathrm{Spec}_{\mathrm{X}}(\mathcal{B})$ consists of an R-point x of X, together with a morphism $x^*\mathcal{B} \to \mathrm{R}$ of R-algebras. The natural map $\mathrm{Spec}_{\mathrm{X}}(\mathcal{B}) \to \mathrm{X}$ is the one remembering the R-point x. (For $\mathrm{X} = \mathrm{Spec}(\mathbf{Z})$, we recover the functor Spec.)

2.4.5. We will prove in Proposition 2.4.6 below that $\mathrm{Spec}_{\mathrm{X}}(\mathcal{B})$ is indeed a scheme, so the functor (2.22) is well-defined. But first, we note that the construction of $\mathrm{Spec}_{\mathrm{X}}(\mathcal{B})$ as a presheaf over X is functorial in X in the following sense.

Given any morphism of schemes $f : \mathrm{Y} \to \mathrm{X}$, we have a Cartesian diagram in PShv:

$$\begin{array}{ccc} \mathrm{Spec}_{\mathrm{Y}}(f^*\mathcal{B}) & \to & \mathrm{Spec}_{\mathrm{X}}(\mathcal{B}) \\ \downarrow & & \downarrow \\ \mathrm{Y} & \xrightarrow{f} & \mathrm{X} \end{array} \tag{2.23}$$

Indeed, given an R-point y of Y with induced R-point $x := f(y)$ of X, the datum of an R-algebra morphism $y^*f^*\mathcal{B} \to \mathrm{R}$ is equivalent to $x^*\mathcal{B} \to \mathrm{R}$.

Proposition 2.4.6. *Let* X *be a scheme and $\mathcal{B}$ be a quasi-coherent $\mathcal{O}_{\mathrm{X}}$-algebra. Then:*

(1) *the presheaf $\mathrm{Spec}_{\mathrm{X}}(\mathcal{B})$ is a scheme;*
(2) *the structural morphism $\mathrm{Spec}_{\mathrm{X}}(\mathcal{B}) \to \mathrm{X}$ is affine.*

Proof. The fact that $\mathrm{Spec}_{\mathrm{X}}(\mathcal{B})$ is a Zariski sheaf follows from descent of QCoh (*cf.* Proposition 2.2.2). For (1), it remains to construct an open cover of $\mathrm{Spec}_{\mathrm{X}}(\mathcal{B})$ by affine schemes.

The Cartesian diagram (2.23) shows that if $f : \mathrm{Y} \to \mathrm{X}$ is an open immersion (respectively, surjective on field-valued points) of schemes, then the same holds for the induced morphism $\mathrm{Spec}_{\mathrm{Y}}(f^*\mathcal{B}) \to \mathrm{Spec}_{\mathrm{X}}(\mathcal{B})$ in Shv.

Now, if $\mathrm{X} = \mathrm{Spec}(\mathrm{A})$ is affine, then $\mathcal{B}$ can be identified with an A-algebra B, and $\mathrm{Spec}_{\mathrm{X}}(\mathcal{B})$ is represented by $\mathrm{Spec}(\mathrm{B})$. Combined with the above observation, this shows that given an open cover $f_i : \mathrm{X}_i \to \mathrm{X}$ ($i \in \mathrm{I}$) where each X_i is an affine scheme, $\mathrm{Spec}_{\mathrm{X}_i}((f_i)^*\mathcal{B}) \to \mathrm{Spec}_{\mathrm{X}}(\mathcal{B})$ ($i \in \mathrm{I}$) is an open cover by affine schemes.

The above paragraph also proves (2). □

Remark 2.4.7 (Total spaces of vector bundles)**.** Let us give an application of Proposition 2.4.6, by representing vector bundles by schemes.

Recall that every $\mathcal{M} \in \mathsf{QCoh}(\mathrm{X})$ has an underlying object of $\mathsf{Shv}_{/\mathrm{X}}$, carrying an R-point x of X ($\mathrm{R} \in \mathsf{Ring}$) to the set underlying $x^*\mathcal{M} \in \mathsf{Mod}_{\mathrm{R}}$.

We claim that when $\mathcal{M}$ is finite locally free, this object of $\mathsf{Shv}_{/\mathrm{X}}$ is representable by a scheme, called the *total space* of $\mathcal{M}$. Indeed, we set:

$$\mathbb{V}(\mathcal{M}) \coloneqq \mathrm{Spec}_{\mathrm{X}}(\mathrm{Sym}\,\mathcal{M}^\vee) \in \mathsf{Sch}_{/\mathrm{X}},$$

where $\mathcal{M}^\vee$ is the $\mathcal{O}_{\mathrm{X}}$-module dual of $\mathcal{M}$. By definition of $\mathrm{Spec}_{\mathrm{X}}$, an R-point of $\mathbb{V}(\mathcal{M})$ consists of an R-point x of X together with a morphism $\mathrm{Sym}_{\mathrm{R}}(x^*\mathcal{M}^\vee) \to \mathrm{R}$ in $\mathsf{CAlg}(\mathsf{Mod}_{\mathrm{R}})$, or equivalently a morphism $x^*\mathcal{M}^\vee \to \mathrm{R}$ in $\mathsf{Mod}_{\mathrm{R}}$, *i.e.* an element of $x^*\mathcal{M}$.

2.4.8. Given a morphism $f : \mathrm{Y} \to \mathrm{X}$ of schemes, the pullback functor $f^* : \mathsf{QCoh}(\mathrm{X}) \to \mathsf{QCoh}(\mathrm{Y})$ is symmetric monoidal. By formal nonsense, this implies that the adjunction (f^*, f_*) lifts to an adjunction on the category of commutative monoids:

$$f^* : \mathsf{CAlg}(\mathsf{QCoh}(\mathrm{X})) \rightleftarrows \mathsf{CAlg}(\mathsf{QCoh}(\mathrm{Y})) : f_* \tag{2.24}$$

Explicitly, the algebra structure on $f_*\mathcal{R}$ (for $\mathcal{R} \in \mathsf{CAlg}(\mathsf{QCoh}(\mathrm{Y}))$) can be described as follows. The mutiplication is the map $f_*\mathcal{R} \otimes_{\mathcal{O}_{\mathrm{X}}} f_*\mathcal{R} \to f_*\mathcal{R}$, which under adjunction, corresponds to the composition:

$$f^*(f_*\mathcal{R} \otimes_{\mathcal{O}_{\mathrm{X}}} f_*\mathcal{R}) \xrightarrow{\simeq} f^*f_*\mathcal{R} \otimes_{\mathcal{O}_{\mathrm{Y}}} f^*f_*\mathcal{R} \to \mathcal{R} \otimes_{\mathcal{O}_{\mathrm{Y}}} \mathcal{R} \to \mathcal{R},$$

and the unit $\mathcal{O}_{\mathrm{X}} \to f_*\mathcal{R}$ corresponds to $f^*\mathcal{O}_{\mathrm{X}} \xrightarrow{\simeq} \mathcal{O}_{\mathrm{Y}} \to \mathcal{R}$.

Remark 2.4.9. If $f : \mathrm{Spec}(\mathrm{B}) \to \mathrm{Spec}(\mathrm{A})$ is a morphism of affine schemes, then (2.24) is the familiar adjunction between A-algebras and B-algebras (viewed as under-categories of Ring), where the right adjoint is the restriction of structure map along $\mathrm{A} \to \mathrm{B}$, and the left adjoint is taking pushout along $\mathrm{A} \to \mathrm{B}$.

2.4.10. We view $\mathcal{O}_{\mathrm{Y}}$ as an object of $\mathsf{CAlg}(\mathsf{QCoh}(\mathrm{Y}))$, so $f_*\mathcal{O}_{\mathrm{Y}}$ is an object of $\mathsf{CAlg}(\mathsf{QCoh}(\mathrm{X}))$. The association from $f : \mathrm{Y} \to \mathrm{X}$ to $f_*\mathcal{O}_{\mathrm{Y}}$ defines a functor:

$$\mathsf{Sch}_{/\mathrm{X}} \to \mathsf{CAlg}(\mathsf{QCoh}(\mathrm{X}))^{\mathrm{op}}, \quad (f : \mathrm{Y} \to \mathrm{X}) \mapsto f_*\mathcal{O}_{\mathrm{Y}}. \tag{2.25}$$

Proposition 2.4.11. *Let* X *be a scheme. Then:*

(1) (2.25) *is the left adjoint of* (2.22)*;*
(2) (2.22) *is fully faithful;*
(3) given a morphism $f : \mathrm{Y} \to \mathrm{X}$ *in* Sch*, the unit of the adjunction:*

$$\mathrm{Y} \to \mathrm{Spec}_{\mathrm{X}}(f_*\mathcal{O}_{\mathrm{Y}}) \tag{2.26}$$

is an isomorphism if and only if f *is affine.*

Proof. Statement (1) means that we have a natural bijection:

$$\mathrm{Hom}_{\mathsf{CAlg}(\mathsf{QCoh}(\mathrm{X}))}(\mathcal{B}, f_*\mathcal{O}_{\mathrm{Y}}) \xrightarrow{\simeq} \mathrm{Hom}_{\mathsf{Sch}_{/\mathrm{X}}}(\mathrm{Y}, \mathrm{Spec}_{\mathrm{X}}(\mathcal{B})), \tag{2.27}$$

for each $\mathcal{B} \in \mathsf{CAlg}(\mathsf{QCoh}(\mathrm{X}))$ and morphism $f : \mathrm{Y} \to \mathrm{X}$ in Sch. If $\mathrm{Y} = \mathrm{Spec}(\mathrm{R})$ is affine, then the bijection (2.27) is a restatement of the definition of $\mathrm{Spec}_{\mathrm{X}}(\mathcal{B})$ as a presheaf. The general case follows from this one, by writing Y as a colimit of affine schemes $\mathrm{Spec}(\mathrm{R})$ indexed by $\mathrm{Y}_{\mathrm{SZar}}$ (*cf.* Corollary 1.6.14, Corollary 2.2.5).

For statement (2), we need to prove that the co-unit for each $\mathcal{B} \in \mathsf{CAlg}(\mathsf{QCoh}(\mathrm{X}))$:

$$\mathcal{B} \to f_* \mathcal{O}_{\mathrm{Y}}, \tag{2.28}$$

where $\mathrm{Y} := \mathrm{Spec}_{\mathrm{X}}(\mathcal{B})$, is an isomorphism. Since f is affine, we may reduce to the case where $\mathrm{X} = \mathrm{Spec}(\mathrm{A})$ is affine (*cf.* Proposition 2.3.9). In this case, $\mathcal{B}$ corresponds to an A-algebra B and (2.28) is the identity on B.

For statement (3), if (2.26) is an isomorphism, then f is clearly affine. Conversely, assume that f is affine. To prove that (2.26) is an isomorphism, we may reduce to the case where $\mathrm{X} = \mathrm{Spec}(\mathrm{A})$ is affine (*cf.* Proposition 2.3.9). In this case, Y corresponds to an A-algebra B and (2.26) is the identity on Spec(B). ☐

Remark 2.4.12. The adjunction of Proposition 2.4.11 shows that every morphism $f : \mathrm{Y} \to \mathrm{X}$ in Sch canonically factors as:

$$\begin{array}{ccc} \mathrm{Y} & \xrightarrow{f'} & \mathrm{X}' \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle g} \\ & & \mathrm{X} \end{array} \quad \text{with } \mathrm{X}' := \mathrm{Spec}_{\mathrm{X}}(f_* \mathcal{O}_{\mathrm{Y}}),$$

where g is affine and f' has the property that the unit $\mathcal{O}_{\mathrm{X}'} \to (f')_* \mathcal{O}_{\mathrm{Y}}$ is an isomorphism in $\mathsf{CAlg}(\mathsf{QCoh}(\mathrm{X}'))$, as it becomes an isomorphism upon applying g_*. This is called the *Stein factorization* of the morphism f.

Corollary 2.4.13. *Let* X *be a scheme. Then the functor* (2.22) *is an equivalence between* $\mathsf{CAlg}(\mathsf{QCoh}(\mathrm{X}))^{\mathrm{op}}$ *and the category of affine morphisms* $\mathrm{Y} \to \mathrm{X}$.

Proof. This is a formal consequence of Proposition 2.4.11. ☐

Corollary 2.4.14 (Affine pushfoward commutes with any base change)**.** *Given a Cartesian square* (2.14) *in* Sch*, where* f *is affine, the base change map* (2.15) *is an isomorphism.*

Proof. Since f is affine, in particular quasi-compact quasi-separated, we reduce to the case where $g : \mathrm{Spec}(\mathrm{A}') \to \mathrm{Spec}(\mathrm{A})$ is a morphism of affine schemes (*cf.* Proposition 2.3.9). Then Y is an affine scheme Spec(B) for an A-algebra B. The functor f_* corresponds to the restriction of scalar functor $\mathsf{Mod}_{\mathrm{B}} \to \mathsf{Mod}_{\mathrm{A}}$, so the base change isomorphism follows from the canonical isomorphism of A′-modules:

$$\mathrm{N} \otimes_{\mathrm{A}} \mathrm{A}' \xrightarrow{\simeq} \mathrm{N} \otimes_{\mathrm{B}} \mathrm{B}',$$

for $\mathrm{B}' := \mathrm{B} \otimes_{\mathrm{A}} \mathrm{A}'$. ☐

Corollary 2.4.15. *Affineness of morphisms in* Sch *is local on the target (*cf. *§1.7.1).*

Proof. We use Lemma 1.5.10 to reduce to the case of standard open covers: Given a standard open cover $\mathrm{Spec}(\mathrm{A}_i) \to \mathrm{Spec}(\mathrm{A})$ $(i \in \mathrm{I})$ and a morphism of schemes $f : \mathrm{Y} \to \mathrm{Spec}(\mathrm{A})$ such that each base change $f_i : \mathrm{Y}_i := \mathrm{Y} \times_{\mathrm{Spec}(\mathrm{A})} \mathrm{Spec}(\mathrm{A}_i) \to \mathrm{Spec}(\mathrm{A}_i)$ is affine, we want to prove that f is affine. Since quasi-compactness and quasi-separatedness are local on the target (*cf.* Lemma 1.7.5, Lemma 1.7.7), we know that f is quasi-compact quasi-separated.

By Proposition 2.4.11, f is affine if and only if the unit morphism $\mathrm{Y} \to \mathrm{Spec}(f_* \mathcal{O}_{\mathrm{Y}})$ is an isomorphism. Since f is quasi-compct quasi-separated, the formation of $f_* \mathcal{O}_{\mathrm{Y}}$ is local on the target (*cf.* Proposition 2.3.9), so the unit morphism is an isomorphism after base change to $\mathrm{Spec}(\mathrm{A}_i)$ for each $i \in \mathrm{I}$. We conclude because being an isomorphism is local on the target (*cf.* Lemma 1.4.8). ☐

2.5. Closed immersions.

2.5.1. A morphism $f : \mathrm{Y} \to \mathrm{X}$ of schemes is a *closed immersion* if for every R-point x of X ($\mathrm{R} \in \mathsf{Ring}$), the fiber product $\mathrm{Y} \times_{\mathrm{X}} \mathrm{Spec(R)}$ is an affine scheme $\mathrm{Spec(R')}$ such that the induced ring map $\mathrm{R} \to \mathrm{R'}$ is surjective.

In particular, closed immersions are affine and monomorphisms. They are closed under composition and stable under base change, and satisfy the permanence property.

We call a morphism $f : \mathrm{Y} \to \mathrm{X}$ of schemes a *locally closed immersion* if it can be factored as $f = j \cdot i$, where i is a closed immersion and j is an open immersion.

2.5.2. Given a scheme X, the equivalence of Corollary 2.4.13 restricts to an equivalence between closed immersions $f : \mathrm{Y} \to \mathrm{X}$ and quotient algebras of $\mathcal{O}_{\mathrm{X}}$. To a closed immersion $f : \mathrm{Y} \to \mathrm{X}$, we may associate a short exact sequence in $\mathsf{QCoh}(\mathrm{X})$:

$$0 \to \mathcal{I} \to \mathcal{O}_{\mathrm{X}} \to f_*\mathcal{O}_{\mathrm{Y}} \to 0, \tag{2.29}$$

where $\mathcal{I}$ is called the "ideal sheaf" associated to Y.

Conversely, we may call an *ideal sheaf* any subobject of $\mathcal{O}_{\mathrm{X}}$ in $\mathsf{QCoh}(\mathrm{X})$, so we also obtain an equivalence between ideal sheaves over X and closed immersions with target X.

Lemma 2.5.3. *The property of being a closed immersion is local on the target.*

Proof. Using Corollary 2.4.15 and Lemma 1.5.10, we reduce to the following assertion: Given a standard open cover $\mathrm{Spec}(\mathrm{A}_i) \to \mathrm{Spec(A)}$ ($i \in \mathrm{I}$) and a morphism $\mathrm{Spec(B)} \to \mathrm{Spec(A)}$ such that each base change $\mathrm{Spec}(\mathrm{B} \otimes_{\mathrm{A}} \mathrm{A}_i) \to \mathrm{Spec}(\mathrm{A}_i)$ is a closed immersion, then so is $\mathrm{Spec(B)} \to \mathrm{Spec(A)}$. This follows from the fact that surjectivity of module maps can be verified over a standard open cover. ☐

2.5.4. *Infinitesimal neighborhoods.* Given a closed immersion $f : \mathrm{Y} \to \mathrm{X}$ corresponding to the ideal sheaf $\mathcal{I} \subset \mathcal{O}_{\mathrm{X}}$ and an integer $n \geq 0$, we obtain an ideal sheaf $\mathcal{I}^{n+1} \subset \mathcal{O}_{\mathrm{X}}$.

More precisely, for any open immersion $x : \mathrm{Spec(R)} \to \mathrm{X}$, we set $\mathcal{I}^{n+1}|_x$ to be the ideal $\mathrm{I}^n \subset \mathrm{R}$ for $\mathrm{I} := \mathcal{I}|_x$. This construction is compatible with localization because $\mathrm{I}^n \subset \mathrm{R}$ is the image of the multiplication map $\mathrm{I} \otimes_{\mathrm{R}} \cdots \otimes_{\mathrm{R}} \mathrm{I} \to \mathrm{R}$ (for $n+1$ copies of I), so $\mathcal{I}^{n+1}$ is well-defined as a subobject of $\mathcal{O}_{\mathrm{X}}$ in $\mathsf{QCoh}(\mathrm{X})$ (*cf.* Remark 2.2.9).

The closed immersion $\mathrm{Y}^{(n+1)} \to \mathrm{X}$ corresponding to the ideal sheaf $\mathcal{I}^{n+1}$ is called the *nth order infinitesimal neighborhood* of Y.

Remark 2.5.5. Let $f : \mathrm{Y} \to \mathrm{X}$ be a closed immersion corresponding to the ideal sheaf $\mathcal{I} \subset \mathcal{O}_{\mathrm{X}}$. Then $f^*\mathcal{I}$ ($\cong \mathcal{I}/\mathcal{I}^2$ viewed as an object of $\mathsf{QCoh}(\mathrm{Y})$) is called the *conormal sheaf* of the closed imersion f.

2.5.6. *Scheme-theoretic images.* Let $f : \mathrm{Y} \to \mathrm{X}$ be a morphism of schemes. Then the category of closed immersions $\mathrm{Z} \to \mathrm{X}$ such that f factors through Z has an initial object, called the *scheme-theoretic image* of f.

Indeed, f induces a morphism $\mathcal{O}_{\mathrm{X}} \to f_*\mathcal{O}_{\mathrm{Y}}$ in $\mathsf{CAlg}(\mathsf{QCoh}(\mathrm{X}))$, and we let Z be closed subscheme of X corresponding to its kernel. Then f factors through Z by the adjunction of Proposition 2.4.11. The universal property of Z is clear.

We can use scheme-theoretic images to express quasi-compact locally closed immersions as an open immersion followed by a closed immersion, *i.e.* swapping the order of the composition.

Lemma 2.5.7. *Let $f : \mathrm{Y} \to \mathrm{X}$ be a quasi-compact locally closed immersion. Then f factors as $\mathrm{Y} \xrightarrow{j} \overline{\mathrm{Y}} \xrightarrow{i} \mathrm{X}$, where j is an open immersion and i is closed immersion.*

Proof. Let $\overline{Y}$ denote the scheme-theoretic image of f. Then clearly i is a closed immersion. It remains to prove that j is an open immersion.

Since f is quasi-compact quasi-separated, the formation of $f_*\mathcal{O}_Y$, hence $\overline{Y}$, is local on X. Write f as a composite $Y \to U \to X$ where $Y \to U$ is an closed immersion and $U \to X$ is an open immersion. The base change $\overline{Y} \times_X U$ is thus the scheme-theoretic image of $Y \cong Y \times_X U \to U$, which is already a closed immersion. Thus $\overline{Y} \times XU \cong Y$, so $Y \to \overline{Y}$ is the base change of $U \to X$, thus an open immersion. □

2.5.8. *Complements.* Next, we relate closed immersions and open immersions (*cf.* §1.5.2). For this, we need the notion of "complements" of a subsheaf.

Let $\mathcal{C}$ be a site such that the image of the Yoneda embedding $\mathcal{C}^{op} \to \mathsf{PShv}(\mathcal{C})$, $c \mapsto \mathrm{Hom}(\cdot, c)$ is contained in $\mathsf{Shv}(\mathcal{C})$. Recall (*cf.* Proposition 1.2.7) that this is indeed the case for the site $\mathsf{Sch}^{\mathrm{aff}}$ of affine schemes equipped with standard open covers.

Let $Y \to X$ be a morphism in $\mathsf{Shv}(\mathcal{C})$. Denote by $X \smallsetminus Y$ the subfunctor of X, where a section $x \in X(c)$ ($c \in \mathcal{C}$) belongs to $(X \smallsetminus Y)(c)$ if $Y \times_X \mathrm{Hom}(\cdot, c)$ is isomorphic to the empty sheaf $\varnothing$. We call $X \smallsetminus Y$ the *complement* of Y in X.

Caution: $(X \smallsetminus Y)(c)$ can be very different from $X(c) \smallsetminus Y(c)$.

Lemma 2.5.9. *Let* $Y \to X$ *be a morphism in* $\mathsf{Shv}(\mathcal{C})$. *Then* $X \smallsetminus Y$ *is a sheaf.*

Proof. Let us begin the proof with some preliminary observations.

Claim: given a cover $c_i \to c$ ($i \in I$), the induced morphism in $\mathsf{Shv}(\mathcal{C})$:

$$\bigsqcup_{i \in I} \mathrm{Hom}(\cdot, c_i) \to \mathrm{Hom}(\cdot, c) \tag{2.30}$$

is an epimorphism. Indeed, it suffices to show that for every morphism $d \to c$, one can find a cover $d_j \to d$ ($j \in J$) such that each $d_j \to c$ factors through some c_i (*cf.* Remark 1.3.18). For this, we may take the cover $d_i \to d$ ($i \in I$) given by $d_i := c_i \times_c d$.

Claim: given an epimorphism $\varnothing \to Z$ for any $Z \in \mathsf{Shv}(\mathcal{C})$, Z is isomorphic to $\varnothing$. Indeed, the morphism $\varnothing \to Z$ is a monomorphism, because $\varnothing \to \varnothing \times_Z \varnothing$ is an isomorphism as one checks for $\mathsf{PShv}(\mathcal{C})$ and uses the fact that sheafification commutes with finite limits (*cf.* Corollary 1.3.21). Thus $\varnothing \to Z$ is both a monomorphism and an epimorphism, hence an isomorphism (*cf.* Remark 1.3.18).

With the above two claims established, let us prove that $X \smallsetminus Y$ is a sheaf. Since it is a subfunctor of X, we only need to prove that given a cover $c_i \to c$ ($i \in I$) such that the fiber product $Y \times_X \mathrm{Hom}(\cdot, c_i) \cong \varnothing$ for each $i \in I$, there holds $Y \times_X \mathrm{Hom}(\cdot, c) \cong \varnothing$. To prove this, we consider the epimorphism (2.30). The hypothesis and universality of colimits (*cf.* Lemma 1.4.2) implies that

$$Y \times_X \Big(\bigsqcup_{i \in I} \mathrm{Hom}(\cdot, c_i)\Big) \cong \varnothing.$$

However, $Y \times_X \mathrm{Hom}(\cdot, c)$ receives an epimorphism from the left-hand-side, because epimorphisms are stable under base change (*cf.* Lemma 1.4.8). The second claim then implies that $Y \times_X \mathrm{Hom}(\cdot, c)$ is isomorphic to $\varnothing$. □

Remark 2.5.10. The formation of complements commutes with base change. More precisely, given morphisms $Y \to X$ and $X' \to X$ in $\mathsf{Shv}(\mathcal{C})$ and setting $Y' := Y \times_X X'$, we have a Cartesian diagram in $\mathsf{Shv}(\mathcal{C})$:

$$\begin{array}{ccc} X' \smallsetminus Y' & \longrightarrow & X' \\ \downarrow & & \downarrow \\ X \smallsetminus Y & \longrightarrow & X \end{array}$$

Indeed, this is because for any $c \in \mathcal{C}$ equipped with a morphism $\mathrm{Hom}(\cdot, c) \to \mathrm{X}'$ (*i.e.* a section of X' at c), we have an isomorphism of sheaves:

$$\mathrm{Y}' \times_{\mathrm{X}'} \mathrm{Hom}(\cdot, c) \xrightarrow{\simeq} \mathrm{Y} \times_{\mathrm{X}} \mathrm{Hom}(\cdot, c).$$

Proposition 2.5.11. *The complement* $\mathrm{X} \smallsetminus \mathrm{Z}$ *of a closed immersion* $i : \mathrm{Z} \to \mathrm{X}$ *of schemes is an open immersion.*[12]

Proof. By Lemma 2.5.9, $\mathrm{X} \smallsetminus \mathrm{Z}$ is a Zariski sheaf. To prove that the monomorphism $\mathrm{X} \smallsetminus \mathrm{Z} \to \mathrm{X}$ is an open immersion, we may reduce to the case where $\mathrm{X} = \mathrm{Spec(A)}$ is affine and i is the closed immersion defined by the ideal $\mathfrak{a} \subset \mathrm{A}$. We claim that the collection of standard opens $\mathrm{Spec}(\mathrm{A}_f) \to \mathrm{Spec(A)}$ $(f \in \mathfrak{a})$ covers $\mathrm{X} \smallsetminus \mathrm{Z}$.

Indeed, each $\mathrm{Spec}(\mathrm{A}_f) \to \mathrm{Spec(A)}$ factors through $\mathrm{X} \smallsetminus \mathrm{Z}$ because $\mathrm{A}_f \otimes_{\mathrm{A}} \mathrm{A}/\mathfrak{a} \cong 0$. Moreover, any field-valued point $\mathrm{Spec(K)} \to \mathrm{X} \smallsetminus \mathrm{Z}$, viewed as a ring map $\mathrm{A} \to \mathrm{K}$, satisfies $\mathrm{K} \otimes_{\mathrm{A}} \mathrm{A}/\mathfrak{a} \cong 0$, so the image of some $f \in \mathfrak{a}$ in K is nonzero, and the map $\mathrm{A} \to \mathrm{K}$ must factor through A_f. ☐

Example 2.5.12. For any $\mathrm{A} \in \mathsf{Ring}$ and $f \in \mathrm{A}$, the morphism $\mathrm{Spec}(\mathrm{A}/f) \to \mathrm{Spec(A)}$ is a closed immersion. Its complement is the standard open $\mathrm{Spec}(\mathrm{A}_f) \to \mathrm{Spec(A)}$. Indeed, given any R-point of Spec(A), the condition $\mathrm{Spec(R)} \times_{\mathrm{Spec(A)}} \mathrm{Spec}(\mathrm{A}/f) = \varnothing$ is equivalent to that $\mathrm{R}/f\mathrm{R} = 0$, or that f is a unit in R. In particular, this shows that every standard open is an open immersion.

For any set I, we have the closed immersion $0 : \mathrm{Spec}(\mathbf{Z}) \to \mathbb{A}^{\mathrm{I}}_{\mathbf{Z}}$ corresponding to the ideal $(x_i)_{i \in \mathrm{I}}$ of $\mathbf{Z}[x_i]_{i \in \mathrm{I}}$. We call this closed immersion the *origin* of the affine space $\mathbb{A}^{\mathrm{I}}_{\mathbf{Z}}$.

Remark 2.5.13. An open immersion can be the complement of many distinct closed immersions. For example, the open immersion $\mathbb{A}^1_{\mathbf{Z}} \smallsetminus 0 \to \mathbb{A}^1_{\mathbf{Z}}$ is the complement of the closed immersions $\mathrm{Spec}(\mathbf{Z}[x]/x^n) \to \mathrm{Spec}(\mathbf{Z}[x]) = \mathbb{A}^1_{\mathbf{Z}}$ for any integer $n \geq 1$.

2.5.14. *Separatedness.* A morphism $f : \mathrm{Y} \to \mathrm{X}$ of schemes is called *separated* if the diagonal $\Delta_f : \mathrm{Y} \to \mathrm{Y} \times_{\mathrm{X}} \mathrm{Y}$ is a closed immersion.

A scheme X is called *separated* if the structural morphism $\mathrm{X} \to \mathrm{Spec}(\mathbf{Z})$ is separated. Note that affiness of the diagonal $\Delta : \mathrm{X} \to \mathrm{X} \times \mathrm{X}$ implies that $\mathrm{U} \times_{\mathrm{X}} \mathrm{V}$ is affine for any affine schemes U, V mapping to X (*cf.* the proof of Lemma 1.7.8).

Lemma 2.5.15. *Separatedness of morphisms in* Sch *is local on the target.*

Proof. This follows from the corresponding property of closed immersions (*cf.* Lemma 2.5.3) as in the proof of Lemma 1.7.7. ☐

Corollary 2.5.16. *Affine morphisms of schemes are separated.*

Proof. Since separatedness is local on the target (*cf.* Lemma 2.5.15), it suffices to prove that affine morphisms with affine targets are separated. This amounts to the assertion that for any ring map $\mathrm{A} \to \mathrm{B}$, the multiplication map $\mathrm{B} \otimes_{\mathrm{A}} \mathrm{B} \to \mathrm{B}$ is surjective. ☐

Remark 2.5.17. In the category Top of topological spaces, $\Delta : \mathrm{T} \to \mathrm{T} \times \mathrm{T}$ is closed if and only if T is Hausdorff. Thus one may view separatedness as an analogue of the Hausdorff condition for schemes.

For example, the scheme $\mathrm{X} \coloneqq \mathrm{X}_1 \sqcup_{\mathrm{U}} \mathrm{X}_2$ obtained by gluing $\mathrm{X}_1 \coloneqq \mathrm{X}_2 \coloneqq \mathbb{A}^1_{\mathbf{Z}}$ along the same open immersion $\mathrm{U} \coloneqq \mathbb{A}^1_{\mathbf{Z}} \smallsetminus 0 \to \mathbb{A}^1_{\mathbf{Z}}$ (*cf.* Example 1.6.3) is not separated. Indeed, write $j_1 : \mathrm{X}_1 \to \mathrm{X}$, $j_2 : \mathrm{X}_2 \to \mathrm{X}$ for the natural morphisms. Then the base change of $\Delta : \mathrm{X} \to \mathrm{X} \times \mathrm{X}$ along $(j_1, j_2) : \mathbb{A}^1_{\mathbf{Z}} \to \mathrm{X} \times \mathrm{X}$ is the morphism $\mathbb{A}^1_{\mathbf{Z}} \smallsetminus 0 \to \mathbb{A}^1_{\mathbf{Z}}$, which is not a closed immersion.

[12] We will prove the converse later on: *Any open immersion* $f : \mathrm{U} \to \mathrm{X}$ *of schemes is the complement of a closed immersion.* Caution: The complement of an open immersion of schemes may not be a closed immersion. In fact, it may not even be a scheme.

3. Constructions

Generally speaking, there are two ways of constructing interesting schemes:

(1) writing down a functor of points and proving that it is a scheme;
(2) taking an existing scheme equipped with some auxiliary data (*e.g.* a group action, a closed subscheme, *etc.*) and construct a new scheme from them.

We shall use the first approach to construct the projective space $\mathbb{P}^n_{\mathbf{Z}}$. Then we discuss quotients by group actions. (This is a delicate topic in general, but we shall confine ourselves to some special cases where the Zariski site is sufficient.) Notably, we will realize $\mathbb{P}^n_{\mathbf{Z}}$ as the quotient of $\mathbb{A}^{n+1}_{\mathbf{Z}} \smallsetminus 0$ by the scaling action of $\mathbb{G}_m := \mathrm{GL}_1$, thus also constructing $\mathbb{P}^n_{\mathbf{Z}}$ following the second approach above.

This second construction of $\mathbb{P}^n_{\mathbf{Z}}$ generalizes to the so-called "Proj" construction. We shall use it to construct the "blow-up" of a scheme along a closed subscheme. We discuss the notions of effective Cartier divisors and more generally, regular closed immersions.

Finally, we explain the construction of a quasi-coherent sheaf canonically associated to a morphism of schemes, called the sheaf of "differential forms". We explain the notion of smoothness and how to perform differential calculus on schemes using infinitesimals.

3.1. The projective space.

3.1.1. Let X be a scheme. Recall the notion of vector bundles (*cf.* §2.1.13). A morphism of vector bundles $\varphi : \mathcal{V}_1 \to \mathcal{V}_2$ is called a *subbundle* if it is injective as a morphism of $\mathsf{QCoh}(\mathrm{X})$ and the quotient $\mathcal{V}_2/\mathcal{V}_1$ is also a vector bundle. If $\mathcal{V}_1$ is of rank one, we also call it a *line subbundle* of $\mathcal{V}_2$.

Subbundles can be pulled back: Given a morphism of schemes $f : \mathrm{Y} \to \mathrm{X}$ and a subbundle $\mathcal{V}_1 \to \mathcal{V}_2$ over X, we obtain a subbundle $f^*\mathcal{V}_1 \to f^*\mathcal{V}_2$. Indeed, this is because the short exact sequence in $\mathsf{QCoh}(\mathrm{X})$:

$$0 \to \mathcal{V}_1 \to \mathcal{V}_2 \to \mathcal{V}_2/\mathcal{V}_1 \to 0$$

remains exact after applying f^*, thanks to the fact that $\mathcal{V}_2/\mathcal{V}_1$ is locally free.

Remark 3.1.2. The pullback of an injective morphism of vector bundles may fail to be injective. For example, let $\mathrm{X} = \mathbb{A}^1_{\mathbf{Z}} \cong \mathrm{Spec}(\mathbf{Z}[x])$ and consider the injective morphism of (free, rank-1) $\mathbf{Z}[x]$-modules:

$$\mathbf{Z}[x] \to \mathbf{Z}[x], \quad f \mapsto xf. \tag{3.1}$$

The pullback of (3.1) to the origin is the zero map on $\mathbf{Z}$.

3.1.3. Fix an integer $n \geq 0$. Define the presheaf:

$$\mathbb{P}^n_{\mathbf{Z}} : \mathsf{Ring} \cong (\mathsf{Sch}^{\mathrm{aff}})^{\mathrm{op}} \to \mathsf{Set}$$

to be the functor assigning to Spec(R) the set of rank-1 subbundles of $\mathrm{R}^{\oplus(n+1)}$, and assiging to a morphism of affine schemes $f : \mathrm{Spec}(\mathrm{R}') \to \mathrm{Spec}(\mathrm{R})$ the pullback map f^*.

For any ring R, we also write $\mathbb{P}^n_{\mathrm{R}} := \mathbb{P}^n_{\mathbf{Z}} \times \mathrm{Spec}(\mathrm{R})$ and call it the *n-dimensional projective space* over R.

Proposition 3.1.4. *The presheaf* $\mathbb{P}^n_{\mathbf{Z}}$ *is a scheme.*

3.1.5. In order to prove that $\mathbb{P}^n_{\mathbf{Z}}$ is a scheme, we need to know that given a morphism of vector bundles, the condition of being an isomorphism is "open on the base".

Here is the precise formulation.

Lemma 3.1.6. *Let* X *be a scheme and* $\varphi : \mathcal{V}_1 \to \mathcal{V}_2$ *be a morphism of vector bundles. The subfunctor* $\mathrm{U} \subset \mathrm{X}$*, whose* R*-points (for* $\mathrm{R} \in \mathsf{Ring}$*) are morphisms* $x : \mathrm{Spec}(\mathrm{R}) \to \mathrm{X}$ *such that* $x^*\varphi : x^*\mathcal{M}_1 \to x^*\mathcal{M}_2$ *is an isomorphism, is an open immersion.*

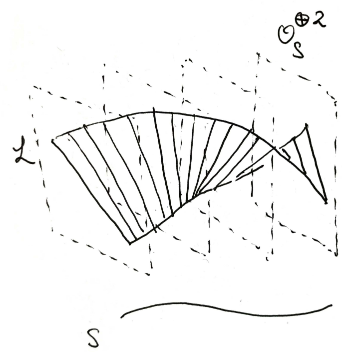

FIGURE 1. In differential geometry, the n-dimensional complex projective space is defined to be the set of 1-dimensional subspaces of $\mathbf{C}^{\oplus(n+1)}$. We are realizing the same idea here: We want an R-point of $\mathbb{P}^n_{\mathbf{Z}}$ to be a family of such parametrized by Spec(R), and this is exactly the data encoded by the line subbundle $\mathcal{L}$ of $\mathrm{R}^{\oplus(n+1)}$.

Proof. Since the property of being an open immersion is local on the target (*cf.* Corollary 1.5.11), we reduce to the case X = Spec(A) and $\varphi : \mathrm{A}^{\oplus r_1} \to \mathrm{A}^{\oplus r_2}$ is a morphism of finite free A-modules. If $r_1 \neq r_2$, then $\mathrm{U} = \varnothing$ is an open subscheme of Spec(A).

If $r_1 = r_2 = r$, then U represents the functor whose R-points (for $\mathrm{R} \in \mathsf{Ring}$) are homomorphisms $\mathrm{A} \to \mathrm{R}$ such that the induced map $\varphi_{\mathrm{R}} : \mathrm{R}^{\oplus r} \to \mathrm{R}^{\oplus r}$ is invertible, *i.e.* $\det(\varphi_{\mathrm{R}})$ is a unit in R. This functor is represented by the localization of A at $\det(\varphi)$. □

Remark 3.1.7 (Vanishing loci)**.** Let X be a scheme and $\mathcal{L}$ be a line bundle over X. Given a global section $s \in \Gamma(\mathrm{X}, \mathcal{L})$, we can use Lemma 3.1.6 to define the *non-vanishing locus* $\mathrm{X}_{s\neq 0}$, *i.e.* the open subscheme of X over which $s : \mathcal{O}_{\mathrm{X}} \to \mathcal{L}$ is an isomorphism.

Note that we can also define the *vanishing locus* $\mathrm{X}_{s=0}$, *i.e.* the closed subscheme $\mathrm{X}_{s=0}$ of X where an R-point x of X belongs to $\mathrm{X}_{s=0}$ if $x^*s = 0$. The fact that $\mathrm{X}_{s=0} \to \mathrm{X}$ is a closed immersion can be verified locally over X, where we may trivialize $\mathcal{L}$ and reduce to Example 2.5.12. The same argument yields an isomorphism of open subschemes of X:

$$\mathrm{X}_{s\neq 0} \xrightarrow{\simeq} \mathrm{X} \smallsetminus \mathrm{X}_{s=0}.$$

3.1.8. We now return to the proof that $\mathbb{P}^n_{\mathbf{Z}}$ is a scheme.

Proof of Proposition 3.1.4. The fact that $\mathbb{P}^n_{\mathbf{Z}}$ is a sheaf follows from descent of quasi-coherent sheaves (*cf.* Proposition 2.2.2) and the fact that being finite locally free can be checked over a standard open cover. It remains to construct an open cover of $\mathbb{P}^n_{\mathbf{Z}}$ by affine schemes.

For each i $(0 \le i \le n)$, let $\mathrm{U}_i \subset \mathbb{P}^n_{\mathbf{Z}}$ denote the subfunctor where an R-point of $\mathbb{P}^n_{\mathbf{Z}}$ belongs to U_i if the inclusion $\mathcal{L} \to \mathrm{R}^{\oplus(n+1)}$ is an isomorphism onto the ith summand:

$$\begin{array}{ccc} \mathcal{L} & \to & \mathrm{R}^{\oplus(n+1)} \\ & \searrow^{\simeq} & \downarrow \pi_i \\ & & \mathrm{R} \end{array} \tag{3.2}$$

Then $\mathrm{U}_i \to \mathbb{P}^n_{\mathbf{Z}}$ is an open immersion by Lemma 3.1.6.

We claim that U_i is an affine scheme. In fact, we shall construct an isomorphism $\mathrm{U}_i \xrightarrow{\simeq} \mathbb{A}^n_{\mathbf{Z}}$. Indeed, for every $\mathrm{R} \in \mathsf{Ring}$, submodules of $\mathrm{R}^{\oplus(n+1)}$ making the composed arrow in diagram

(3.2) an isomorphism are in bijection with morphisms $\mathrm{R} \to \mathrm{R}^{\oplus(n+1)}$ which are identity on the ith summand. The latter are in bijection with tuples a_j $(0 \le j \le n,\ j \neq i)$ of elements of R, *i.e.* R-points of $\mathbb{A}^n_{\mathbf{Z}} \cong \mathrm{Spec}(\mathbf{Z}[x_0, \cdots, \widehat{x_i}, \cdots, x_n])$. □

3.1.9. *Homogeneous coordinates.* Let $\mathcal{O}_{\mathbb{P}^n_{\mathbf{Z}}}(1)$ denote the line bundle over $\mathbb{P}^n_{\mathbf{Z}}$ whose value at any R-point $\mathcal{L}$ ($\mathrm{R} \in \mathsf{Ring}$) is the line bundle $\mathcal{L}^\vee \in \mathsf{Mod}_{\mathrm{R}}$.[13] We also write $\mathcal{O}(1) \coloneqq \mathcal{O}_{\mathbb{P}^n_{\mathbf{Z}}}(1)$ to ease the notation and set $\mathcal{O}(d) \coloneqq \mathcal{O}(1)^{\otimes d}$ for any integer d.

Dualizing the inclusion $\mathcal{L} \to \mathrm{R}^{\oplus(n+1)}$ yields a surjection $\mathrm{R}^{\oplus(n+1)} \to \mathcal{L}^\vee$ of R-modules. Restriction to each coordinate yields a morphism $\mathrm{R} \to \mathcal{L}^\vee$ $(0 \le i \le n)$. These morphisms are functorial in R, so they define global sections:

$$\mathrm{X}_i \in \Gamma(\mathbb{P}^n_{\mathbf{Z}}, \mathcal{O}(1)), \quad 0 \le i \le n. \tag{3.3}$$

We call the sections (3.3) the *homogeneous coordinates* on $\mathbb{P}^n_{\mathbf{Z}}$. The intuitive meaning is that they are the $(n+1)$ coordinate functions on the line $\mathcal{L}$.

Remark 3.1.10. In terms of the homogeneous coordinates (3.3), the proof of Proposition 3.1.4 yields the following information. For each $0 \le i \le n$, it constructs an isomorphism:

$$(\mathbb{P}^n_{\mathbf{Z}})_{\mathrm{X}_i \neq 0} \xrightarrow{\simeq} \mathbb{A}^n_{\mathbf{Z}}. \tag{3.4}$$

If we write $\mathbb{A}^n_{\mathbf{Z}} \cong \mathrm{Spec}(\mathbf{Z}[x_0, \cdots, \widehat{x_i}, \cdots, x_n])$, then (3.4) carries each x_j $(j \neq i)$ to the global section $\mathrm{X}_i^{-1}\mathrm{X}_j \in \Gamma((\mathbb{P}^n_{\mathbf{Z}})_{\mathrm{X}_i \neq 0}, \mathcal{O})$, which is well-defined as $\mathrm{X}_i : \mathcal{O} \to \mathcal{O}(1)$ is an isomorphism over $(\mathbb{P}^n_{\mathbf{Z}})_{\mathrm{X}_i \neq 0}$. We view (3.4) as the "standard affine charts" of $\mathbb{P}^n_{\mathbf{Z}}$.

3.1.11. *Global generation.* Let X be a scheme.

Given a morphism $f : \mathrm{X} \to \mathbb{P}^n_{\mathbf{Z}}$, we obtain a line bundle $\mathcal{Q} \coloneqq f^*\mathcal{O}(1)$ over X equipped with global sections $s_i \coloneqq f^*\mathrm{X}_i$ of $\mathcal{Q}$ $(0 \le i \le n)$. The induced morphism in $\mathsf{QCoh}(\mathrm{X})$:

$$(s_0, \cdots, s_n) : \mathcal{O}_{\mathrm{X}}^{\oplus(n+1)} \to \mathcal{Q} \tag{3.5}$$

is surjective, as it pulls back to the surjection $\mathrm{R}^{\oplus(n+1)} \to \mathcal{L}^\vee$ over any R-point of X.

Conversely, given a line bundle $\mathcal{Q}$ over X and global sections s_i of $\mathcal{Q}$ $(0 \le i \le n)$, we say that $s_0, \cdots, s_n$ *globally generate* $\mathcal{Q}$ if the induced map (3.5) is surjective. Such data $(\mathcal{Q}, s_0, \cdots, s_n)$ induce a morphism $f : \mathrm{X} \to \mathbb{P}^n_{\mathbf{Z}}$ sending every R-point x of X to the dual of the pullback of (3.5) to Spec(R), which yields a line subbundle $\mathcal{L} \coloneqq \mathcal{Q}^\vee$ of $\mathrm{R}^{\oplus(n+1)}$.

Lemma 3.1.12. *Let* X *be a scheme. Let* $\mathcal{Q}$ *be a line bundle over* X *and* $s_0, \cdots, s_n \in \Gamma(\mathrm{X}, \mathcal{Q})$ *globally generate* $\mathcal{Q}$. *The following are equivalent:*

1. *the induced morphism* $f : \mathrm{X} \to \mathbb{P}^n_{\mathbf{Z}}$ *is a closed immersion;*
2. *for each* $0 \le i \le n$, *the nonvanishing locus* $\mathrm{X}_{s_i \neq 0}$ *is affine and the induced ring map:*

$$\mathbf{Z}[x_0, \cdots, \widehat{x_i}, \cdots, x_n] \to \Gamma(\mathrm{X}_{s_i \neq 0}, \mathcal{O}), \quad x_j \mapsto s_i^{-1}s_j$$

is surjective.

Proof. By Lemma 2.5.3, f is a closed immersion if and only if its base change to each $(\mathbb{P}^n_{\mathbf{Z}})_{\mathrm{X}_i \neq 0}$ is a closed immersion. Since the formation of nonvanishing loci is compatible with base change, we have:

$$\mathrm{X}_{s_i \neq 0} \xrightarrow{\simeq} \mathrm{X} \times_{\mathbb{P}^n_{\mathbf{Z}}} (\mathbb{P}^n_{\mathbf{Z}})_{\mathrm{X}_i \neq 0}.$$

Condition (2) is a restatement that $\mathrm{X}_{s_i \neq 0} \to (\mathbb{P}^n_{\mathbf{Z}})_{\mathrm{X}_i \neq 0}$ is a closed immersion for each $0 \le i \le n$ (*cf.* Remark 3.1.10). □

[13]Note the dualization!

Example 3.1.13 (Segre embedding)**.** Given integers $m, n \geq 0$, we consider the line bundle over $\mathbb{P}^m_{\mathbf{Z}} \times \mathbb{P}^n_{\mathbf{Z}}$ given by the external tensor product:

$$\mathcal{O}(1) \boxtimes \mathcal{O}(1) := p_1^*\mathcal{O}(1) \otimes p_2^*\mathcal{O}(1), \tag{3.6}$$

where p_1, p_2 are the projections of $\mathbb{P}^m_{\mathbf{Z}} \times \mathbb{P}^n_{\mathbf{Z}}$ onto its two factors.

The line bundle (3.6) is globally generated by sections $\mathrm{X}_i\mathrm{Y}_j$ for homogeneous coordinates X_i over $\mathbb{P}^m_{\mathbf{Z}}$ and Y_j over $\mathbb{P}^n_{\mathbf{Z}}$ ($0 \leq i \leq m$, $0 \leq j \leq n$). We argue that the induced morphism:

$$\mathbb{P}^m_{\mathbf{Z}} \times \mathbb{P}^n_{\mathbf{Z}} \to \mathbb{P}^r_{\mathbf{Z}}, \quad r := (m+1)(n+1) - 1,$$

is a closed immersion using Lemma 3.1.12.

Indeed, for each i, j, the nonvanishing locus of $\mathrm{X}_i\mathrm{Y}_j$ is identified with $(\mathbb{P}^m_{\mathbf{Z}})_{\mathrm{X}_i \neq 0} \times (\mathbb{P}^n_{\mathbf{Z}})_{\mathrm{Y}_j \neq 0}$ which is affine. Its ring of functions is the polynomial ring in generators $(\mathrm{X}_i)^{-1}\mathrm{X}_{i'}$ and $(\mathrm{Y}_j)^{-1}\mathrm{Y}_{j'}$ ($i' \neq i$, $j' \neq j$), which receives a surjection from the polynomial ring in $\mathrm{Z}_{i'j'}$ ($(i', j') \neq (i, j)$) sending each $\mathrm{Z}_{i'j'}$ to $(\mathrm{X}_i\mathrm{Y}_j)^{-1}\mathrm{X}_{i'}\mathrm{Y}_{j'}$. For example, we find $\mathrm{X}_i^{-1}\mathrm{X}_{i'}$ with the choice $i' \neq i$, $j' = j$ and $\mathrm{Y}_j^{-1}\mathrm{Y}_{j'}$ with $i' = i$, $j' \neq j$.

3.1.14. We shall now calculate the global section of $\mathcal{O}(d)$ over $\mathbb{P}^n_{\mathrm{R}}$ for any ring R.

Note that by taking tensor product and sums of the homogeneous coordinates (3.3), any *homogeneous* polynomial $\mathrm{P}(\mathrm{X}_0, \cdots, \mathrm{X}_n) \in \mathrm{R}[\mathrm{X}_0, \cdots, \mathrm{X}_n]$ of degree d defines a section:

$$\mathrm{P}(\mathrm{X}_0, \cdots, \mathrm{X}_n) \in \Gamma(\mathbb{P}^n_{\mathrm{R}}, \mathcal{O}(d)). \tag{3.7}$$

Denote by $\mathrm{R}[\mathrm{X}_0, \cdots, \mathrm{X}_n]_d$ the R-module of homogeneous polynomials of degree d in the variables $\mathrm{X}_0, \cdots, \mathrm{X}_n$. The sections (3.7) induce a map of R-modules:

$$\mathrm{R}[\mathrm{X}_0, \cdots, \mathrm{X}_n]_d \to \Gamma(\mathbb{P}^n_{\mathrm{R}}, \mathcal{O}(d)). \tag{3.8}$$

Proposition 3.1.15. *The map* (3.8) *is an isomorphism for any* $n \geq 0$ *and* $d \in \mathbf{Z}$.

Proof. We shall describe $\mathcal{O}(d)$ by its descent data along the cover:

$$\mathbb{A}^n_{\mathrm{R}} \cong (\mathbb{P}^n_{\mathrm{R}})_{\mathrm{X}_i \neq 0} \to \mathbb{P}^n_{\mathrm{R}} \quad (0 \leq i \leq n).$$

Indeed, the restriction $\mathcal{O}(d)|_{\mathrm{U}_i}$ of the line bundle $\mathcal{O}(d)$ to $\mathrm{U}_i := (\mathbb{P}^n_{\mathrm{R}})_{\mathrm{X}_i \neq 0}$ is trivialized by the section $\mathrm{X}_i^d : \mathcal{O} \to \mathcal{O}(d)$. For each $0 \leq i, j \leq n$, the identity on $\mathcal{O}(d)|_{\mathrm{U}_{ij}}$ corresponds to the isomorphism of the trivial line bundle over $\mathrm{U}_{ij} := \mathrm{U}_i \times_{\mathbb{P}^n_{\mathrm{R}}} \mathrm{U}_j$:

$$\varphi_{ij} : \mathcal{O}_{\mathrm{U}_i}|_{\mathrm{U}_{ij}} \xrightarrow{\simeq} \mathcal{O}_{\mathrm{U}_j}|_{\mathrm{U}_{ij}}, \quad f \mapsto f \cdot (\mathrm{X}_j^{-1}\mathrm{X}_i)^d.$$

We now identify each U_i with $\mathbb{A}^n_{\mathrm{R}} \cong \operatorname{Spec}(\mathrm{R}[x_0, \cdots, \widehat{x_i}, \cdots, x_n])$ by sending x_j to the global section $\mathrm{X}_i^{-1}\mathrm{X}_j$ over U_i ($j \neq i$). By the description of the descent data of $\mathcal{O}(d)$, an element of $\Gamma(\mathbb{P}^n_{\mathrm{R}}, \mathcal{O}(d))$ consists of polynomials:

$$f_i \in \mathrm{R}[\mathrm{X}_i^{-1}\mathrm{X}_0, \cdots, \widehat{\mathrm{X}_i^{-1}\mathrm{X}_i}, \cdots, \mathrm{X}_i^{-1}\mathrm{X}_n], \quad \text{with } f_i \cdot \left(\mathrm{X}_j^{-1}\mathrm{X}_i\right)^d = f_j.$$

We can compare f_i and f_j by viewing them as elements of $\mathrm{R}[\mathrm{X}_0, \cdots, \mathrm{X}_n]_{\mathrm{X}_i\mathrm{X}_j}$. This proves that $\deg(f_i) \leq d$ for each i and all f_i's are determined by f_0, which corresponds to the homogeneous polynomial $f_0 \cdot \mathrm{X}_0^d$ of degree d. □

Example 3.1.16 (Hypersurfaces)**.** Fix a field k. For any integer $d \geq 0$, we call a closed subscheme X of $\mathbb{P}^n_k$ a *degree-d hypersurface* if it is the vanishing locus of some nonzero element $f \in \mathrm{H}^0(\mathbb{P}^n_k, \mathcal{O}(d))$. Degree-1 hypersurfaces are also called *hyperplanes*.

Note that two nonzero elements $f_1, f_2 \in \mathrm{H}^0(\mathbb{P}^n_k, \mathcal{O}(d))$ define the same hypersurface if and only if they differ by multiplication by some $\lambda \in k \setminus 0$. Indeed, by working on each chart $(\mathbb{P}^n_k)_{\mathrm{X}_i \neq 0}$, this reduces to the statement that two nonzero elements of $k[x_0, \cdots, \widehat{x_i}, \cdots, x_n]$ generate the same ideal if and only if they are scalar multiples of one another (by unique factorization).

3.2. Torsors.

3.2.1. In this subsection, we shall make sense of the following two statements (both natural from the point of view of differential geometry):

(1) the category of rank-r vector bundles with isomorphisms on a scheme X is equivalent to the category of "GL_r-principal bundles";
(2) the projective space $\mathbb{P}^n_{\mathbf{Z}}$ is the "quotient" of the affine space without origin by the scaling action of $\mathbb{G}_m \coloneqq \mathrm{GL}_1$:

$$\mathbb{P}^n_{\mathbf{Z}} \xrightarrow{\simeq} (\mathbb{A}^{n+1}_{\mathbf{Z}} \smallsetminus 0)/\mathbb{G}_m.$$

To do so, we need to the notion of "torsors". These are general concepts which make sense on any site, so we shall explain them in this generality.

3.2.2. Denote by Grp the category of groups.

Let $\mathcal{C}$ be a site. A *sheaf of groups* on $\mathcal{C}$ is a functor $\mathcal{C}^{\mathrm{op}} \to \mathsf{Grp}$ satisfying the sheaf axiom, *cf.* §1.3.3. (Since the forgetful functor $\mathsf{Grp} \to \mathsf{Set}$ preserves limits, this condition is equivalent to the underlying Set-valued presheaf being a sheaf.).

It is also helpful to think of sheaves of groups on $\mathcal{C}$ as group objects in $\mathsf{Shv}(\mathcal{C})$, *i.e.* $\mathrm{G} \in \mathsf{Shv}(\mathcal{C})$ equipped with maps $\mathrm{G} \times \mathrm{G} \to \mathrm{G}$ and $* \to \mathrm{G}$ satisfying the group axioms.

Remark 3.2.3. Since the sheafification functor $\mathsf{PShv}(\mathcal{C}) \to \mathsf{Shv}(\mathcal{C})$ commutes with finite limits (*cf.* Corollary 1.3.21), the sheafification of a presheaf of groups yields a sheaf of groups.

In particular, given a morphism $f : \mathrm{H} \to \mathrm{G}$ of sheaves of groups, we can define its image $f(\mathrm{H}) \subset \mathrm{G}$ via sheafification of the presheaf image, and this yields a subsheaf of groups of G. Consequently, the notion of exact sequences makes sense for sheaves of groups.

3.2.4. Given a sheaf of groups G and a (Set-valued) sheaf X on $\mathcal{C}$, a G-*action* on X consists of a morphism in $\mathsf{Shv}(\mathcal{C})$:

$$\mathrm{X} \times \mathrm{G} \to \mathrm{X}, \quad (x, g) \mapsto x \cdot g \tag{3.9}$$

satisfying $x \cdot e = x$ and $(x \cdot g_1) \cdot g_2 = x \cdot (g_1 \cdot g_2)$, where g_1, g_2 (respectively, x) are elements of $\mathrm{G}(c)$ (respectively, $\mathrm{X}(c)$) at any $c \in \mathcal{C}$, and $e \in \mathrm{G}(c)$ is the unit.

Note that the action map (3.9) induces a map in $\mathsf{Shv}(\mathcal{C})$:

$$\mathrm{X} \times \mathrm{G} \to \mathrm{X} \times \mathrm{X}, \quad (x, g) \mapsto (x, x \cdot g). \tag{3.10}$$

We say that the G-action on X is:

(1) *free* if (3.10) is a monomorphism;
(2) *simply transitive* if (3.10) is an isomorphism.

Remark 3.2.5. We have defined a right G-action. One can also define a *left* G-action, but the two notions are equivalent: The category of $\mathrm{X} \in \mathsf{Shv}(\mathcal{C})$ equipped with a left G-action is equivalent to that of $\mathrm{X} \in \mathsf{Shv}(\mathcal{C})$ equipped with a right G-action, compatibly with the forgetful functors to $\mathsf{Shv}(\mathcal{C})$.

3.2.6. A G-*torsor* on $\mathcal{C}$ is an object $\mathrm{X} \in \mathsf{Shv}(\mathcal{C})$ equipped with a G-action such that:

(1) the G-action on X is simply transitive;
(2) for any $c \in \mathcal{C}$, there exists a cover $c_i \to c$ ($i \in \mathrm{I}$) such that $\mathrm{X}(c_i) \neq \varnothing$ for all $i \in \mathrm{I}$.

(Elements of $\mathrm{X}(c)$ are typically called *sections* of X over $c \in \mathcal{C}$.)

A morphism of G-torsors is a morphism $f : \mathrm{Y} \to \mathrm{X}$ of sheaves which is G-*equivariant*, *i.e.* $f(y \cdot g) = f(y) \cdot g$ for any $y \in \mathrm{Y}(c)$, $g \in \mathrm{G}(c)$, $c \in \mathcal{C}$. Thus, any morphism of G-torsors is automatically an isomorphism, so G-torsors on $\mathcal{C}$ form a groupoid.

Example 3.2.7. The sheaf of groups G, equipped with the G-action defined by multiplication from the right, is a G-torsor. It is called the *trivial* G-torsor.

Note that a G-torsor X is trivial (*i.e.* isomorphic to the trivial G-torsor) if and only if $\Gamma(\mathcal{C}, \mathrm{X})$ is nonempty. Indeed, any $x \in \Gamma(\mathcal{C}, \mathrm{X})$ exhibits an isomorphism $\mathrm{G} \xrightarrow{\simeq} \mathrm{X}$, $g \mapsto x \cdot g$ of G-torsors. We shall also say that x *trivializes* the G-torsor X.

Remark 3.2.8. For any $\mathrm{X} \in \mathsf{Shv}(\mathcal{C})$, G restricts to a sheaf of groups on the site $\mathcal{C}_{/\mathrm{X}}$.

We shall call a G-torsor on $\mathcal{C}_{/\mathrm{X}}$ a G-*torsor over* X. It may be regarded as an object $\mathrm{Y} \in \mathsf{Shv}(\mathcal{C})$ equipped with a structure morphism $\mathrm{Y} \to \mathrm{X}$ admitting local sections and a G-action inducing an isomorphism over X:

$$\mathrm{Y} \times \mathrm{G} \xrightarrow{\simeq} \mathrm{Y} \times_{\mathrm{X}} \mathrm{Y}, \quad (y, g) \mapsto (y, y \cdot g).$$

Given a morphism $f : \mathrm{X}' \to \mathrm{X}$ in $\mathsf{Shv}(\mathcal{C})$ and a G-torsor $\mathrm{Y} \to \mathrm{X}$, the pullback $f^*\mathrm{Y} := \mathrm{Y} \times_{\mathrm{X}} \mathrm{X}'$ is naturally a G-torsor over X'. The association from $\mathrm{X} \in \mathsf{Shv}(\mathcal{C})$ to the groupoid of G-torsors satisfies descent along epimorphisms, by Proposition 1.4.13.

3.2.9. Given a morphism $f : \mathrm{H} \to \mathrm{G}$ of sheaves of groups on $\mathcal{C}$ and an H-torsor Y, we obtain a G-torsor $\mathrm{Y} \times^{\mathrm{H}} \mathrm{G}$ as sheafification of the presheaf assigning to $c \in \mathcal{C}$ equivalence classes of pairs (y, g) ($y \in \mathrm{Y}(c)$, $g \in \mathrm{G}(c)$) subject to the equivalence relation:

$$(y \cdot h, g) \simeq (y, h \cdot g), \text{ for any } h \in \mathrm{H}(c).$$

The G-action on $\mathrm{Y} \times^{\mathrm{H}} \mathrm{G}$ is defined by multiplication from the right on the G-factor.

We call $\mathrm{Y} \times^{\mathrm{H}} \mathrm{G}$ the G-torsor *induced* by Y along the morphism f. We shall also call it *change of structure groups.*

Remark 3.2.10. If G is a sheaf of *abelian* groups on $\mathcal{C}$, then the groupoid of G-torsors on $\mathcal{C}$ inherits a symmetric monoidal structure. The monoidal product is given by

$$\mathrm{X}_1 \otimes \mathrm{X}_2 := (\mathrm{X}_1 \times \mathrm{X}_2) \times^{\mathrm{G} \times \mathrm{G}} \mathrm{G},$$

i.e. the G-torsor induced from the $\mathrm{G} \times \mathrm{G}$-torsor $\mathrm{X}_1 \times \mathrm{X}_2$ along the map $\mathrm{G} \times \mathrm{G} \to \mathrm{G}$, $(g_1, g_2) \mapsto g_1 g_2$. (This is a group homomorphism because G is abelian.) The monoidal unit is given by the trivial G-torsor.

3.2.11. *Quotients.* Quotients of *free* group actions fall under the paradigm of §1.4.3. Namely, consider a site $\mathcal{C}$, sheaf of groups G on $\mathcal{C}$, and an object $\mathrm{X} \in \mathsf{Shv}(\mathcal{C})$ equipped with a G-action. If the action is free, then (3.10) defines an equivalence relation:

$$\mathrm{R} := \mathrm{X} \times \mathrm{G} \subset \mathrm{X} \times \mathrm{X}.$$

In this case, we shall write the quotient as $\mathrm{X}/\mathrm{G} \in \mathsf{Shv}(\mathcal{C})$ (instead of X/R.)

Here is how quotients and torsors are related: The morphism $\mathrm{X} \to \mathrm{X}/\mathrm{G}$ realizes X as a G-torsor over X/G (*cf.* Remark 3.2.8). Indeed, the G-action on X is as given. It is simply transitive (over X/G) by the Cartesian diagram (1.18) in $\mathsf{Shv}(\mathcal{C})$, which reads as follows in our situation:

$$\begin{array}{ccc} \mathrm{X} \times \mathrm{G} & \longrightarrow & \mathrm{X} \\ \downarrow & & \downarrow \\ \mathrm{X} & \longrightarrow & \mathrm{X}/\mathrm{G} \end{array}$$

The fact that $\mathrm{X} \to \mathrm{X}/\mathrm{G}$ admits local sections follows from the fact that it is an epimorphism in $\mathsf{Shv}(\mathcal{C})$ (*cf.* Remark 1.3.18).

From this description, we obtain the following characterization of X/G.

Lemma 3.2.12. *Let $\mathcal{C}$ be a site, G be a sheaf of groups on $\mathcal{C}$, and $X \in \mathsf{Shv}(\mathcal{C})$ be equipped with a free G-action. Then for every $Y \in \mathsf{Shv}(\mathcal{C})$, the following groupoids are canonically equivalent:*

(1) the (discrete) groupoid of morphisms $Y \to X/G$;
(2) the groupoid of pairs (P, f), where P is a G-torsor on Y and $f : P \to X$ is a G-equivariant morphism.

Proof. The functor (1) → (2) sends $Y \to X/G$ to $P := Y \times_{X/G} X$, *i.e.* the base change of the G-torsor $X \to X/G$ to Y, and $f : P \to X$ the projection onto the second factor.

The inverse functor (2) → (1) is constructed as follows. Given (P, f), the morphism $P \to Y$ is an epimorphism in $\mathsf{Shv}(\mathcal{C})$, so Y is identified with the quotient of the equivalence relation $P \times_Y P \subset P \times P$ (*cf.* Lemma 1.4.6). We obtain the morphism $Y \to X/G$ by passing to the quotients of the following morphism of equivalence relations:

$$\begin{array}{ccc} P \times_Y P & \xrightarrow{(f,f)} & X \times_{X/G} X \\ \downarrow & & \downarrow \\ P \times P & \xrightarrow{(f,f)} & X \times X \end{array}$$

We omit verifying that the two functors are mutual inverses. □

Remark 3.2.13. Here is a direct proof that the groupoid of pairs (P, f) in Lemma 3.2.12(2) is discrete: Any automorphism $\alpha : P \xrightarrow{\simeq} P$ with $f = f \cdot \alpha$ must be the identity, as $\alpha(x) = x \cdot g$ implies that $f(x) = (f \cdot \alpha)(x) = f(x) \cdot g$, so $g = e$ by freeness of the G-action.

3.2.14. *Torsors on schemes.* Let us now return to scheme theory.

We consider the site $\mathcal{C} := \mathsf{Sch}^{\mathrm{aff}}$ of affine schemes where covers are given by standard Zariski covers. Sheaves on this site are the Zariski sheaves introduced in §1.2. For a sheaf of groups G and $X \in \mathsf{Shv}$, we may consider the category of G-torsors over X. If we wish to emphasize the role played by standard Zariski covers, we shall say "Zariski G-torsors" over X rather than simply "G-torsors".

3.2.15. *Group schemes.* A group object of Sch is called a *group scheme*. Equivalently, a group scheme is a Grp-valued presheaf $\mathsf{Ring} \to \mathsf{Grp}$ whose underlying Set-valued presheaf is a scheme. A group scheme is an *affine group scheme* if the underlying scheme is affine. Thus, an affine group scheme is a group object of $\mathsf{Sch}^{\mathrm{aff}}$.

For an affine group scheme $G \cong \operatorname{Spec}(R)$, we may write the unit $\operatorname{Spec}(\mathbf{Z}) \to G$ and the multiplication morphisms $G \times G \to G$ in terms of the ring R and obtain the *co-unit* and *co-multiplication* morphisms in Ring:

$$\epsilon : R \to \mathbf{Z}, \quad \mu : R \to R \otimes R,$$

satisfying the duals of the unit, associativity, and inverse axioms. A ring R equipped with such structures is called a (commutative) *Hopf algebra*.

Thus, we have a tautological anti-equivalence between the category of affine group schemes and the category of Hopf algebras.

Example 3.2.16. For $r \in \mathbf{Z}_{\geq 0}$, consider the presheaf GL_r whose R-points ($R \in \mathsf{Ring}$) is the group of invertible r-by-r matrices with coefficients in R. We claim that GL_r is an affine group scheme.

Indeed, the presheaf M_r whose R-points is the set of all r-by-r matrices is represented by $\mathbb{A}^{r^2}_{\mathbf{Z}}$. There is a morphism $\det : M_r \to \mathbb{A}^1_{\mathbf{Z}}$ carrying an r-by-r matrix to its determinant, and a Cartesian square:

$$\begin{array}{ccc} \mathrm{GL}_r & \longrightarrow & \mathrm{M}_r \\ \downarrow & & \downarrow \scriptstyle\det \\ \mathbb{A}^1_{\mathbf{Z}} \smallsetminus 0 & \rightarrow & \mathbb{A}^1_{\mathbf{Z}} \end{array}$$

where $0 : \mathrm{Spec}(\mathbf{Z}) \to \mathbb{A}^1_{\mathbf{Z}}$ is the closed subscheme corresponding to the origin. Thus, GL_r is an affine scheme and may be viewed as a sheaf of groups on $\mathsf{Sch}^{\mathrm{aff}}$.

The affine group scheme GL_r is called the *general linear group* of rank r. The special case $\mathbb{G}_m := \mathrm{GL}_1$ is also called the *multiplicative group.*

Lemma 3.2.17. *Let* G *be an affine group scheme. Let* P *be a Zariski* G*-torsor over* $\mathrm{X} \in \mathsf{Shv}$. *If* X *is a scheme, then so is* P.

Proof. Note that P is Zariski sheaf by definition. It remains to construct an open cover of P by affine schemes.

Let $f_i : \mathrm{X}_i \to \mathrm{X}$ $(i \in \mathrm{I})$ be an open cover by affine schemes such that each $(f_i)^*\mathrm{P}$ is trivial. Such a cover exists by the Zariski local triviality of P. Hence we have a Cartesian square:

$$\begin{array}{ccc} \bigsqcup_{i\in \mathrm{I}} \mathrm{X}_i \times \mathrm{G} & \rightarrow & \mathrm{P} \\ \downarrow & & \downarrow \\ \bigsqcup_{i\in \mathrm{I}} \mathrm{X}_i & \longrightarrow & \mathrm{X} \end{array}$$

where each $\mathrm{X}_i \times \mathrm{G}$ is an affine scheme (since both X_i and G are) and $\mathrm{X}_i \times \mathrm{G} \to \mathrm{P}$ $(i \in \mathrm{I})$ is an open cover. □

3.2.18. Given any scheme X, we construct a functor from the category $\mathrm{Bun}_r(\mathrm{X})$ of rank-r vector bundles with isomorphisms to the category $\mathrm{Tors}_{\mathrm{GL}_r}(\mathrm{X})$ of GL_r-torsors on X:

$$\mathrm{Bun}_r(\mathrm{X}) \to \mathrm{Tors}_{\mathrm{GL}_r}(\mathrm{X}). \tag{3.11}$$

Namely, it sends $\mathcal{M} \in \mathrm{Bun}_r(\mathrm{X})$ to the presheaf $\mathcal{I}som(\mathcal{O}_{\mathrm{X}}^{\oplus r}, \mathcal{M}) \to \mathrm{X}$ whose value at an R-point x of X ($\mathrm{R} \in \mathsf{Ring}$) is the set of isomorphisms $\mathrm{R}^{\oplus r} \xrightarrow{\simeq} \mathcal{M}$ in $\mathsf{Mod}_{\mathrm{R}}$:

$$\mathcal{I}som(\mathcal{O}_{\mathrm{X}}^{\oplus r}, \mathcal{M})(\mathrm{R}) := \{\mathrm{R}^{\oplus r} \xrightarrow{\simeq} \mathcal{M} \text{ in } \mathsf{Mod}_{\mathrm{R}}\}.$$

This is a sheaf by descent of QCoh. The sheaf $\mathcal{I}som(\mathcal{O}_{\mathrm{X}}^{\oplus r}, \mathcal{M})$ is acted on by GL_r, for any $g \in \mathrm{GL}_r(\mathrm{R})$ carries an isomorphism $y : \mathrm{R}^{\oplus r} \xrightarrow{\simeq} \mathcal{M}$ to the composition $y \cdot g$. This action is clearly simply transitive. The structure morphism $\mathcal{I}som(\mathcal{O}_{\mathrm{X}}^{\oplus r}, \mathcal{M}) \to \mathrm{X}$ admits local sections because $\mathcal{M}$ is locally free of rank r.

Proposition 3.2.19. *The functor* (3.11) *is an equivalence of groupoids.*

Proof. We shall construct the inverse of (3.11) using descent of QCoh along epimorphisms in Shv (*cf.* Proposition 2.2.2).

First, observe that given a GL_r-torsor $f : \mathrm{Y} \to \mathrm{X}$, the pullback functor f^* defines an equivalence:

$$\mathsf{QCoh}(\mathrm{X}) \xrightarrow{\simeq} \mathsf{QCoh}(\mathrm{Y}/\mathrm{X}),$$

where we used the existence of local sections to see that f is an epimorphism in Shv. Note that $\mathcal{M} \in \mathsf{QCoh}(\mathrm{X})$ is a rank-r vector bundle if and only if $f^*\mathcal{M}$ is a rank-r vector bundle, since the property of being locally free is Zariski local.

Under the isomorphism $\mathrm{Y} \times \mathrm{GL}_r \xrightarrow{\simeq} \mathrm{Y} \times_{\mathrm{X}} \mathrm{Y}$, $(y, g) \mapsto (y, y \cdot g)$, the two projections maps p_1, p_2 from $\mathrm{Y} \times_{\mathrm{X}} \mathrm{Y}$ to Y correspond to the projection and the action maps p, a from $\mathrm{Y} \times \mathrm{GL}_r$ to Y. Thus, descent data along $\mathrm{Y} \to \mathrm{X}$ can be expressed as pairs $(\mathcal{N}, \varphi)$ where:

(1) $\mathcal{N} \in \mathsf{QCoh}(\mathrm{Y})$; and
(2) $\varphi : p^*\mathcal{N} \xrightarrow{\simeq} a^*\mathcal{N}$ is an isomorphism in $\mathsf{QCoh}(\mathrm{Y}\times\mathrm{GL}_r)$ satisfying the cocycle condition.

We can make the cocycle condition explicit: for each R-point (y,g) of $\mathrm{Y}\times\mathrm{GL}_r$ ($\mathrm{R}\in\mathsf{Ring}$), write $\varphi_{y,g} : y^*\mathcal{N} \xrightarrow{\simeq} (y\cdot g)^*\mathcal{N}$ for the pullback of φ. Then the cocycle condition asserts the equality $\varphi_{y,g_1g_2} = \varphi_{yg_1,g_2}\cdot\varphi_{y,g_1}$ for every R-point (y,g_1,g_2) of $\mathrm{Y}\times\mathrm{GL}_r\times\mathrm{GL}_r$.

Let us construct the functor inverse to (3.11):

$$\mathrm{Tors}_{\mathrm{GL}_r}(\mathrm{X}) \to \mathrm{Bun}_r(\mathrm{X}). \tag{3.12}$$

Namely, it sends Y to the descent datum $(\mathcal{O}_{\mathrm{Y}}^{\oplus r},\varphi)$ along $\mathrm{Y}\to\mathrm{X}$, where φ is the isomorphism $p^*\mathcal{O}_{\mathrm{Y}}^{\oplus r} \xrightarrow{\simeq} a^*\mathcal{O}_{\mathrm{Y}}^{\oplus r}$ given by the automorphism $g : \mathrm{R}^{\oplus r} \xrightarrow{\simeq} \mathrm{R}^{\oplus r}$ at any R-point (y,g) of $\mathrm{Y}\times\mathrm{GL}_r$ ($\mathrm{R}\in\mathsf{Ring}$). This descent datum yields a quasi-coherent $\mathcal{O}_{\mathrm{X}}$-module $\mathcal{M}$, which is a rank-r vector bundle because $\mathcal{O}_{\mathrm{Y}}^{\oplus r}$ is. □

Example 3.2.20 (The Picard group)**.** For $r = 1$, both sides of (3.11) have natural symmetric monoidal structures. Indeed, $\mathrm{Bun}_1(\mathrm{X})$ has a symmetric monoidal structure given by tensor product of line bundles, and $\mathrm{Tors}_{\mathbb{G}_m}(\mathrm{X})$ has the symmetric monoidal structure constructed in Remark 3.2.10. The equivalence (3.11) respects these symmetric monoidal structures.

The abelian group of their isomorphism calsses is called the *Picard group* of X:

$$\mathrm{Pic}(\mathrm{X}) \coloneqq \mathrm{Bun}_1(\mathrm{X})/\simeq\ \xrightarrow{\simeq} \mathrm{Tors}_{\mathbb{G}_m}(\mathrm{X})/\simeq.$$

3.2.21. We now address the second problem of this subsection: Express the projective space as a quotient of the affine space without origin.

For $n\in\mathbf{Z}_{\geq 0}$, we shall define a morphism:

$$\begin{array}{c} \mathbb{A}^{n+1}_{\mathbf{Z}} \smallsetminus 0 \\ \big\downarrow \pi \\ \mathbb{P}^n_{\mathbf{Z}} \end{array} \tag{3.13}$$

Given an R-point of $\mathbb{A}^{n+1}_{\mathbf{Z}}$, corresponding to elements $\mathrm{X}_0,\cdots,\mathrm{X}_n\in\mathrm{R}$, we consider the following morphism in $\mathsf{Mod}_{\mathrm{R}}$:

$$\mathrm{R}\to\mathrm{R}^{\oplus(n+1)},\quad 1\mapsto(\mathrm{X}_0,\cdots,\mathrm{X}_n). \tag{3.14}$$

Note that the R-point factors through $\mathbb{A}^{n+1}_{\mathbf{Z}}\smallsetminus 0$ if and only if (3.14) admits a retract. Indeed, the first condition is equivalent to that $\mathrm{X}_0,\cdots,\mathrm{X}_n$ generates R as an ideal, which is equivalent to the existence of $f_1,\cdots,f_n\in\mathrm{R}$ such that the map $\mathrm{R}^{\oplus(n+1)}\to\mathrm{R}$, sending the ith basis to f_i, provides a retract of (3.14). In this case, the image of (3.14) is a line subbundle, *i.e.* an R-point of $\mathbb{P}^n_{\mathbf{Z}}$.

The association from $\mathrm{X}_0,\cdots,\mathrm{X}_n$ to the image of the induced map (3.14) is functorial in $\mathrm{R}\in\mathsf{Ring}$. This concludes the construction of (3.13).

3.2.22. Next, we observe that $\mathbb{G}_m$ acts on $\mathbb{A}^{n+1}_{\mathbf{Z}}\smallsetminus 0$, where an R-point $\lambda\in\mathbb{G}_m(\mathrm{R})\cong\mathrm{R}^\times$ carries $(\mathrm{X}_0,\cdots,\mathrm{X}_n)$ to $(\mathrm{X}_0\cdot\lambda,\cdots,\mathrm{X}_n\cdot\lambda)$.

Clearly, this action does not change the image of the induced map (3.14), so we have an induced morphism in Shv:

$$(\mathbb{A}^{n+1}_{\mathbf{Z}}\smallsetminus 0)/\mathbb{G}_m \to \mathbb{P}^n_{\mathbf{Z}}. \tag{3.15}$$

Proposition 3.2.23. *The morphism* (3.15) *is an isomorphism.*

Proof. The assertion is equivalent to that (3.13) is a $\mathbb{G}_m$-torsor. For simple transitivity, this means that any two injections $R \to R^{\oplus(n+1)}$ in Mod_R whose images are the same direct summand of $R^{\oplus(n+1)}$ differ by multiplication by element of $R^{\times}$; this is clear. To construct local sections of (3.13), it suffices to observe that any line subbundle $\mathcal{L} \subset R^{\oplus(n+1)}$ is free on a standard Zariski cover of Spec(R); this follows by definition. □

Remark 3.2.24. For any scheme S, (3.15) induces an isomorphism:

$$(\mathbb{A}^{n+1}_S \smallsetminus 0)/\mathbb{G}_m \xrightarrow{\simeq} \mathbb{P}^n_S.$$

Indeed, this is because the formation of quotients is universal (*cf.* Lemma 1.4.2).

3.2.25. *Torsors under quasi-coherent sheaves.* As the last topic of this subsection, we shall study torsors under quasi-coherent sheaves on schemes.

Let X be a scheme. Then any $\mathcal{M} \in \mathsf{QCoh}(X)$ has an underlying sheaf of abelian groups on $(\mathsf{Sch}^{\mathrm{aff}})_{/X}$ (where covers are given by standard Zariski covers). In particular, we may consider the groupoid $\mathrm{Tors}_{\mathcal{M}}(X)$ of $\mathcal{M}$-torsors over X.

For any $\mathcal{M}, \mathcal{N} \in \mathsf{QCoh}(X)$, we consider the groupoid $\mathsf{Ext}^1(\mathcal{N}, \mathcal{M})$ of short exact sequences:

$$0 \to \mathcal{M} \to \mathcal{E} \to \mathcal{N} \to 0, \tag{3.16}$$

where a morphism from $0 \to \mathcal{M} \to \mathcal{E} \to \mathcal{N} \to 0$ to $0 \to \mathcal{M} \to \mathcal{E}' \to \mathcal{N} \to 0$ is an isomorphism $\mathcal{E} \xrightarrow{\simeq} \mathcal{E}'$ in $\mathsf{QCoh}(X)$ which induces the identity maps on $\mathcal{M}$ and $\mathcal{N}$.

There is a functor:

$$\mathsf{Ext}^1(\mathcal{O}_X, \mathcal{M}) \to \mathrm{Tors}_{\mathcal{M}}(X), \tag{3.17}$$

which sends a short exact sequence $0 \to \mathcal{M} \to \mathcal{E} \xrightarrow{p} \mathcal{O}_X \to 0$ to the torsor of its splittings—the sheaf which assigns to $\mathrm{Spec}(R) \to X$ ($R \in \mathsf{Ring}$) the set of R-linear maps $s : R \to \mathcal{E}|_{\mathrm{Spec}(R)}$ such that $p \cdot s = \mathrm{id}_R$; it is acted on simply transitively by $\mathcal{M}|_{\mathrm{Spec}(R)}$ via addition. The local triviality follows from the fact that R is a projective R-module.

Proposition 3.2.26. *The functor* (3.17) *is an equivalence of groupoids.*

Proof. We first prove that (3.17) is fully faithful. Given objects $0 \to \mathcal{M} \to \mathcal{E} \to \mathcal{O}_X \to 0$ and $0 \to \mathcal{M} \to \mathcal{E}' \to \mathcal{O}_X \to 0$ of $\mathsf{Ext}^1(\mathcal{O}_X, \mathcal{M})$, the set of morphisms between them is acted on simply transitively by $\mathrm{Hom}(\mathcal{O}_X, \mathcal{M}) \xrightarrow{\simeq} \Gamma(X, \mathcal{M})$. The map on Hom-sets defined by the functor (3.17) is $\Gamma(X, \mathcal{M})$-equivariant, thus bijective.

To prove that (3.17) is essentially surjective, we let P be any $\mathcal{M}$-torsor over X. By local triviality, we find an open cover $X_i \to X$ ($i \in I$) over which $P_i := P \times_X X_i$ is the trivial $\mathcal{M}$-torsor. Thus, P may be described by the descent data $f_{ij} \in \mathcal{M}_{ij}$, for $\mathcal{M}_{ij} := \Gamma(X_i \cap X_j, \mathcal{M})$ ($i, j \in I$). We construct a short exact sequence using descent of QCoh:

$$0 \to \mathcal{M} \to \mathcal{E} \to \mathcal{O}_X \to 0, \tag{3.18}$$

by setting it to be the split extension $\mathcal{M}_i \oplus \mathcal{O}_{X_i}$ over X_i (with $\mathcal{M}_i := \Gamma(X_i, \mathcal{M})$), glued using the following isomorphism over $X_{ij} := X_i \cap X_j$:

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{M}_{ij} & \to & \mathcal{M}_{ij} \oplus \mathcal{O}_{X_{ij}} & \to & \mathcal{O}_{X_{ij}} & \to & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \varphi_{ij}} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \to & \mathcal{M}_{ij} & \to & \mathcal{M}_{ij} \oplus \mathcal{O}_{X_{ij}} & \to & \mathcal{O}_{X_{ij}} & \to & 0 \end{array} \qquad \varphi_{ij} := \begin{pmatrix} \mathrm{id} & 0 \\ f_{ij} & \mathrm{id} \end{pmatrix}$$

Then the image of (3.18) under the functor (3.17) is isomorphic to P. □

Corollary 3.2.27. *Let* X *be a scheme and* $\mathcal{M} \in \mathsf{QCoh}(X)$. *If* X *is affine, then any* $\mathcal{M}$*-torsor over* X *is trivial.*

Proof. By Proposition 3.2.26, it suffices to prove that given a ring A and $\mathrm{M} \in \mathsf{Mod}_{\mathrm{A}}$, any extension of A by M splits. This holds because A is projective as an A-module. □

Remark 3.2.28. When the structure sheaf of $\mathrm{Spec}(\mathbf{Z})$ is viewed as a sheaf of abelian groups on $\mathsf{Sch}^{\mathrm{aff}}$, we also denote it by $\mathbb{G}_a$ and call it the "additive group". Of course, the underlying scheme of $\mathbb{G}_a$ is $\mathbb{A}^1_{\mathbf{Z}} \cong \mathrm{Spec}(\mathbf{Z}[x])$. The equivalence (3.17) specializes to:

$$\mathsf{Ext}^1(\mathcal{O}_{\mathrm{X}}, \mathcal{O}_{\mathrm{X}}) \xrightarrow{\simeq} \mathrm{Tors}_{\mathbb{G}_a}(\mathrm{X}).$$

In other words, $\mathbb{G}_a$-torsors over any scheme X are equivalent to extensions of $\mathcal{O}_{\mathrm{X}}$ by itself.

3.3. The "Proj" construction.

3.3.1. Let $\mathrm{X} = \mathrm{Spec}(\mathrm{A})$ be an affine scheme and $\mathrm{G} = \mathrm{Spec}(\mathrm{R})$ be an affine group scheme. The datum of a G-action on X can be rewritten in ring-theoretic terms as follows.

The identity and multiplication morphisms on G correspond to morphisms of rings $\epsilon : \mathrm{R} \to \mathbf{Z}$ and $\mu : \mathrm{R} \to \mathrm{R} \otimes \mathrm{R}$. The action morphism $\mathrm{X} \times \mathrm{G} \to \mathrm{X}$ corresponds to a morphism of rings $\rho : \mathrm{A} \to \mathrm{A} \otimes \mathrm{R}$, called *co-action*, and the unit and cocycle conditions translate into the following commutative diagrams:

$$\begin{array}{ccc} \mathrm{A} & \xrightarrow{\rho} & \mathrm{A} \otimes \mathrm{R} \\ & {}_{\mathrm{id}_{\mathrm{A}}}\searrow & \downarrow{\scriptstyle \mathrm{id}_{\mathrm{A}} \otimes \epsilon} \\ & & \mathrm{A} \end{array} \qquad \begin{array}{ccc} \mathrm{A} & \xrightarrow{\quad\rho\quad} & \mathrm{A} \otimes \mathrm{R} \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \mathrm{id}_{\mathrm{A}} \otimes \mu} \\ \mathrm{A} \otimes \mathrm{R} & \xrightarrow{\rho \otimes \mathrm{id}_{\mathrm{R}}} & \mathrm{A} \otimes \mathrm{R} \otimes \mathrm{R} \end{array}$$

3.3.2. Recall that a $\mathbf{Z}$-*graded ring* is a monoid in the category of $\mathbf{Z}$-graded abelian groups. More explicitly, it is a $\mathbf{Z}$-graded abelian group $\mathrm{A} = \bigoplus_{d \in \mathbf{Z}} \mathrm{A}_d$ equipped with a multiplication map $\mathrm{A} \otimes \mathrm{A} \to \mathrm{A}$ carrying $\mathrm{A}_d \otimes \mathrm{A}_{d'}$ into $\mathrm{A}_{d+d'}$ and a multiplicative unit $1 \in \mathrm{A}_0$.

Let A be a ring. We claim that the following data are equivalent:

(1) a $\mathbb{G}_m$-action on $\mathrm{Spec}(\mathrm{A})$;
(2) a structure of a $\mathbf{Z}$-graded ring $\mathrm{A} = \bigoplus_{d \in \mathbf{Z}} \mathrm{A}_d$.

Indeed, given a $\mathbb{G}_m$-action on $\mathrm{Spec}(\mathrm{A})$, the co-action morphism is a map $\rho : \mathrm{A} \to \mathrm{A}[x, x^{-1}]$. We set $\mathrm{A}_d := \rho^{-1}(\mathrm{A}x^d)$ to obtain a $\mathbf{Z}$-graded ring structure on A. Conversely, given a $\mathbf{Z}$-graded ring structure $\mathrm{A} = \bigoplus_{d \in \mathbf{Z}} \mathrm{A}_d$, we define $\rho : \mathrm{A} \to \mathrm{A}[x, x^{-1}]$ to be the map sending $a \in \mathrm{A}$ to $\sum_{d \in \mathbf{Z}} a_d x^d \in \mathrm{A}[x, x^{-1}]$, where $a_d \in \mathrm{A}_d$ is the degree-d component of a.

3.3.3. Given a $\mathbf{Z}_{\geq 0}$-graded ring $\mathrm{A} = \bigoplus_{d \geq 0} \mathrm{A}_d$, the quotient map $\mathrm{A} \to \mathrm{A}_0$ is compatible with the grading, so it defines a $\mathbb{G}_m$-equivariant morphism:

$$0 : \mathrm{Spec}(\mathrm{A}_0) \to \mathrm{Spec}(\mathrm{A}),$$

where $\mathbb{G}_m$ acts trivially on $\mathrm{Spec}(\mathrm{A}_0)$.

Let f be an element of $\mathrm{A}_{\geq 1} := \bigoplus_{d \geq 1} \mathrm{A}_d$. Then the open immersion $\mathrm{Spec}(\mathrm{A}_f) \to \mathrm{Spec}(\mathrm{A})$ factors through $\mathrm{Spec}(\mathrm{A}) \smallsetminus 0$. If f is furthermore homogenous, then A_f inherits a $\mathbf{Z}$-grading such that $\mathrm{A} \to \mathrm{A}_f$ is a morphism of $\mathbf{Z}$-graded rings. In this case, we obtain a $\mathbb{G}_m$-equivariant open immersion:

$$\mathrm{Spec}(\mathrm{A}_f) \to \mathrm{Spec}(\mathrm{A}) \smallsetminus 0.$$

Lemma 3.3.4. *Let* A *be a* $\mathbf{Z}_{\geq 0}$*-graded ring. Suppose that* A *is generated by* A_1 *as an* A_0*-algebra. Then for any* $f \in \mathrm{A}_1$*, there is a canonical* $\mathbb{G}_m$*-equivariant isomorphism:*

$$\mathrm{Spec}(\mathrm{A}_f) \xrightarrow{\simeq} \mathrm{Spec}((\mathrm{A}_f)_0) \times \mathbb{G}_m,$$

where $(\mathrm{A}_f)_0$ *is the degree-*0 *component of* A_f *and* $\mathbb{G}_m$ *acts on the right-hand-side by multiplication on the* $\mathbb{G}_m$*-factor.*

Proof. In ring-theoretic terms, the assertion means that there is a canonical isomorphism of $\mathbf{Z}$-graded rings:

$$\mathrm{A}_f \xrightarrow{\simeq} (\mathrm{A}_f)_0[x, x^{-1}],$$

where x is of degree 1. Let us construct this isomorphism.

Since $f \in (\mathrm{A}_f)_1$ is invertible, multiplication by f^d defines an isomorphism of $(\mathrm{A}_f)_0$-modules $(\mathrm{A}_f)_0 \xrightarrow{\simeq} (\mathrm{A}_f)_d$ for every $d \in \mathbf{Z}$.

We first apply this observation to $d = 1$: by assumption, A_f is generated by $(\mathrm{A}_f)_1$ and f^{-1} over $(\mathrm{A}_f)_0$. Since every element of $(\mathrm{A}_f)_1$ is an $(\mathrm{A}_f)_0$-multiple of f, A_f is in fact generated by f and f^{-1} as an $(\mathrm{A}_f)_0$-algebra. This defines a map of $\mathbf{Z}$-graded $(\mathrm{A}_f)_0$-algebras:

$$(\mathrm{A}_f)_0[x, x^{-1}] \to \mathrm{A}_f, \quad x \mapsto f.$$

To see that it is bijective on each component, we apply the above observation again. □

Proposition 3.3.5. *Let* A *be a* $\mathbf{Z}_{\geq 0}$*-graded ring. Suppose that* A *is generated by* A_1 *as an* A_0*-algebra. Then:*

(1) $\mathbb{G}_m$ *acts freely on* $\mathrm{Spec(A)} \smallsetminus 0$*;*
(2) the following object of Shv *is a scheme:*

$$\mathrm{Proj(A)} \coloneqq (\mathrm{Spec(A)} \smallsetminus 0)/\mathbb{G}_m.$$

Proof. We begin with a general observation: Given a scheme X equipped with the action of a group scheme G, if X admits a G-stable open cover $\mathrm{X}_i \to \mathrm{X}$ ($i \in \mathrm{I}$) such that the G-action on X_i is free, then the G-action on X is free. Indeed, we need to prove that the morphism:

$$\mathrm{X} \times \mathrm{G} \to \mathrm{X} \times \mathrm{X}, \quad (x, g) \mapsto (x, x \cdot g)$$

is a monomorphism. This can be checked after base change along $\mathrm{X}_i \times \mathrm{X} \to \mathrm{X} \times \mathrm{X}$ for each $i \in \mathrm{I}$ (*cf.* Lemma 1.4.8). The fact that X_i is G-stable means that the base change factors as:

$$\mathrm{X}_i \times \mathrm{G} \to \mathrm{X}_i \times \mathrm{X}_i \to \mathrm{X}_i \times \mathrm{X},$$

where the first morphism is a monomorphism since the G-action on X_i is free, and the second morphism is a monomorphism since it is an open immersion.

Since A is generated by A_1 as an A_0-algebra, the collection of $\mathbb{G}_m$-equivariant maps:

$$\mathrm{Spec}(\mathrm{A}_f) \to \mathrm{Spec(A)} \smallsetminus 0, \quad f \in \mathrm{A}_1$$

forms an open cover. Moreoever, by Lemma 3.3.4, each $\mathrm{Spec}(\mathrm{A}_f)$ is $\mathbb{G}_m$-equivariantly isomorphic to $\mathrm{Spec}((\mathrm{A}_f)_0) \times \mathbb{G}_m$. In particular, $\mathbb{G}_m$ acts freely on $\mathrm{Spec}(\mathrm{A}_f)$. This implies that $\mathbb{G}_m$ acts freely on $\mathrm{Spec(A)} \smallsetminus 0$, using the observation above. Statement (1) is proved.

To prove statement (2), we note that the induced morphisms:

$$\mathrm{Spec}((\mathrm{A}_f)_0) \xrightarrow{\simeq} \mathrm{Spec}(\mathrm{A}_f)/\mathbb{G}_m \to (\mathrm{Spec(A)} \smallsetminus 0)/\mathbb{G}_m, \quad f \in \mathrm{A}_1$$

form an open cover, where we used Corollary 1.5.11 for descent of open immersions and Lemma 3.3.4 for the first isomorphism. This provides the Zariski sheaf $(\mathrm{Spec(A)} \smallsetminus 0)/\mathbb{G}_m$ with an open cover by affine schemes. □

3.3.6. The "Proj" construction is compatible with base change in A_0. More precisely, given a $\mathbf{Z}_{\geq 0}$-graded ring A which is generated by A_1 as an A_0-algebra and a morphism of rings $\mathrm{A}_0 \to \mathrm{B}_0$, we may form $\mathrm{B} \coloneqq \mathrm{A} \otimes_{\mathrm{A}_0} \mathrm{B}_0$. Then B is a $\mathbf{Z}_{\geq 0}$-graded ring with $\mathrm{B}_d \coloneqq \mathrm{A}_d \otimes_{\mathrm{A}_0} \mathrm{B}_0$ and is generated by B_1 as a B_0-algebra.

Moreover, we have an isomorphism:

$$\mathrm{Proj(B)} \xrightarrow{\simeq} \mathrm{Proj(A)} \times_{\mathrm{Spec}(\mathrm{A}_0)} \mathrm{Spec}(\mathrm{B}_0).$$

Indeed, this follows from the fact that complements and quotients are both compatible with base change.

3.3.7. Let S be a scheme and $\mathcal{A} = \bigoplus_{d\in\mathbf{Z}_{\geq 0}} \mathcal{A}_d$ be a quasi-coherent $\mathbf{Z}_{\geq 0}$-graded $\mathcal{O}_S$-algebra which is generated by $\mathcal{A}_1$ as an $\mathcal{A}_0$-algebra. By the functoriality of Proj noted in §3.3.6, we obtain a scheme:

$$\mathrm{Proj}_S(\mathcal{A}) \coloneqq (\mathrm{Spec}_S(\mathcal{A}) \smallsetminus 0)/\mathbb{G}_m$$

whose base change to any affine scheme $s : \mathrm{Spec}(R) \to S$ is $\mathrm{Proj}(s^*\mathcal{A})$, where $s^*\mathcal{A}$ is viewed as a $\mathbf{Z}_{\geq 0}$-graded R-algebra.

The $\mathbb{G}_m$-torsor $\mathrm{Spec}_S(\mathcal{A}) \smallsetminus 0$ over $\mathrm{Proj}_S(\mathcal{A})$ corresponds to a line bundle, which we denote by $\mathcal{O}_{\mathrm{Proj}_S(\mathcal{A})}(-1)$. For each $d \in \mathbf{Z}$, its $(-d)$th tensor power is denoted by $\mathcal{O}_{\mathrm{Proj}_S(\mathcal{A})}(d)$.

Example 3.3.8. Let S be a scheme and $\mathcal{M} \in \mathsf{QCoh}(S)$. Then $\mathrm{Sym}_{\mathcal{O}_S}(\mathcal{M})$ has the structure of a $\mathbf{Z}_{\geq 0}$-graded $\mathcal{O}_S$-algebras with degree-d component $\mathrm{Sym}^d_{\mathcal{O}_S}(\mathcal{M})$. In this case, we write:

$$\mathbb{P}(\mathcal{M}) \coloneqq \mathrm{Proj}_S(\mathrm{Sym}_{\mathcal{O}_S} \mathcal{M}).$$

If $\mathcal{M}$ is isomorphic to $\mathcal{O}_S^{\oplus r+1}$ for some $r \in \mathbf{Z}_{\geq 0}$, then $\mathbb{P}(\mathcal{O}_S^{\oplus r+1})$ is isomorphic to $\mathbb{P}^r_S$ (*cf.* Proposition 3.2.23) and the line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{O}_S^{\oplus r+1})}(1)$ defined above coincides with $\mathcal{O}_{\mathbb{P}^r_S}(1)$, the pullback of $\mathcal{O}_{\mathbb{P}^r_{\mathbf{Z}}}(1)$ defined in §3.1.9.

More generally, if $\mathcal{M}$ is a vector bundle, then $\mathrm{Spec}_S(\mathrm{Sym}_{\mathcal{O}_S} \mathcal{M})$ is isomorphic to the total space $\mathbb{V}(\mathcal{M}^\vee)$ of $\mathcal{M}^\vee$ (*cf.* Remark 2.4.7). In other words, $\mathbb{P}(\mathcal{M})$ parametrizes line subbundles of $\mathcal{M}^\vee$, *i.e.* rank-1 quotients of $\mathcal{M}$.

3.3.9. The following result generalizes the comparison between the two descriptions of the projective space.

Proposition 3.3.10. *Let* S *be a scheme and* $\mathcal{A} = \bigoplus_{d\in\mathbf{Z}_{\geq 0}} \mathcal{A}_d$ *be a quasi-coherent* $\mathbf{Z}_{\geq 0}$*-graded* $\mathcal{O}_S$*-algebra generated by* $\mathcal{A}_1$ *as an* $\mathcal{A}_0$*-algebra. Given a morphism of schemes* $f : X \to S$*, the following (discrete) groupoids are canonically equivalent:*

1. *the (discrete) groupoid of morphisms* $X \to \mathrm{Proj}_S(\mathcal{A})$ *over* S*;*
2. *the groupoid of line bundles* $\mathcal{Q}$ *over* X *equipped with a morphism of quasi-coherent* $\mathbf{Z}_{\geq 0}$*-graded* $\mathcal{O}_X$*-algebras:*
$$f^*\mathcal{A} \to \bigoplus_{d\geq 0} \mathcal{Q}^{\otimes d},$$
such that $f^*\mathcal{A}_1 \to \mathcal{Q}$ *is surjective.*

Proof. By pulling back $\mathcal{A}$ along f, we may assume that $\mathcal{A}$ is a quasi-coherent $\mathbf{Z}_{\geq 0}$-graded $\mathcal{O}_X$-algebra and $f = \mathrm{id}_X$.

A morphism $X \to \mathrm{Proj}_X(\mathcal{A})$ over X is equivalent to a $\mathbb{G}_m$-torsor $\widetilde{X} \to X$ together with a $\mathbb{G}_m$-equivariant morphism $\widetilde{X} \to \mathrm{Spec}_X(\mathcal{A}) \smallsetminus 0$ over X (*cf.* Lemma 3.2.12).

Let $\mathcal{Q}$ be the *dual* of the line bundle associated to $\widetilde{X}$, so $\widetilde{X}$ is isomorphic as a scheme over X to $\mathbb{V}(\mathcal{Q}^\vee) \smallsetminus 0 \xrightarrow{\simeq} \mathrm{Spec}_X(\bigoplus_{d\in\mathbf{Z}} \mathcal{Q}^{\otimes d})$. Hence a $\mathbb{G}_m$-equivariant morphism $\widetilde{X} \to \mathrm{Spec}_X(\mathcal{A}) \smallsetminus 0$ over X is equivalent to a morphism of quasi-coherent $\mathbf{Z}$-graded $\mathcal{O}_X$-algebras:

$$\mathcal{A} \to \bigoplus_{d\in\mathbf{Z}} \mathcal{Q}^{\otimes d}, \tag{3.19}$$

whose base change along $\mathcal{A} \to \mathcal{A}_0$ vanishes.

Since $\mathcal{A}$ is $\mathbf{Z}_{\geq 0}$-graded, the morphism (3.19) is equivalent to a morphism into $\bigoplus_{d\geq 0} \mathcal{Q}^{\otimes d}$. In other words, $\mathbb{G}_m$-equivariant morphisms $\mathbb{V}(\mathcal{Q}^\vee) \smallsetminus 0 \to \mathrm{Spec}_X(\mathcal{A})$ extend uniquely to $\mathbb{G}_m$-equivariant morphisms $\mathbb{V}(\mathcal{Q}^\vee) \to \mathrm{Spec}_X(\mathcal{A})$.

It remains to prove that $(\bigoplus_{d\in\mathbf{Z}}\mathcal{Q}^{\otimes d})\otimes_{\mathcal{A}}\mathcal{A}_0$ vanishes if and only if the map $\mathcal{A}_1\to\mathcal{Q}$ is surjective. The "$\Leftarrow$" direction is clear. To prove the "$\Rightarrow$" direction, note that since $\mathcal{Q}$ is a finite $\mathcal{O}_{\mathrm{X}}$-module, it suffices to show that the fiber of the map $\mathcal{A}_1\to\mathcal{Q}$ at any field-valued point of X is nonzero (Nakayama). Upon base change, we may assume that X is the spectrum of a field. If $\mathcal{A}_1\to\mathcal{Q}$ vanishes, the fact that $\mathcal{A}$ is generated by $\mathcal{A}_1$ over $\mathcal{A}_0$ implies that (3.19) factors through $\mathcal{A}_0$, but then its base change along $\mathcal{A}\to\mathcal{A}_0$ must not vanish. □

Remark 3.3.11. Under the equivalent of Proposition 3.3.10, the line bundle $\mathcal{Q}$ defined by a morphism $\mathrm{X}\to\mathrm{Proj}_{\mathrm{S}}(\mathcal{A})$ over S is the pullback of $\mathcal{O}_{\mathrm{Proj}_{\mathrm{S}}(\mathcal{A})}(1)$.

3.3.12. Let $f:\mathrm{X}\to\mathrm{S}$ be a quasi-compact, quasi-separated morphism of schemes. Let $\mathcal{Q}$ be a line bundle over X.

We say that $\mathcal{Q}$ is *f-very ample* if the canonical morphism $f^*f_*\mathcal{Q}\to\mathcal{Q}$ is surjective and the morphism $\mathrm{X}\to\mathbb{P}(f_*\mathcal{Q})$ induced from the equivalence of Proposition 3.3.10 is a locally closed immersion.

3.3.13. Let $f:\mathrm{X}\to\mathrm{S}$ be a quasi-compact, quasi-separated morphism of schemes. Let $\mathcal{A}=\bigoplus_{d\in\mathbf{Z}_{\geq 0}}\mathcal{A}_d$ be a quasi-coherent $\mathbf{Z}_{\geq 0}$-graded $\mathcal{O}_{\mathrm{S}}$-algebra such that $\mathcal{A}$ is generated by $\mathcal{A}_1$ as an $\mathcal{A}_0$-algebra.

Suppose that f factors through a closed immersion $i:\mathrm{X}\to\mathrm{Proj}_{\mathrm{S}}(\mathcal{A})$ over S. In this case, we can realize X itself as $\mathrm{Proj}_{\mathrm{S}}(\mathcal{B})$ for a quasi-coherent $\mathbf{Z}_{\geq 0}$-graded $\mathcal{O}_{\mathrm{S}}$-algebra $\mathcal{B}=\bigoplus_{d\in\mathbf{Z}_{\geq 0}}\mathcal{B}_d$ such that $\mathcal{B}$ is generated by $\mathcal{B}_1$ as a $\mathcal{B}_0$-algebra.

Construction. The morphism i corresponds to a map $f^*\mathcal{A}\to\bigoplus_{d\geq 0}\mathcal{Q}^{\otimes d}$ of $\mathbf{Z}_{\geq 0}$-graded $\mathcal{O}_{\mathrm{X}}$-algebras under Proposition 3.3.10. Applying adjunction, we obtain a map $\mathcal{A}\to\bigoplus_{d\geq 0}f_*(\mathcal{Q}^{\otimes d})$ of $\mathbf{Z}_{\geq 0}$-graded $\mathcal{O}_{\mathrm{S}}$-algebras. Let $\mathcal{B}$ be the image of this morphism:

$$\mathcal{A}\twoheadrightarrow\mathcal{B}\subset\bigoplus_{d\geq 0}f_*(\mathcal{Q}^{\otimes d}).$$

Then $\mathcal{B}$ has the structure of a $\mathbf{Z}_{\geq 0}$-graded $\mathcal{O}_{\mathrm{S}}$-algebra. Since $\mathcal{A}$ is generated by $\mathcal{A}_1$ as an $\mathcal{A}_0$-algebra, $\mathcal{B}$ is also generated by $\mathcal{B}_1$ as a $\mathcal{B}_0$-algebra.

Taking relative spectra, we have the following $\mathbb{G}_m$-equivariant morphisms over S:

$$\mathbb{V}(\mathcal{Q}^\vee)\xrightarrow{\simeq}\mathrm{Spec}_{\mathrm{X}}(\bigoplus_{d\geq 0}\mathcal{Q}^{\otimes d})\to\mathrm{Spec}_{\mathrm{S}}(\mathcal{B})\hookrightarrow\mathrm{Spec}_{\mathrm{S}}(\mathcal{A}). \tag{3.20}$$

By construction, $\mathrm{Spec}_{\mathrm{S}}(\mathcal{B})$ is the scheme-theoretic image of the morphism $\mathbb{V}(\mathcal{Q}^\vee)\to\mathrm{Spec}_{\mathrm{S}}(\mathcal{A})$ (*cf.* §2.5.6), so its formation commutes with pullbacks by open immersions. The base change of (3.20) to $\mathrm{Spec}_{\mathrm{S}}(\mathcal{A})\smallsetminus 0$ yields the $\mathbb{G}_m$-equivariant morphism:

$$\mathbb{V}(\mathcal{Q}^\vee)\smallsetminus 0\to\mathrm{Spec}_{\mathrm{S}}(\mathcal{B})\smallsetminus 0\to\mathrm{Spec}_{\mathrm{S}}(\mathcal{A})\smallsetminus 0, \tag{3.21}$$

where $\mathrm{Spec}_{\mathrm{S}}(\mathcal{B})\smallsetminus 0$ is the scheme-theoretic image of the composition. However, this composition is a closed immersion, since it is identified with the base change of i along $\mathrm{Spec}_{\mathrm{S}}(\mathcal{A})\smallsetminus 0\to\mathrm{Proj}_{\mathrm{S}}(\mathcal{A})$. This implies that the first morphism in (3.21) is an isomorphism. So we obtain isomorphisms of schemes over S:

$$\mathrm{X}\xrightarrow{\simeq}(\mathbb{V}(\mathcal{Q}^\vee)\smallsetminus 0)/\mathbb{G}_m\xrightarrow{\simeq}(\mathrm{Spec}_{\mathrm{S}}(\mathcal{B})\smallsetminus 0)/\mathbb{G}_m,$$

where the last term is by definition $\mathrm{Proj}_{\mathrm{S}}(\mathcal{B})$. □

Remark 3.3.14. In particular, the construction of §3.3.13 shows that the "Proj" construction exhausts all closed subschemes of $\mathbb{P}^n_{\mathrm{S}}$, $n\in\mathbf{Z}_{\geq 0}$.

3.4. Around the Rees algebra.

3.4.1. Let X be a scheme. A closed subscheme $D \to X$ is called an *effective Cartier divisor* if its corresponding ideal sheaf $\mathcal{I}$ is a line bundle.

Note that this is equivalent to the condition that X admits an open cover $\mathrm{Spec}(A_i) \to X$ ($i \in I$) such that $D_i := D \times_X \mathrm{Spec}(A_i)$ is defined by the ideal $(f_i) \subset A_i$ generated by a non-zero-divisor $f_i \in A_i$. Indeed, this follows from the fact that an ideal $\mathfrak{a} \subset A$ of a ring A is free of rank-1 if and only if $\mathfrak{a} = (f)$ for a non-zero-divisor $f \in A$.

3.4.2. Given a scheme X and an effective Cartier divisor D with ideal sheaf $\mathcal{I}$, we obtain a morphism of line bundles $\mathcal{I} \to \mathcal{O}_X$. Dualizing, we find a morphism of line bundles:

$$\mathcal{O}_X \to \mathcal{O}_X(D) := \mathcal{I}^{\otimes -1}.$$

Thus, $\mathcal{O}_X(D)$ may be regarded as a line bundle over X equipped with a global section $1_D \in \Gamma(X, \mathcal{O}_X(D))$.

Given a line bundle $\mathcal{L}$ over X, a section $f \in \Gamma(X, \mathcal{L})$ is called *regular* if the induced map $f : \mathcal{O}_X \to \mathcal{L}$ in $\mathsf{QCoh}(X)$ is injective. The section 1_D associated to an effective Cartier divisor is regular since it is locally given by multiplication by a non-zero-divisor.

Lemma 3.4.3. *Let* X *be a scheme. The association* $D \mapsto (\mathcal{O}_X(D), 1_D)$ *defines an equivalence between the following (discrete) groupoids:*

1. *the (discrete) groupoid of effective Cartier divisors on* X*;*
2. *the groupoid of pairs* $(\mathcal{L}, f)$ *where* $\mathcal{L}$ *is a line bundle over* X *and* $f \in \Gamma(X, \mathcal{L})$ *is a regular section.*

Proof. The inverse (2) ⇒ (1) assigns to a pair $(\mathcal{L}, f)$ the closed subscheme corresponding to $\mathcal{I} := \mathcal{L}^\vee$, realized as an ideal sheaf via the dual of f. □

Remark 3.4.4. The effective Cartier divisor D associated to the pair $(\mathcal{L}, f)$ under Lemma 3.4.3 is isomorphic to the vanishing locus $X_{f=0}$ defined in Remark 3.1.7.

3.4.5. Given a scheme X and effective Cartier divisors D_1, D_2 with ideal sheaves $\mathcal{I}_1$, $\mathcal{I}_2$, we define $D_1 + D_2$ to be the closed subscheme of X associated to the ideal sheaf $\mathcal{I}_1\mathcal{I}_2 \subset \mathcal{O}_X$.

Using flatness of either $\mathcal{I}_1$ or $\mathcal{I}_2$ over $\mathcal{O}_X$, we see that the natural map $\mathcal{I}_1 \otimes_{\mathcal{O}_X} \mathcal{I}_2 \to \mathcal{O}_X$ is injective, so $\mathcal{I}_1\mathcal{I}_2$ is identified wtih $\mathcal{I}_1 \otimes_{\mathcal{O}_X} \mathcal{I}_2$. In particular, $D_1 + D_2$ is also an effective Cartier divisor on X.

The sum operation thus defined turns the set $\mathrm{Div}_+(X)$ of effective Cartier divisors on X into a monoid. The unit is the effective Cartier divisor $\varnothing \to X$. The above identification of ideal sheaves shows that we have a map of monoids:

$$\mathrm{Div}_+(X) \to \mathrm{Pic}(X), \quad D \mapsto \mathcal{O}_X(D),$$

where Pic(X) is the group of isomorphism classes of line bundles over X (*cf.* Example 3.2.20).

3.4.6. We shall now describe a procedure which replaces every closed subscheme by an effective Cartier divisor. This procedure is called "blow up".

Let X be a scheme and $Z \to X$ be a closed immersion. Consider the category of schemes Y over X such that $Z \times_X Y \to Y$ is an effective Cartier divisor. The terminal object of this category is called the *blow-up* of X along Z.

On the other hand, write $\mathcal{I}$ for the ideal sheaf corresponding to Z. We may form a quasi-coherent $\mathbf{Z}_{\geq 0}$-graded $\mathcal{O}_X$-algebra $\bigoplus_{d \geq 0} \mathcal{I}^d$, called the *Rees algebra* of $\mathcal{I}$. By construction, it is generated by the degree-1 component $\mathcal{I}$ over the degree-0 component $\mathcal{I}^0 \cong \mathcal{O}_X$.

Proposition 3.4.7. *Let* X *be a scheme and* $Z \to X$ *be a closed immersion corresponding to the ideal sheaf* $\mathcal{I}$*. Then:*

(1) the blow up $\mathrm{Bl_Z\,X}$ *of* X *along* Z *is represented by the following scheme over* X*:*

$$\mathrm{Bl_Z\,X} \xrightarrow{\simeq} \mathrm{Proj_X}(\bigoplus_{d\geq 0} \mathcal{I}^d);$$

(2) writing $\mathrm{E} := \mathrm{Bl_Z\,X} \times_{\mathrm{X}} \mathrm{Z}$*, we have a canonical isomorphism of schemes over* Z*:*

$$\mathrm{E} \xrightarrow{\simeq} \mathrm{Proj_Z}(\bigoplus_{d\geq 0} \mathcal{I}^d/\mathcal{I}^{d+1}).$$

(The closed subscheme E *of* $\mathrm{Bl_Z\,X}$ *is called the* exceptional divisor*.)*

3.4.8. We shall prove Proposition 3.4.7 by relating the Rees algebra to another construction, called "deformation to the normal cone". Namely, we consider the quasi-coherent $\mathbf{Z}$-graded $\mathcal{O}_X$-algebra $\bigoplus_{d\in\mathbf{Z}} \mathcal{I}^d$ where we set $\mathcal{I}^d := \mathcal{O}_X$ for all $d \leq 0$. This algebra contains the Rees algebra as a $\mathbf{Z}_{\geq 0}$-graded subalgebra.

The relative spectrum $\mathrm{Spec_X}(\bigoplus_{d\geq 0} \mathcal{I}^d)$ contains the zero section $i : \mathrm{X} \to \mathrm{Spec_X}(\bigoplus_{d\geq 0} \mathcal{I}^d)$ defined by the ideal $\bigoplus_{d\geq 1} \mathcal{I}^d$. The pullback of i to $\mathrm{Spec_X}(\bigoplus_{d\in\mathbf{Z}} \mathcal{I}^d)$ is the following $\mathbb{G}_m$-equivariant closed immersion:

$$\tilde{i} : \mathrm{Z} \times \mathbb{A}^{1,-} \to \mathrm{Spec_X}(\bigoplus_{d\in\mathbf{Z}} \mathcal{I}^d), \tag{3.22}$$

where $\mathbb{A}^{1,-}$ indicates a copy of $\mathbb{A}^1_{\mathbf{Z}}$ on which $\mathbb{G}_m$ acts by $a, x \mapsto a^{-1}x$ for every R-point (a, x) of $\mathbb{G}_m \times \mathbb{A}^{1,-}$ ($\mathrm{R} \in \mathsf{Ring}$). Indeed, the closed immersion $\tilde{i}$ corresponds to the quotient map $\bigoplus_{d\in\mathbf{Z}} \mathcal{I}^d \to \bigoplus_{d\leq 0} \mathcal{O}_X/\mathcal{I}$.

The induced morphism on complements is a $\mathbb{G}_m$-equivariant isomorphism:

$$\mathrm{Spec_X}(\bigoplus_{d\in\mathbf{Z}} \mathcal{I}^d) \smallsetminus (\mathrm{Z} \times \mathbb{A}^{1,-}) \xrightarrow{\simeq} \mathrm{Spec_X}(\bigoplus_{d\geq 0} \mathcal{I}^d) \smallsetminus \mathrm{X}.$$

Indeed, by the open cover constructed in the proof of Proposition 3.3.5, it suffices to show that the inclusion $\bigoplus_{d\geq 0} \mathcal{I}^d \to \bigoplus_{d\in\mathbf{Z}} \mathcal{I}^d$ becomes an isomorphism upon localizing at any $f \in \mathcal{I}$ lying in degree 1. This holds because the cokernel $\bigoplus_{d\leq -1} \mathcal{I}^d$ consists of f-torsion elements.

3.4.9. Moreover, the $\mathbf{Z}$-graded $\mathcal{O}_X$-algebra $\bigoplus_{d\in\mathbf{Z}} \mathcal{I}^d$ has a canonical section t given by $1 \in \mathcal{O}_X$ placed in degree -1. This section corresponds to a $\mathbb{G}_m$-equivariant morphism:

$$t : \mathrm{Spec_X}(\bigoplus_{d\in\mathbf{Z}} \mathcal{I}^d) \to \mathbb{A}^{1,-}. \tag{3.23}$$

Its composite with (3.22) is the projection $\mathrm{Z} \times \mathbb{A}^{1,-} \to \mathbb{A}^{1,-}$ onto the second factor.

The fiber of (3.23) at 0 is isomorphic to $\mathrm{C_{Z/X}} := \mathrm{Spec_X}(\bigoplus_{d\geq 0} \mathcal{I}^d/\mathcal{I}^{d+1})$, called the *normal cone* of Z. The fiber of (3.23) at $\mathbb{A}^{1,-} \smallsetminus 0$ is isomorphic to $\mathrm{Spec_X}(\bigoplus_{d\in\mathbf{Z}} \mathcal{O}_X) \xrightarrow{\simeq} \mathrm{X} \times (\mathbb{A}^{1,-} \smallsetminus 0)$. The morphism (3.23) is called the *deformation to the normal cone* because it realizes a degeneration of the closed immersion $\mathrm{Z} \to \mathrm{X}$ to the closed immersion $\mathrm{Z} \to \mathrm{C_{Z/X}}$.

3.4.10. The following diagram summarizes the relations among the relevant schemes:

$$\begin{array}{ccccccccc}
\mathrm{Z} & \longrightarrow & \mathrm{C_{Z/X}} & \longleftarrow & \mathrm{C_{Z/X}} \smallsetminus \mathrm{Z} & \xrightarrow{\mathbb{G}_m} & \mathrm{Proj_Z}(\bigoplus_{d\geq 0} \mathcal{I}^d/\mathcal{I}^{d+1}) & \to & \mathrm{Z} \\
\big\downarrow & & \big\downarrow{\scriptstyle 0} & & \big\downarrow & & \big\downarrow & & \big\downarrow \\
\mathrm{Z} \times \mathbb{A}^{1,-} & \xrightarrow{\tilde{i}} & \mathrm{Spec_X}(\bigoplus_{d\in\mathbf{Z}} \mathcal{I}^d) & \xleftarrow{\tilde{j}} & \mathrm{Spec_X}(\bigoplus_{d\geq 0} \mathcal{I}^d) \smallsetminus \mathrm{X} & \xrightarrow{\mathbb{G}_m} & \mathrm{Proj_X}(\bigoplus_{d\geq 0} \mathcal{I}^d) & \longrightarrow & \mathrm{X} \\
 & & \big\downarrow{\scriptstyle t} & & & & & & \\
 & & \mathbb{A}^{1,-} & & & & & &
\end{array} \tag{3.24}$$

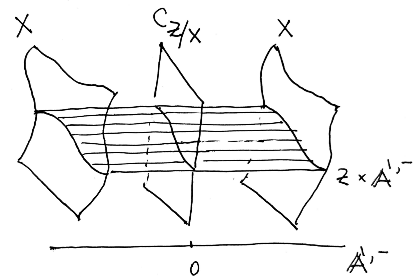

FIGURE 2. The deformation to the normal cone associated to a closed immersion $\mathrm{Z}\to\mathrm{X}$. The total space of the deformation is $\mathrm{Spec}_{\mathrm{X}}(\bigoplus_{d\in\mathbf{Z}}\mathcal{O}_{\mathrm{X}})$, which contains $\mathrm{Z}\times\mathbb{A}^{1,-}$ as a closed subscheme. The generic fiber over $\mathbb{A}^{1,-}$ is the closed immersion $\mathrm{Z}\to\mathrm{X}$. The special fiber at 0 is the closed immersion $\mathrm{Z}\to\mathrm{C}_{\mathrm{Z}/\mathrm{X}}$.

Here, $\tilde{j}$ is the complement of $\tilde{i}$, and the squares are Cartesian. The morphisms labeled by "$\mathbb{G}_m$" are $\mathbb{G}_m$-torsors.

Proof of Proposition 3.4.7. To prove (1), we define $\mathrm{Bl}_{\mathrm{Z}}\mathrm{X}:=\mathrm{Proj}_{\mathrm{X}}(\bigoplus_{d\geq 0}\mathcal{I}^d)$ and prove that it satisfies the universal property of the blow-up of X at Z. First, we need to prove that:

$$\mathrm{E}:=\mathrm{Bl}_{\mathrm{Z}}\mathrm{X}\times_{\mathrm{X}}\mathrm{Z}\xrightarrow{\simeq}\mathrm{Proj}_{\mathrm{Z}}(\bigoplus_{d\geq 0}\mathcal{I}^d/\mathcal{I}^{d+1})$$

is an effective Cartier divisor. This isomorphism would then yield (2). By the Cartesian squares in (3.24), this reduces to showing that $0:\mathrm{C}_{\mathrm{Z}/\mathrm{X}}\to\mathrm{Spec}_{\mathrm{X}}(\bigoplus_{d\in\mathbf{Z}}\mathcal{I}^d)$ is an effective Cartier divisor. This holds because the section t of $\bigoplus_{d\in\mathbf{Z}}\mathcal{I}^d$ is a non-zero-divisor. (Multiplication by t is the operation of shifting the grading by one unit, hence injective.)

To prove that $\mathrm{Bl}_{\mathrm{Z}}\mathrm{X}$ satisfies the universal property of the blow-up, we let $f:\mathrm{Y}\to\mathrm{X}$ be any morphism of schemes such that $f^{-1}\mathrm{Z}:=\mathrm{Y}\times_{\mathrm{X}}\mathrm{Z}$ is an effective Cartier divisor in Y. Let $\mathcal{I}_{f^{-1}\mathrm{Z}}$ denote the ideal sheaf associated to $f^{-1}\mathrm{Z}$. By definition, it is a line bundle and we have a surjective morphism $f^*\mathcal{I}\to\mathcal{I}_{f^{-1}\mathrm{Z}}$ in $\mathsf{QCoh}(\mathrm{Y})$ over $\mathcal{O}_{\mathrm{Y}}$. This morphism extends uniquely to a morphism of quasi-coherent $\mathbf{Z}_{\geq 0}$-graded $\mathcal{O}_{\mathrm{Y}}$-algebras:

$$f^*(\bigoplus_{d\geq 0}\mathcal{I}^d)\to\bigoplus_{d\geq 0}(\mathcal{I}_{f^{-1}\mathrm{Z}})^{\otimes d}.$$

By Proposition 3.3.10, this corresponds to a morphism $\mathrm{Y}\to\mathrm{Bl}_{\mathrm{Z}}\mathrm{X}$ of schemes over X. This is the unique morphism $\mathrm{Y}\to\mathrm{Bl}_{\mathrm{Z}}\mathrm{X}$ of schemes over X since any morphism is determined by its restriction off an effective Cartier divisor. □

Remark 3.4.11. Since the fiber of (3.23) over $\mathbb{A}^{1,-}\smallsetminus 0$ is $\mathbb{G}_m$-equivariantly isomorphic to $\mathrm{X}\times(\mathbb{A}^{1,-}\smallsetminus 0)$, we obtain an isomorphism:

$$\mathrm{Bl}_{\mathrm{Z}}\mathrm{X}\smallsetminus\mathrm{E}\xrightarrow{\simeq}(\mathrm{X}\smallsetminus\mathrm{Z})\times(\mathbb{A}^{1,-}\smallsetminus 0)/\mathbb{G}_m\xrightarrow{\simeq}\mathrm{X}\smallsetminus\mathrm{Z}.$$

Remark 3.4.12. From (3.24), we see that the line bundle $\mathcal{O}(\mathrm{E})$ associated to the effective Cartier divisor $\mathrm{E}\to\mathrm{Bl}_{\mathrm{Z}}\mathrm{X}$ is canonically isomorphic to $\mathcal{O}_{\mathrm{Bl}_{\mathrm{Z}}\mathrm{X}}(-1)$. (Caution: This sign is *opposite* to the hyperplane divisor in $\mathbb{P}^n_{\mathbf{Z}}$ which corresponds to $\mathcal{O}_{\mathbb{P}^n_{\mathbf{Z}}}(1)$.)

Indeed, $\mathrm{Spec}_{\mathrm{X}}(\bigoplus_{d\geq 0}\mathcal{I}^d)\smallsetminus\mathrm{X}$ is the $\mathbb{G}_m$-torsor associated to the line bundle $\mathcal{O}_{\mathrm{Bl}_{\mathrm{Z}}\mathrm{X}}(-1)$. The composite $t\cdot\tilde{j}$, whose vanishing locus is $\mathrm{C}_{\mathrm{Z}/\mathrm{X}}\smallsetminus\mathrm{Z}$, is a $\mathbb{G}_m$-equivariant map:

$$t\cdot\tilde{j}:\mathrm{Spec}_{\mathrm{X}}(\bigoplus_{d\geq 0}\mathcal{I}^d)\smallsetminus\mathrm{X}\to\mathbb{A}^{1,-}.$$

By inverting the $\mathbb{G}_m$-action, it corresponds to a $\mathbb{G}_m$-equivariant morphism from the $\mathbb{G}_m$-torsor associated to $\mathcal{O}_{\mathrm{Bl_Z\,X}}(1)$ to $\mathbb{A}^1_{\mathbf{Z}}$, *i.e.* a section of $\mathcal{O}_{\mathrm{Bl_Z\,X}}(-1)$.

3.4.13. Let X be a scheme and $\mathrm{Z} \to \mathrm{X}$ be a closed immersion. Let $\mathrm{X}' \to \mathrm{X}$ be a morphism of schemes. We form $\mathrm{Z}' := \mathrm{Z} \times_{\mathrm{X}} \mathrm{X}'$. Let $\mathrm{Bl_Z\,X}$ (respectively $\mathrm{Bl_{Z'}\,X'}$) be the blow-up of X (respectively X′) along Z (respectively Z′), so we have a commutative diagram:

$$\begin{array}{ccc} \mathrm{Bl_{Z'}\,X'} & \to & \mathrm{Bl_Z\,X} \\ \downarrow & & \downarrow \\ \mathrm{X}' & \longrightarrow & \mathrm{X} \end{array} \tag{3.25}$$

by the universal property defining $\mathrm{Bl_Z\,X}$. The scheme $\mathrm{Bl_{Z'}\,X'}$ is called the *strict transform* (or *proper transform*) of X′ in the blow-up $\mathrm{Bl_Z\,X}$. It is most commonly applied when $\mathrm{X}' \to \mathrm{X}$ is a closed immersion which nontrivial intersection with Z.

Emphatically, (3.25) is *not* a Cartesian square. Rather, we have an induced morphism:

$$\mathrm{Bl_{Z'}\,X'} \to \mathrm{X}' \times_{\mathrm{X}} \mathrm{Bl_Z\,X}. \tag{3.26}$$

Since the exceptial divisor $\mathrm{E} \subset \mathrm{Bl_Z\,X}$ is locally defined by a principal ideal, its preimage $\mathrm{E}' := \mathrm{X}' \times_{\mathrm{X}} \mathrm{E}$ is also locally defined by a principal ideal. However, E′ may not be an effective Cartier divisor in $\mathrm{X}' \times_{\mathrm{X}} \mathrm{Bl_Z\,X}$, *i.e.* the defining ideal may not be generated by a non-zero-divisor.

Lemma 3.4.14 (Blow-up closure lemma)**.** *The morphism* (3.26) *is the scheme-theoretic image of the open immersion (*cf. *§2.5.6):*

$$(\mathrm{X}' \times_{\mathrm{X}} \mathrm{Bl_Z\,X}) \smallsetminus \mathrm{E}' \to \mathrm{X}' \times_{\mathrm{X}} \mathrm{Bl_Z\,X}. \tag{3.27}$$

Proof. Denote by W the scheme-theoretic image of (3.27). Let $f : \mathrm{Y} \to \mathrm{X}'$ be a morphism of schemes such that $f^{-1}\mathrm{Z}' := \mathrm{Z}' \times_{\mathrm{X}'} \mathrm{Y}$ is an effective Cartier divisor on Y. We need to show that there is a unique morphism from Y to the closed subscheme W.

By the universal property defining $\mathrm{Bl_Z\,X}$, there is a unique morphism $\mathrm{Y} \to \mathrm{X}' \times_{\mathrm{X}} \mathrm{Bl_Z\,X}$. It remains to prove that the latter factors through W.

Since $\mathcal{I}_{\mathrm{E}'}$ is locally principal, this reduces to the following statement: Given a ring A and $a \in \mathrm{A}$, any morphism $\mathrm{A} \to \mathrm{B}$ for which the image of a is a non-zero-divisor factors through the image of $\mathrm{A} \to \mathrm{A}[\frac{1}{a}]$, but the latter is precisely the quotient of A by the ideal of elements annihilated by a power of a. □

3.4.15. We shall now describe a class of closed immersions for which the blow-up is particularly simple. These can be seen as a generalization of effective Cartier divisors.

Recall that given a ring A, a sequence of elements $f_1, \cdots, f_n \in \mathrm{A}$ is called *regular* if for each $1 \le i \le n-1$, f_{i+1} is a non-zero-divisor in $\mathrm{A}/(f_1, \cdots, f_i)$.

On the other hand, given $f_1, \cdots, f_n \in \mathrm{A}$, we may form the *Koszul complex*, which is the following complex of free A-modules situated in cohomological degrees $[-n, 0]$:

$$\bigwedge^n(\mathrm{A}^{\oplus n}) \xrightarrow{d^{-n}} \cdots \xrightarrow{d^{-2}} \bigwedge^1(\mathrm{A}^{\oplus n}) \xrightarrow{d^{-1}} \mathrm{A}, \tag{3.28}$$

where $d^{-k}(e_{i_1} \wedge \cdots \wedge e_{i_k}) = \sum_{\alpha=1}^k (-1)^{\alpha-1} f_{i_\alpha} e_{i_1} \wedge \cdots \wedge \hat{e}_{i_\alpha} \wedge \cdots \wedge e_{i_k}$ for $1 \le k \le n$ and the canonical basis $e_1, \cdots, e_n$ of $\mathrm{A}^{\oplus n}$.

If $f_1, \cdots, f_n$ is a regular sequence, then (3.28) is a free resolution of $\mathrm{A}/(f_1, \cdots, f_n)$: This follows by induction on n and expressing (3.28) as the tensor product of the complexes $f_i : \mathrm{A} \to \mathrm{A}$ concentrated in cohomological degrees $[-1, 0]$. The converse holds if A is a Noetherian local ring and $f_1, \cdots, f_n$ belong to the maximal ideal (*cf.* [Sta18, 09CC]).

In particular, when $f_1, \cdots, f_n$ is a regular sequence, the ideal $\mathfrak{a} := (f_1, \cdots, f_n)$ is the cokernel of the map:

$$d^{-2} : \bigwedge^2 (\mathrm{A}^{\oplus n}) \to \mathrm{A}^{\oplus n}, \quad e_i \wedge e_j \mapsto f_i e_j - f_j e_i. \tag{3.29}$$

By base change to $\mathrm{A}/\mathfrak{a}$, we see that $\mathfrak{a}/\mathfrak{a}^2$ is a free $\mathrm{A}/\mathfrak{a}$-module of rank n.

Lemma 3.4.16. *Let* A *be a ring and* $f_1, \cdots, f_n \in \mathrm{A}$ *be a regular sequence. Write* $\mathfrak{a} := (f_1, \cdots, f_n)$*. Then the canonical map of* $\mathbf{Z}_{\geq 0}$*-graded* A*-algebras is an isomorphism:*

$$\mathrm{Sym}_{\mathrm{A}}(\mathfrak{a}) \xrightarrow{\simeq} \bigoplus_{d \geq 0} \mathfrak{a}^d. \tag{3.30}$$

Proof. See [Bou07, §9, §§7, Théorème 1]. □

Remark 3.4.17. In the context of Lemma 3.4.16, we obtain an isomorphism of $\mathbf{Z}_{\geq 0}$-graded $\mathrm{A}/\mathfrak{a}$-algebras upon base change to $\mathrm{A}/\mathfrak{a}$:

$$\mathrm{Sym}_{\mathrm{A}/\mathfrak{a}}(\mathfrak{a}/\mathfrak{a}^2) \xrightarrow{\simeq} \bigoplus_{d \geq 0} \mathfrak{a}^d/\mathfrak{a}^{d+1}.$$

3.4.18. Let X be a scheme. A closed immersion $\mathrm{Z} \to \mathrm{X}$ is called *regular* (or *local complete intersection*) if there is an open cover $\mathrm{Spec}(\mathrm{A}_i) \to \mathrm{X}$ ($i \in \mathrm{I}$) such that for each $i \in \mathrm{I}$, the closed subscheme $\mathrm{Z}_i := \mathrm{Z} \times_{\mathrm{X}} \mathrm{Spec}(\mathrm{A}_i)$ is defined by an ideal in A_i generated by a regular sequence.

Let $\mathcal{I}$ denote the ideal sheaf of a regular closed immersion $i : \mathrm{Z} \to \mathrm{X}$. From the ring-theoretic statements of §3.4.15, we find:

(1) the conormal sheaf $i^*\mathcal{I}$ (isomorphic to $\mathcal{I}/\mathcal{I}^2$) is a vector bundle over Z;
(2) we have canonical isomorphisms (*cf.* Lemma 3.4.16, Remark 3.4.17):

$$\mathrm{Sym}_{\mathcal{O}_{\mathrm{X}}}(\mathcal{I}) \xrightarrow{\simeq} \bigoplus_{d \geq 0} \mathcal{I}^d, \quad \mathrm{Sym}_{\mathcal{O}_{\mathrm{Z}}}(\mathcal{I}/\mathcal{I}^2) \xrightarrow{\simeq} \bigoplus_{d \geq 0} \mathcal{I}^d/\mathcal{I}^{d+1}.$$

In particular, the blow-up of X along Z is identified with $\mathrm{Proj}_{\mathrm{X}}(\mathrm{Sym}_{\mathcal{O}_{\mathrm{X}}} \mathcal{I})$ and the normal cone $\mathrm{C}_{\mathrm{Z}/\mathrm{X}}$ is identified with the total space of the *normal bundle* $\mathcal{N}_{\mathrm{Z}/\mathrm{X}} := (\mathcal{I}/\mathcal{I}^2)^\vee$.

Example 3.4.19. Let us calculate the blow-up $\mathrm{Bl}_0 \mathbb{A}^2_{\mathbf{Z}}$ of $\mathbb{A}^2_{\mathbf{Z}}$ at the origin 0, defined by the ideal $\mathfrak{a} := (x, y)$ in $\mathrm{A} := \mathbf{Z}[x, y]$. Since $\mathfrak{a}$ is generated by the regular sequence x, y, the Rees algebra is identified with the symmetric algebra (3.30):

$$\bigoplus_{d \geq 0} \mathfrak{a}^d \xrightarrow{\simeq} \mathrm{Sym}_{\mathrm{A}}(\mathfrak{a}) \xrightarrow{\simeq} \mathrm{Sym}_{\mathrm{A}}(\mathrm{A}^{\oplus 2}) \otimes_{\mathrm{Sym}_{\mathrm{A}}(\wedge^2(\mathrm{A}^{\oplus 2}))} \mathrm{A}, \tag{3.31}$$

where the second isomorphism uses the fact that $\mathfrak{a}$ is the cokernel of the differential $d^{-2} : \wedge^2(\mathrm{A}^{\oplus 2}) \to \mathrm{A}^{\oplus 2}$ in the Koszul complex. Denote the generators of $\mathrm{A}^{\oplus 2}$ by X, Y. The differential d^{-2} maps the generator $\mathrm{X} \wedge \mathrm{Y}$ to $x\mathrm{Y} - y\mathrm{X}$. Thus, the Rees algebra (3.31) is:

$$\bigoplus_{d \geq 0} \mathfrak{a}^d \xrightarrow{\simeq} \mathrm{A}[\mathrm{X}, \mathrm{Y}]/(x\mathrm{Y} - y\mathrm{X}), \quad \deg(\mathrm{X}) = \deg(\mathrm{Y}) = 1.$$

In other words, $\mathrm{Bl}_0 \mathbb{A}^2_{\mathbf{Z}}$ is the closed subscheme of $\mathbb{A}^2_{\mathbf{Z}} \times \mathbb{P}^1_{\mathbf{Z}} \cong \mathbb{P}^1_{\mathrm{A}}$ defined by the section $x\mathrm{Y} - y\mathrm{X}$ of $\mathcal{O}_{\mathbb{P}^1_{\mathrm{A}}}(1)$, where X, Y are the homogeneous coordinates on $\mathbb{P}^1_{\mathrm{A}}$.

The exceptional divisor $\mathrm{E} \to \mathrm{Bl}_0 \mathbb{A}^2_{\mathbf{Z}}$ is the fiber at 0 of the projection $p : \mathrm{Bl}_0 \mathbb{A}^2_{\mathbf{Z}} \to \mathbb{A}^2_{\mathbf{Z}}$. We can describe it on the open affine cover U_0, U_1 of $\mathrm{Bl}_0 \mathbb{A}^2_{\mathbf{Z}}$ defined by $\mathrm{X} \neq 0$ and $\mathrm{Y} \neq 0$, with:

$$\mathrm{U}_0 \xrightarrow{\simeq} \mathrm{Spec}(\mathbf{Z}[x, \frac{\mathrm{Y}}{\mathrm{X}}]), \quad \mathrm{U}_1 \xrightarrow{\simeq} \mathrm{Spec}(\mathbf{Z}[y, \frac{\mathrm{X}}{\mathrm{Y}}]).$$

The closed subscheme $\mathrm{E} \cap \mathrm{U}_0 \to \mathrm{U}_0$ is defined by the ideal (x) and the closed subscheme $\mathrm{E} \cap \mathrm{U}_1 \to \mathrm{U}_1$ is defined by the ideal (y).

Example 3.4.20. Let us calculate the strict transform $\widetilde{X}$ of the cuspidal curve $X \to \mathbb{A}^2_{\mathbf{Z}}$ under the blow-up $p : \widetilde{\mathbb{A}}^2_{\mathbf{Z}} := \mathrm{Bl}_0\,\mathbb{A}^2_{\mathbf{Z}} \to \mathbb{A}^2_{\mathbf{Z}}$ of Example 3.4.19. Here, X is the closed subscheme of $\mathbb{A}^2_{\mathbf{Z}}$ defined by the ideal $(y^2 - x^3)$ of $\mathbf{Z}[x, y]$.

We shall do so using the blow-up closure lemma (*cf.* Lemma 3.4.14). Namely, $\widetilde{X}$ is the scheme-theoretic image of $p^{-1}X \setminus E \to p^{-1}X$.

Over the open affine subscheme U_0, the closed subscheme $p^{-1}X \cap U_0$ is defined by:

$$p^{-1}X \cap U_0 \xrightarrow{\simeq} \mathrm{Spec}(\mathbf{Z}[x, \frac{Y}{X}]/(x \cdot \frac{Y}{X})^2 - x^3).$$

The ideal of x-power torsion elements is generated by $(Y/X)^2 - x$. Hence $\widetilde{X} \cap U_0$ is the spectrum of $\mathrm{Spec}(\mathbf{Z}[x, Y/X]/(Y/X)^2 - x)$. It intersects the exceptional divisor at a "double point", *i.e.* a subscheme isomorphic to $\mathrm{Spec}(\mathbf{Z}[\epsilon]/\epsilon^2)$.

Over the open affine subscheme U_1, the closed subscheme $p^{-1}X \cap U_1$ is defined by:

$$p^{-1}X \cap U_1 \xrightarrow{\simeq} \mathrm{Spec}(\mathbf{Z}[y, \frac{X}{Y}]/(y^2 - (y \cdot \frac{X}{Y})^3)).$$

The ideal of y-power torsion elements is generated by $1 - y \cdot (X/Y)^3$. Hence $\widetilde{X} \cap U_1$ is the spectrum of $\mathrm{Spec}(\mathbf{Z}[y, X/Y]/(1 - y \cdot (X/Y)^3))$. It does not intersect the exceptional divisor.

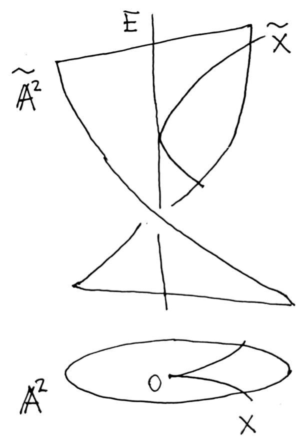

FIGURE 3. The blow-up of a cuspidal curve X embedded in $\mathbb{A}^2_{\mathbf{Z}}$ (Example 3.4.20). The strict transform $\widetilde{X}$ is tangential to the exceptional divisor E.

3.5. Differential calculus.

3.5.1. Let R be a ring and M be an R-module. Consider the R-module $R \oplus M$ equipped with the ring structure $(f_1, \epsilon_1) \cdot (f_2, \epsilon_2) := (f_1 f_2, f_1\epsilon_2 + f_2\epsilon_1)$ for $f_1, f_2 \in R$ and $\epsilon_1, \epsilon_2 \in M$. In particular, M is an ideal of $R \oplus M$ satisfying $M^2 = 0$.

We have ring maps $R \to R \oplus M$, $f \mapsto (f, 0)$ and $R \oplus M \to R$, $(f, \epsilon) \mapsto f$. They correspond to morphisms of affine schemes:

$$\mathrm{Spec}(R) \xrightarrow{i} \mathrm{Spec}(R \oplus M) \xrightarrow{p} \mathrm{Spec}(R). \tag{3.32}$$

The affine scheme $\mathrm{Spec}(R \oplus M)$, equipped with structural morphisms i and p, is called the *split square-zero extension* of Spec(R) by M: we think of i as defining the extension and p as the splitting. Its formation is functorial in the pair M, in that given a morphism

$M_1 \to M_2$ in Mod_R, we have a morphism $\mathrm{Spec}(R \oplus M_2) \to \mathrm{Spec}(R \oplus M_1)$ compatible with the structural morphisms.

3.5.2. Let A be a ring and B be an A-algebra. An A-*linear derivation* on B consists of $N \in \mathsf{Mod}_B$ equipped with a morphism $d : B \to N$ satisfying the following conditions:

(1) $d(ab) = ad(b)$ for any $a \in A$, $b \in B$;
(2) $d(b_1 b_2) = b_1 d(b_2) + b_2 d(b_1)$ for any $b_1, b_2 \in B$ (the *Leibniz rule*).

Let $\Omega_{B/A}$, $d : B \to \Omega_{B/A}$ denote the universal A-linear derivation on B, so $\Omega_{B/A}$ is generated by symbols $d(b)$, $b \in B$, subject to the relations (1) & (2) above. Elements of the B-module $\Omega_{B/A}$ are called *Kähler differentials* on B relative to A.

Lemma 3.5.3. *Let* A *be a ring and* B *be an* A*-algebra. Denote by* $f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *the induced morphism of affine schemes. For any* R*-point* y *of* $\mathrm{Spec}(B)$ *(*$R \in \mathsf{Ring}$*) and* $M \in \mathsf{Mod}_R$*, the following sets are in canonical bijection:*

(1) R*-linear maps* $\Omega_{B/A} \otimes_B R \to M$*;*
(2) morphisms $\tilde{y} : \mathrm{Spec}(R \oplus M) \to \mathrm{Spec}(B)$ *making the following diagram commute:*

$$\begin{array}{ccc} \mathrm{Spec}(R) & \xrightarrow{y} & \mathrm{Spec}(B) \\ \downarrow i & \overset{\tilde{y}}{\nearrow} & \downarrow f \\ \mathrm{Spec}(R \oplus M) & \xrightarrow{f(y) \cdot p} & \mathrm{Spec}(A) \end{array} \tag{3.33}$$

where i *and* p *are as in* (3.32).

Proof. We view y as a morphism of A-algebras $y : B \to R$. The datum of $\tilde{y}$ rendering (3.34) commutative is equivalent to a morphism of A-algebras $\tilde{y} : B \to R \oplus M$ which reduces to y modulo M. Note that the morphism $\tilde{y}$ can be written as (y, d), where $d : B \to M$ is a morphism in Mod_A which satisfies:

$$d(b_1 b_2) = y(b_1) d(b_2) + y(b_2) d(b_1), \quad b_1, b_2 \in B.$$

This is equivalent to a B-linear map $\Omega_{B/A} \to M$, where M is viewed as a B-module via y. □

Remark 3.5.4. Under the bijection of Lemma 3.5.3, the abelian group structure on the set of R-linear maps $\Omega_{B/A} \otimes_B R \to M$ corresponds to an abelian group structure on the set of $\tilde{y}$ making (3.33) commute.

This abelian group structure can be described intrinsically as follows. The zero element corresponds to the morphism $y \cdot p$. The sum is given by pre-composition with the morphism:

$$\begin{aligned} \mathrm{Spec}(R \oplus M) &\to \mathrm{Spec}(R \oplus M \oplus M) \\ &\xrightarrow{\simeq} \mathrm{Spec}(R \oplus M) \sqcup_{\mathrm{Spec}(R)} \mathrm{Spec}(R \oplus M), \end{aligned}$$

where the first morphism is induced from the map $M \oplus M \to M$, $(\epsilon_1, \epsilon_2) \mapsto \epsilon_1 + \epsilon_2$, and the push-out is taken in the category of affine schemes.

3.5.5. Let $f : Y \to X$ be a morphism of schemes.

For any R-point y of Y ($R \in \mathsf{Ring}$), we denote by $y^* \Omega_f$ the R-module co-representing the functor assigning to $M \in \mathsf{Mod}_R$ the set of morphisms $\tilde{y} : \mathrm{Spec}(R \oplus M) \to Y$ making the following diagram commute:

$$\begin{array}{ccc} \mathrm{Spec}(R) & \xrightarrow{y} & Y \\ \downarrow i & \overset{\tilde{y}}{\nearrow} & \downarrow f \\ \mathrm{Spec}(R \oplus M) & \xrightarrow{f(y) \cdot p} & X \end{array} \tag{3.34}$$

By Lemma 3.5.3 and descent of QCoh, the R-module $y^*\Omega_f$ exists. It enjoys the following functoriality: Given a morphism $\varphi : \mathrm{Spec}(\mathrm{R}') \to \mathrm{Spec}(\mathrm{R})$, there is a canonical isomorphism $\varphi^*(y^*\Omega_f) \xrightarrow{\simeq} (y')^*\Omega_f$, where $y' := y \cdot \varphi$.

Thus, we may define $\Omega_f \in \mathsf{QCoh}(\mathrm{Y})$ to be the object whose value at any R-point $y \in \mathrm{Y}$ ($\mathrm{R} \in \mathsf{Ring}$) is the R-module $y^*\Omega_f$, subject to the above functoriality in R. The object Ω_f is called the *sheaf of differential forms* on Y relative to X. We shall also denote it by $\Omega_{\mathrm{Y}/\mathrm{X}}$, or simply Ω_{Y} if the base scheme X is clear from the context.

If $f : \mathrm{Spec}(\mathrm{B}) \to \mathrm{Spec}(\mathrm{A})$ is a morphism of affine schemes, then $\Omega_{\mathrm{Y}/\mathrm{X}}$ may be identified with the B-module $\Omega_{\mathrm{B}/\mathrm{A}}$ (Lemma 3.5.3).

Remark 3.5.6. The formation of the sheaf of differential forms is compatible with base change. More precisely, let $f : \mathrm{Y} \to \mathrm{X}$ be a morphism of schemes. Given a morphism of schemes $\mathrm{X}' \to \mathrm{X}$, we write $f' : \mathrm{Y}' := \mathrm{Y} \times_{\mathrm{X}} \mathrm{X}' \to \mathrm{X}'$ for the base change of f. Then there is a canonical isomorphism:

$$\Omega_f|_{\mathrm{Y}'} \xrightarrow{\simeq} \Omega_{f'}.$$

This follows immediately from the definition of Ω_f.

3.5.7. Let S be a scheme and $f : \mathrm{Y} \to \mathrm{X}$ be a morphism of schemes over S. We write $\Omega_{\mathrm{Y}} := \Omega_{\mathrm{Y}/\mathrm{S}}$ and $\Omega_{\mathrm{X}} := \Omega_{\mathrm{X}/\mathrm{S}}$. Then there is a canonical exact sequence in $\mathsf{QCoh}(\mathrm{Y})$:

$$f^*\Omega_{\mathrm{X}} \to \Omega_{\mathrm{Y}} \to \Omega_f \to 0, \tag{3.35}$$

where the first map df is called the *differential* of f.

The differential df is defined as follows: given an R-point y of Y ($\mathrm{R} \in \mathsf{Ring}$) and $\mathrm{M} \in \mathsf{Mod}_{\mathrm{R}}$, a morphism $y^*\Omega_{\mathrm{Y}} \to \mathrm{M}$ induces a morphism $y^*f^*\Omega_{\mathrm{X}} \to \mathrm{M}$ by composing the corresponding lift $\tilde{y} : \mathrm{Spec}(\mathrm{R} \oplus \mathrm{M}) \to \mathrm{Y}$ over S with f.

To identify the cokernel of df, we note that the lift $\tilde{y} : \mathrm{Spec}(\mathrm{R} \oplus \mathrm{M}) \to \mathrm{Y}$ over S corresponding to a morphism $y^*\Omega_{\mathrm{Y}} \to \mathrm{M}$ renders the lower triangle in (3.34) commutative if and only if $f \cdot \tilde{y}$ is the zero lift of $f \cdot y$, *i.e.* the induced morphism $y^*f^*\Omega_{\mathrm{X}} \to \mathrm{M}$ vanishes.

Example 3.5.8. Let R be a ring. For each $n \in \mathbf{Z}_{\geq 0}$, we write $\Omega_{\mathbb{A}^n_{\mathrm{R}}} := \Omega_{\mathbb{A}^n_{\mathrm{R}}/\mathrm{Spec}(\mathrm{R})}$. Then $\Omega_{\mathbb{A}^n_{\mathrm{R}}}$ is the free $\mathrm{R}[x_1, \cdots, x_n]$-module on generators $dx_1, \cdots, dx_n$, where each dx_i is the image of $x_i \in \mathrm{R}[x_1, \cdots, x_n]$ under the universal R-linear derivation.

For $m, n \in \mathbf{Z}_{\geq 0}$ and a morphism $f : \mathbb{A}^n_{\mathrm{R}} \to \mathbb{A}^m_{\mathrm{R}}$, corresponding to $f_1, \cdots, f_m \in \mathrm{R}[x_1, \cdots, x_n]$, the differential df is the $\mathrm{R}[x_1, \cdots, x_n]$-linear map:

$$df : \bigoplus_{j=1}^m \mathrm{R}[x_1, \cdots, x_n] dy_j \to \bigoplus_{i=1}^n \mathrm{R}[x_1, \cdots, x_n] dx_i, \quad dy_j \mapsto \sum_{i=1}^n \frac{\partial f_j}{\partial x_i} dx_i.$$

Moreover, (3.35) identifies Ω_f as the cokernel of df. This gives us a way to calculate $\Omega_{\mathrm{X}} := \Omega_{\mathrm{X}/\mathrm{Spec}(\mathrm{R})}$ for any affine scheme $\mathrm{X} = \mathrm{Spec}(\mathrm{A})$, where A is a finitely presented R-algebra. Namely, we choose a presentation $\mathrm{A} \cong \mathrm{R}[x_1, \cdots, x_n]/(f_1, \cdots, f_m)$, and the base change property (Remark 3.5.6) implies that Ω_{X} is the A-module generated by dx_i ($1 \leq i \leq n$) subject to the relations $\sum_{i=1}^n (\partial f_j/\partial x_i) dx_i = 0$ for each $1 \leq j \leq m$.

3.5.9. Next, we shall relate the sheaf of differential forms to the conormal sheaf associated to a closed immersion. Let us begin with the case of affine schemes.

Let A be a ring and $\mathfrak{a} \subset \mathrm{A}$ be an ideal with $\mathrm{B} := \mathrm{A}/\mathfrak{a}$. Then the conormal sheaf is the B-module $\mathfrak{a} \otimes_{\mathrm{A}} \mathrm{B} \cong \mathfrak{a}/\mathfrak{a}^2$. If A is moreover an R-algebra ($\mathrm{R} \in \mathsf{Ring}$), then the universal R-linear derivation d induces an B-*linear* map:

$$\mathfrak{a} \otimes_{\mathrm{A}} \mathrm{B} \to \Omega_{\mathrm{A}/\mathrm{R}} \otimes_{\mathrm{A}} \mathrm{B}. \tag{3.36}$$

Indeed, the restriction of $d : \mathrm{A} \to \Omega_{\mathrm{A/R}}$ to $\mathfrak{a}$ defines an A-linear map $\mathfrak{a} \to \Omega_{\mathrm{A/R}} \otimes_{\mathrm{A}} \mathrm{B}$ because of the Leibniz rule, and we obtain (3.36) by adjunction.

Furthermore, (3.36) is surjective onto the kernel of the differential $\Omega_{\mathrm{A/R}} \otimes_{\mathrm{A}} \mathrm{B} \to \Omega_{\mathrm{B/R}}$. To see this, it suffices to observe that an R-linear derivation $d : \mathrm{A} \to \mathrm{N}$, for $\mathrm{N} \in \mathsf{Mod}_{\mathrm{B}}$, factors through an R-linear derivation $\mathrm{B} \to \mathrm{N}$ if and only if it annihilates $\mathfrak{a}$.

These observations globalize: Given a scheme S and a closed immersion $\mathrm{Y} \to \mathrm{X}$ of schemes over S, we have an exact sequence in $\mathsf{QCoh}(\mathrm{Y})$:

$$\check{\mathcal{N}}_{\mathrm{Y/X}} \to \Omega_{\mathrm{X}}|_{\mathrm{Y}} \to \Omega_{\mathrm{Y}} \to 0. \tag{3.37}$$

Here, $\check{\mathcal{N}}_{\mathrm{Y/X}} := \mathcal{I} \otimes_{\mathcal{O}_{\mathrm{X}}} \mathcal{O}_{\mathrm{Y}}$ is the conormal sheaf of Y, with $\mathcal{I}$ being the ideal sheaf.

Remark 3.5.10 (Exact sequence of conormal sheaves)**.** Let A be a ring and and $\mathfrak{a} \subset \mathfrak{b} \subset \mathrm{A}$ be two ideals, with $\mathrm{B} := \mathrm{A}/\mathfrak{a}$ and $\mathrm{C} := \mathrm{A}/\mathfrak{b}$. Then we have $\bar{\mathfrak{b}} := \mathfrak{b}/\mathfrak{a} \subset \mathrm{B}$ the induced ideal. Tensoring to C yields an exact sequence:

$$\mathfrak{a} \otimes_{\mathrm{A}} \mathrm{C} \to \mathfrak{b} \otimes_{\mathrm{A}} \mathrm{C} \to \bar{\mathfrak{b}} \otimes_{\mathrm{B}} \mathrm{C} \to 0, \tag{3.38}$$

where the first term is identified with $(\mathfrak{a} \otimes_{\mathrm{A}} \mathrm{B}) \otimes_{\mathrm{B}} \mathrm{C}$, *i.e.* the base change of the conormal sheaf of $\mathrm{A} \to \mathrm{B}$ to C.

The short exact sequence (3.38) globalizes: Given closed immersions $\mathrm{Z} \to \mathrm{Y} \to \mathrm{X}$ of schemes corresponding to ideal sheaves $\mathcal{I}_{\mathrm{Y}} \subset \mathcal{I}_{\mathrm{Z}} \subset \mathcal{O}_{\mathrm{X}}$, we have an exact sequence in $\mathsf{QCoh}(\mathrm{Z})$:

$$\check{\mathcal{N}}_{\mathrm{Y/X}}|_{\mathrm{Z}} \to \check{\mathcal{N}}_{\mathrm{Z/X}} \to \check{\mathcal{N}}_{\mathrm{Z/Y}} \to 0. \tag{3.39}$$

3.5.11. *Module of imperfection.* Given a ring map $\mathrm{R} \to \mathrm{A}$, we may choose a free R-algebra $\mathrm{R}[x_i]_{i\in\mathrm{I}}$ together with a surjection $\mathrm{R}[x_i]_{i\in\mathrm{I}} \to \mathrm{A}$. In other words, we factor the morphism $\mathrm{X} := \mathrm{Spec}(\mathrm{A}) \to \mathrm{Spec}(\mathrm{R})$ as the composition of a closed immersion $\mathrm{X} \to \mathbb{A}^{\mathrm{I}}_{\mathrm{R}}$ and the projection to Spec(R). The morphism (3.36) specializes to the following morphism in $\mathsf{QCoh}(\mathrm{X}) \cong \mathsf{Mod}_{\mathrm{A}}$:

$$d : \check{\mathcal{N}}_{\mathrm{X}/\mathbb{A}^{\mathrm{I}}_{\mathrm{R}}} \to \Omega_{\mathbb{A}^{\mathrm{I}}_{\mathrm{R}}}|_{\mathrm{X}}. \tag{3.40}$$

Note that the cokernel of (3.40) is identified with $\Omega_{\mathrm{A/R}}$, *cf.* §3.5.9. We define $\mathrm{H}^{-1}\mathsf{L}_{\mathrm{A/R}}$ to be the kernel of (3.40). It *a priori* depends on the choice of the surjection $\mathrm{R}[x_i]_{i\in\mathrm{I}} \to \mathrm{A}$. This is however not the case: Different choices induce canonically isomorphic A-module $\mathrm{H}^{-1}\mathsf{L}_{\mathrm{A/R}}$, see [Sta18, 00S1] for details. We call $\mathrm{H}^{-1}\mathsf{L}_{\mathrm{A/R}}$ the *module of imperfection* of the ring map $\mathrm{R} \to \mathrm{A}$. We also denote it by $\mathrm{H}^{-1}\mathsf{L}_{\mathrm{A}}$ when the base ring R is clear from the context.

The formation of $\mathrm{H}^{-1}\mathsf{L}_{\mathrm{A/R}}$ is local on Spec(A). More precisely, the formation of the two-term complex (3.40) is compatible with localization at any $f \in \mathrm{A}$ in the following sense. Writing $\mathrm{X}_f := \mathrm{Spec}(\mathrm{A}_f)$, we have a closed immersion $\mathrm{X}_f \to \mathbb{A}^{\mathrm{I}\sqcup\{1\}}_{\mathrm{R}}$ where the last coordinate is defined by $f^{-1} \in \mathrm{A}_f$. Then we may form the corresponding two-term complex:

$$d : \check{\mathcal{N}}_{\mathrm{X}_f/\mathbb{A}^{\mathrm{I}\sqcup\{1\}}_{\mathrm{R}}} \to \Omega_{\mathbb{A}^{\mathrm{I}\sqcup\{1\}}_{\mathrm{R}}}|_{\mathrm{X}_f}. \tag{3.41}$$

Now, (3.41) is canonically the sum of the localization of (3.40) at $f \in \mathrm{A}$ with the two-term complex $\mathrm{id} : \mathrm{A}_f \xrightarrow{\cong} \mathrm{A}_f$, see [Sta18, 08JZ] for details. In particular, this implies that $\mathrm{H}^{-1}\mathsf{L}_{\mathrm{A}_f/\mathrm{R}}$ is canonically isomorphic to the localization of $\mathrm{H}^{-1}\mathsf{L}_{\mathrm{A/R}}$ at $f \in \mathrm{A}$.[14]

Remark 3.5.12. If $\mathrm{R} \to \mathrm{A}$ is surjective with kernel $\mathfrak{a}$, then $\mathrm{H}^{-1}\mathsf{L}_{\mathrm{A/R}}$ is isomorphic to the conormal sheaf $\mathfrak{a} \otimes_{\mathrm{R}} \mathrm{A}$ of the closed immersion $\mathrm{Spec}(\mathrm{A}) \to \mathrm{Spec}(\mathrm{R})$. Indeed, this follows from the definition using the closed immersion $\mathrm{Spec}(\mathrm{A}) \to \mathbb{A}^0_{\mathrm{R}} \cong \mathrm{Spec}(\mathrm{R})$.

[14]This observation allows us to form the quasi-coherent sheaf $\mathrm{H}^{-1}\mathsf{L}_{\mathrm{X/S}} \in \mathsf{QCoh}(\mathrm{X})$ for every morphism of schemes $\mathrm{X} \to \mathrm{S}$ using descent of QCoh, but we will not use it.

3.5.13. We shall use extend the short exact sequence (3.35) to the left.

The following assertion does so in the case of ring morphisms, extending (3.35) by two steps. It is sufficient for our purpose but is not the best formulation of this phenomenon.[15]

Proposition 3.5.14 (The Jacobi-Zariski sequence)**.** *Let* R *be a ring and* A → B *be a morphism of* R*-algebras. There exists an exact sequence of* B*-modules:*

$$\mathrm{H}^{-1}\mathsf{L}_{\mathrm{B}/\mathrm{R}} \to \mathrm{H}^{-1}\mathsf{L}_{\mathrm{B}/\mathrm{A}} \to \Omega_{\mathrm{A}/\mathrm{R}} \otimes_{\mathrm{A}} \mathrm{B} \to \Omega_{\mathrm{B}/\mathrm{R}} \to \Omega_{\mathrm{B}/\mathrm{A}} \to 0. \tag{3.42}$$

Proof. Choose closed immersions $\mathrm{Spec(A)} \to \mathbb{A}^{\mathrm{I}}_{\mathrm{R}}$ and $\mathrm{Spec(B)} \to \mathbb{A}^{\mathrm{J}}_{\mathrm{A}}$, so we have a commutative diagram of affine schemes:

$$\begin{array}{ccccc} \mathrm{Y} := \mathrm{Spec(B)} & \hookrightarrow & \mathbb{A}^{\mathrm{J}}_{\mathrm{A}} & \hookrightarrow & \mathbb{A}^{\mathrm{I+J}}_{\mathrm{R}} \\ & \searrow & \downarrow & & \downarrow \\ \mathrm{X} := & & \mathrm{Spec(A)} & \hookrightarrow & \mathbb{A}^{\mathrm{I}}_{\mathrm{R}} \\ & & & \searrow & \downarrow \\ & & & & \mathrm{Spec(R)} \end{array}$$

where the square is Cartesian.

Since the formation of conormal sheaves commutes with base change, we obtain a map of exact sequences (using Remark 3.5.10 for the top row):

$$\begin{array}{ccccccccc} & & \check{\mathcal{N}}_{\mathrm{X}/\mathbb{A}^{\mathrm{I}}_{\mathrm{R}}}|_{\mathrm{Y}} & \longrightarrow & \check{\mathcal{N}}_{\mathrm{Y}/\mathbb{A}^{\mathrm{I+J}}_{\mathrm{R}}} & \longrightarrow & \check{\mathcal{N}}_{\mathrm{Y}/\mathbb{A}^{\mathrm{J}}_{\mathrm{A}}} & \longrightarrow & 0 \\ & & \downarrow d & & \downarrow d & & \downarrow d & & \\ 0 & \longrightarrow & \Omega_{\mathbb{A}^{\mathrm{I}}_{\mathrm{R}}}|_{\mathrm{Y}} & \longrightarrow & \Omega_{\mathbb{A}^{\mathrm{I+J}}_{\mathrm{R}}}|_{\mathrm{Y}} & \longrightarrow & \Omega_{\mathbb{A}^{\mathrm{J}}_{\mathrm{R}}}|_{\mathrm{Y}} & \longrightarrow & 0 \end{array}$$

The exact sequence (3.42) now follows from the snake lemma. □

3.5.15. Let S be a scheme. We say that a morphism X → S of schemes is *formally smooth* (or that X is formally smooth *over* S) if for every affine scheme Spec(B) over S with an ideal $\mathfrak{b} \subset \mathrm{B}$ satisfying $\mathfrak{b}^2 = 0$, every morphism $\mathrm{Spec(B/\mathfrak{b})} \to \mathrm{X}$ over S extend to a morphism Spec(B) → X over S.

This condition can be expressed by saying that a dotted morphism exists in the following diagram, rendering both triangles commutative:

$$\begin{array}{ccc} \mathrm{Spec(B/\mathfrak{b})} & \to & \mathrm{X} \\ \downarrow & \nearrow & \downarrow \\ \mathrm{Spec(B)} & \longrightarrow & \mathrm{S} \end{array} \tag{3.43}$$

Closed immersions of the form $\mathrm{Spec(B/\mathfrak{b})} \to \mathrm{Spec(B)}$ with $\mathfrak{b}^2 = 0$ are called *square-zero extensions.* The split square-zero extensions (*cf.* §3.5.1) are precisely the square-zero extensions equipped with a retract $\mathrm{Spec(B)} \to \mathrm{Spec(B/\mathfrak{b})}$.

Clearly, for any $\mathrm{R} \in \mathsf{Ring}$ and $\mathrm{I} \in \mathsf{Set}$, the morphism $\mathbb{A}^{\mathrm{I}}_{\mathrm{R}} \to \mathrm{Spec(R)}$ is formally smooth.

Lemma 3.5.16. *Let* R → A *be a ring map. The following are equivalent:*

(1) the morphism Spec(A) → Spec(R) *is formally smooth;*
(2) $\mathrm{H}^{-1}\mathsf{L}_{\mathrm{A}/\mathrm{R}} = 0$ *and* $\Omega_{\mathrm{A}/\mathrm{R}}$ *is a projective* A*-module.*

[15]The relevant theory is that of the "cotangent complex" $\mathsf{L}_{\mathrm{Y}/\mathrm{X}}$ of an arbitrary morphism Y → X of schemes. Here we are only studying its cohomology groups in degrees H^{-1} and H^0 in an *ad hoc* manner.

Proof. Let us factor $X := \mathrm{Spec}(A) \to \mathrm{Spec}(R)$ by a closed immersion $X \to \mathbb{A}^I_R$ followed by the projection $\mathbb{A}^I_R \to \mathrm{Spec}(R)$. Denote by $X^{(2)} \to \mathbb{A}^I_R$ the first-order infinitesimal neighborhood of X, *i.e.* if X is defined by the ideal $\mathfrak{a} \subset R[x_i]_{i\in I}$, then $X^{(2)} := \mathrm{Spec}(A^{(2)})$ is defined by $\mathfrak{a}^2$. In particular, $X \to X^{(2)}$ is a square-zero extension of schemes over Spec(R).

Claim: (1) is equivalent to the condition that $X \to X^{(2)}$ admits a retract. Indeed, if X is formally smooth over Spec(R), then the identity on X extends to a retract $X^{(2)} \to X$. Conversely, given a retract $X^{(2)} \to X$ and a square-zero extension $\mathrm{Spec}(B/\mathfrak{b}) \to \mathrm{Spec}(B)$ over Spec(R) equipped with a morphism $\mathrm{Spec}(B/\mathfrak{b}) \to \mathbb{A}^I_R$, an extension $\mathrm{Spec}(B) \to \mathbb{A}^I_R$ exists by the formal smoothness of $\mathbb{A}^I_R$ over Spec(R). Since $\mathfrak{b}^2 = 0$, it factors through $X^{(2)}$. Its composition with the retract $X^{(2)} \to X$ provides the desired extension $\mathrm{Spec}(B) \to X$.

Next, we shall use the closed immersion $X \to \mathbb{A}^I_R$ to present $H^{-1}\mathsf{L}_{A/R}$ and $\Omega_{A/R}$ as the kernel and cokernel of the morphism of A-modules:

$$d : \check{\mathcal{N}}_{X/\mathbb{A}^I_R} \to \Omega_{\mathbb{A}^I_R}|_X.$$

We note that this complex can be computed after replacing $\mathbb{A}^I_R$ by $X^{(2)}$:

$$\begin{array}{ccc} \check{\mathcal{N}}_{X/\mathbb{A}^I_R} & \xrightarrow{d} & \Omega_{\mathbb{A}^I_R}|_X \\ \downarrow\cong & & \downarrow\cong \\ \check{\mathcal{N}}_{X/X^{(2)}} & \xrightarrow{d} & \Omega_{X^{(2)}}|_X \end{array}$$

Indeed, the left vertical isomorphism is clear. The right vertical morphism is an isomorphism because the morphism $\mathcal{N}_{X^{(2)}/\mathbb{A}^I_R} \to \Omega_{\mathbb{A}^I_R}|_{X^{(2)}}$ vanishes upon base change to X (Leibniz rule) and we use the exact sequence (3.37).

It remains to prove that $X \to X^{(2)}$ admits a retract if and only if:

$$d : \check{\mathcal{N}}_{X/X^{(2)}} \to \Omega_{X^{(2)}}|_X \tag{3.44}$$

admits a retract. Indeed, a retract $X^{(2)} \to X$ exhibits $X^{(2)}$ as the split square-zero extension associated to $\check{\mathcal{N}}_{X/X^{(2)}}$ viewed as an A-module, so a retract of (3.44) is supplied by the morphism $\Omega_{X^{(2)}}|_X \to \check{\mathcal{N}}_{X/X^{(2)}}$ classifying the identity on $X^{(2)}$. Conversely, if (3.44) admits a retract, then we have an isomorphism in Mod_A:

$$\Omega_{X^{(2)}}|_X \xrightarrow{\simeq} \check{\mathcal{N}}_{X/X^{(2)}} \oplus \Omega_X. \tag{3.45}$$

Since $A^{(2)} \to A$ is surjective with kernel $\check{\mathcal{N}}_{X/X^{(2)}}$, we can lift any $f \in A$ to an element $\tilde{f} \in A^{(2)}$ uniquely characterized by the equality $d\tilde{f} = (0, df)$ under the isomorphism (3.45). This supplies a retract of $X \to X^{(2)}$. □

Remark 3.5.17. We may strengthen the notion of formal smoothness as follows. Let S be a scheme, we say that a morphism $X \to S$ of schemes is *formally étale* if every solid diagram (3.43) of schemes over S admits a *unique* extension $\mathrm{Spec}(B) \to X$ over S.

Then the proof of Lemma 3.5.16 establishes the following variant: a morphism $\mathrm{Spec}(A) \to \mathrm{Spec}(R)$ of affine schemes is formally étale if and only if $H^{-1}\mathsf{L}_{A/R} = \Omega_{A/R} = 0$.

In particular, a morphism $X \to S$ of schemes is formally étale if and only if it is formally smooth and $\Omega_{X/S} = 0$.

3.5.18. Given a formally smooth morphism $X \to S$ and a square-zero extension $\mathrm{Spec}(B_0) \to \mathrm{Spec}(B)$ of affine schemes over S equipped with a morphism $\mathrm{Spec}(B_0) \to X$ over S, how to describe the set of all extensions $\mathrm{Spec}(B) \to X$ over S?

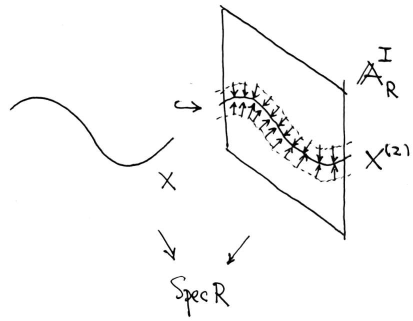

FIGURE 4. In the proof of Lemma 3.5.16, we see that $X := \mathrm{Spec}(A) \to \mathrm{Spec}(R)$ is formally smooth if and only if for every closed immersion of X into an affine space $\mathbb{A}^I_R$, the first-order infinitesimal neighborhood $X^{(2)}$ admits a retract onto X.

Denote by $\mathfrak{b}$ the kernel of $B \to B_0$, so $\mathfrak{b}^2 = 0$ and $\mathfrak{b}$ acquires a B_0-module structure. *Claim*: the set of all such extensions is acted on simply transitively by the abelian group:

$$\mathrm{Hom}_{B_0}(\Omega_{X/S}|_{\mathrm{Spec}(B_0)}, \mathfrak{b}).$$

Indeed, by base change along $\mathrm{Spec}(B) \to S$, we may assume that S is affine. By covering X with open affine subschemes and using descent of QCoh, we reduce to the case $X = \mathrm{Spec}(A)$ is affine. The result now follows from the identification of affine schemes:

$$\mathrm{Spec}(B) \sqcup_{\mathrm{Spec}(B_0)} \mathrm{Spec}(B) \xrightarrow{\simeq} \mathrm{Spec}(B \oplus \mathfrak{b}),$$

and the defining property of $\Omega_{X/S}$.

Lemma 3.5.19 (Formal smoothness is a local property)**.** *Let $f : X \to S$ be a morphism of schemes. Given open affine covers $\mathrm{Spec}(A_i) \to X$, $\mathrm{Spec}(R_i) \to S$ ($i \in I$) such that the restriction of f to each $\mathrm{Spec}(A_i)$ factors through $\mathrm{Spec}(R_i)$ and the induced map $\mathrm{Spec}(A_i) \to \mathrm{Spec}(R_i)$ is formally smooth, then f is formally smooth.*

Proof. Given a square-zero extension $\mathrm{Spec}(B_0) \to \mathrm{Spec}(B)$ of affine schemes over S equipped with a morphism $\mathrm{Spec}(B_0) \to X$ over S, we need to prove that an extension $\mathrm{Spec}(B) \to X$ over S exists.

Let us consider the sheaf on the standard Zariski site of $\mathrm{Spec}(B_0)$ assigning to each standard open $\mathrm{Spec}((B_0)_g) \to \mathrm{Spec}(B_0)$ ($g \in B$) the set of extensions $\mathrm{Spec}(B_g) \to X$ over S. By the hypothesis and §3.5.18, this sheaf is a torsor under the quasi-coherent sheaf $\mathcal{H}om_{B_0}(\Omega_{X/S}|_{\mathrm{Spec}(B_0)}, \mathfrak{b})$. We conclude because B_0 is affine and $\mathcal{H}om_{B_0}(\Omega_{X/S}|_{\mathrm{Spec}(B_0)}, \mathfrak{b})$ is quasi-coherent (Corollary 3.2.27). □

3.5.20. Finally, we gather some consequences of formal smoothness.

Corollary 3.5.21. *Let S be a scheme. Let $f : Y \to X$ be a morphism of schemes over S.*

(1) If f is formally smooth, then (3.35) *is exact on the left, giving a short exact sequence:*

$$0 \to \Omega_X|_Y \to \Omega_Y \to \Omega_{Y/X} \to 0;$$

(2) If the structural morphism $Y \to S$ is formally smooth, then (3.37) *is exact on the left, giving a short exact sequence:*

$$0 \to \check{\mathcal{N}}_{Y/X} \to \Omega_X|_Y \to \Omega_Y \to 0. \tag{3.46}$$

Proof. Both statements reduce to the case where S, X, Y are all affine schemes. Then they follow from Lemma 3.5.16 and the Jacobi–Zariski sequence (Proposition 3.5.14). □

3.6. Smoothness.

3.6.1. Let $f : \mathrm{X} \to \mathrm{S}$ be a morphism of schemes. We say that f is:

(1) *smooth* if it is formally smooth (*cf.* 3.5.15) and locally of finite presentation (*cf.* Lemma 1.8.11);
(2) *étale* if it is formally étale (*cf.* Remark 3.5.17) and locally of finite presentation.

In particular, a morphism $f : \mathrm{X} \to \mathrm{S}$ of schemes is étale if and only if it is smooth and $\Omega_{\mathrm{X/S}} = 0$. The analogue of Lemma 3.5.19 holds for the properties of being smooth and being étale. In other words, they are local properties of f.

It is also clear that smoothness (respectively, étale-ness) is stable under composition and base change.

If $f : \mathrm{X} \to \mathrm{S}$ is a smooth morphism, then $\Omega_{\mathrm{X/S}}$ is a vector bundle (Lemma 3.5.16, Example 3.5.8). In this case, we say that f is smooth of *relative dimension* r ($r \in \mathbf{Z}_{\geq 0}$) if $\Omega_{\mathrm{X/S}}$ has rank r. Thus, an étale morphism is precisely a smooth morphism of relative dimension 0.

It is clear that open immersions are étale. The morphism $\mathbb{A}^n_{\mathbf{Z}} \to \mathrm{Spec}(\mathbf{Z})$ (for $n \in \mathbf{Z}_{\geq 0}$) is smooth of relative dimension n.

3.6.2. A morphism of rings $\mathrm{R} \to \mathrm{A}$ is called *standard smooth* if there exists a presentation $\mathrm{A} \cong \mathrm{R}[x_1, \cdots, x_n]/(f_1, \cdots, f_m)$ (with $m \leq n \in \mathbf{Z}_{\geq 0}$) such that $(\partial f_j / \partial x_i)_{1 \leq i,j \leq m}$, viewed as an m-by-m matrix with coefficients in A, is invertible.

Note that when $\mathrm{R} \to \mathrm{A}$ is standard smooth, we can factor $\mathrm{Spec}(\mathrm{A}) \to \mathrm{Spec}(\mathrm{R})$ as $\mathrm{Spec}(\mathrm{A}) \to \mathbb{A}^{n-m}_{\mathrm{R}} \to \mathrm{Spec}(\mathrm{R})$ where the second morphism is the projection and the first morphism is standard smooth with a presentation $\mathrm{A} \cong \mathrm{R}'[x_1, \cdots, x_m]/(f_1, \cdots, f_m)$ such that $(\partial f_j / \partial x_i)_{1 \leq i,j \leq m}$ is an invertible m-by-m matrix; here, $\mathrm{R}' \coloneqq \mathrm{R}[x_{m+1}, \cdots, x_n]$.

Proposition 3.6.3 (Jacobian criterion for smoothness)**.** *Let* $f : \mathrm{X} \to \mathrm{S}$ *be a morphism of schemes. The following are equivalent:*

(1) f *is smooth;*
(2) there exist open affine covers $\mathrm{Spec}(\mathrm{A}_i) \to \mathrm{X}$, $\mathrm{Spec}(\mathrm{R}_i) \to \mathrm{S}$ *(*$i \in \mathrm{I}$*) such that the restriction of* f *to each* $\mathrm{Spec}(\mathrm{A}_i)$ *factors through* $\mathrm{Spec}(\mathrm{R}_i)$ *and the induced map* $\mathrm{R}_i \to \mathrm{A}_i$ *is standard smooth.*

Proof. (2) ⇒ (1). Since smoothness is a local property, it suffices to prove that given a standard smooth morphism $\mathrm{R} \to \mathrm{A}$ of rings, the induced morphism $\mathrm{Spec}(\mathrm{A}) \to \mathrm{Spec}(\mathrm{R})$ is smooth. It is clearly of finite presentation. To check that it is formally smooth, we may assume $\mathrm{A} \cong \mathrm{R}[x_1, \cdots, x_m]/(f_1, \cdots, f_m)$ where $(\partial f_j / \partial x_i)_{1 \leq i,j \leq m}$ is an invertible m-by-m matrix, via the factorization in §3.6.2. Then we consider the lifting problem for a square-zero extension $\mathrm{Spec}(\mathrm{B}/\mathfrak{b}) \to \mathrm{Spec}(\mathrm{B})$ over $\mathrm{Spec}(\mathrm{R})$:

$$\begin{array}{ccc} \mathrm{Spec}(\mathrm{B}/\mathfrak{b}) & \to & \mathrm{Spec}(\mathrm{A}) \\ \downarrow & \nearrow & \downarrow \\ \mathrm{Spec}(\mathrm{B}) & \longrightarrow & \mathrm{Spec}(\mathrm{R}) \end{array}$$

The morphism $\mathrm{Spec}(\mathrm{B}/\mathfrak{b}) \to \mathrm{A}$ amounts to the choice of elements $\bar{b}_1, \cdots, \bar{b}_m \in \mathrm{B}/\mathfrak{b}$ such that $f_j(\bar{b}_1, \cdots, \bar{b}_m) = 0$ for each j. We want to construct their lifts $b_1, \cdots, b_m \in \mathrm{B}$ such that $f_j(b_1, \cdots, b_m) = 0$ for each j. First, we find arbitrary lifts $b_1, \cdots, b_m \in \mathrm{B}$, so each $f_j(b_1, \cdots, b_m)$

is some element of $\mathfrak{b}$. Then we want to find "adjustments" $\epsilon_i \in \mathfrak{b}$ $(1 \le i \le m)$ so that:

$$f_j(b_1+\epsilon_1,\cdots,b_m+\epsilon_m) = f_j(b_1,\cdots,b_m) + \sum_{i=1}^{m} \frac{\partial f_j}{\partial x_i}(b_1,\cdots,b_m)\epsilon_i$$

vanishes for each j. This is possible because $(\partial f_j/\partial x_i)(b_1,\cdots,b_m)_{1\le i,j\le m}$ is invertible.

(1) ⇒ (2). It suffices to prove that given a ring morphism $\mathrm{R}\to\mathrm{A}$ whose induced morphism $\mathrm{X} := \mathrm{Spec}(\mathrm{A}) \to \mathrm{Spec}(\mathrm{R})$ is smooth, there exists a standard open cover $\mathrm{Spec}(\mathrm{A}_i) \to \mathrm{X}$ $(i \in \mathrm{I})$ such that each $\mathrm{R} \to \mathrm{A}_i$ is standard smooth. Since $\mathrm{R}\to\mathrm{A}$ is of finite presentation, we may choose a presentation $\mathrm{A} \cong \mathrm{R}[x_1,\cdots,x_n]/(f_1,\cdots,f_m)$. In particular, we have a closed immersion $\mathrm{X} \to \mathbb{A}^n_{\mathrm{R}}$ and the associated two-term complex:

$$d : \check{\mathcal{N}}_{\mathrm{X}/\mathbb{A}^n_{\mathrm{R}}} \to \Omega_{\mathbb{A}^n_{\mathrm{R}}}|_{\mathrm{X}}, \tag{3.47}$$

cf. §3.5.11. By Lemma 3.5.16, (3.47) is injective and its cokernel $\Omega_{\mathrm{A}/\mathrm{R}}$ is finite projective, so we have a short exact sequence of vector bundles over X:

$$0 \to \check{\mathcal{N}}_{\mathrm{X}/\mathbb{A}^n_{\mathrm{R}}} \to \Omega_{\mathbb{A}^n_{\mathrm{R}}}|_{\mathrm{X}} \to \Omega_{\mathrm{A}/\mathrm{R}} \to 0. \tag{3.48}$$

By localizing on X and the compatibility between (3.47) with localization indicated in §3.5.11, we may assume that each term in (3.48) is a finite free A-module.

Let $g_1,\cdots,g_r \in \mathfrak{a} := (f_1,\cdots,f_m)$ be elements whose images in $\check{\mathcal{N}}_{\mathrm{X}/\mathbb{A}^n_{\mathrm{R}}} \cong \mathfrak{a}\otimes_{\mathrm{A}} \mathrm{A}/\mathfrak{a}$ form a basis. The inclusion of ideals $(g_1,\cdots,g_r) \subset \mathfrak{a}$ becomes an isomorphism upon localizing at some $a \in 1+\mathfrak{a}$ (Nakayama's lemma). Thus we have isomorphisms:

$$\mathrm{A} \xrightarrow{\simeq} \mathrm{A}_a \xrightarrow{\simeq} (\mathrm{R}[x_1,\cdots,x_n]/(g_1,\cdots,g_r))_a \xrightarrow{\simeq} \mathrm{R}[x_1,\cdots,x_{n+1}]/(g_1,\cdots,g_{r+1}),$$

with $g_{r+1} := a\cdot x_{n+1} - 1$. In this presentation, $\check{\mathcal{N}}_{\mathrm{X}/\mathbb{A}^n_{\mathrm{R}}}$ is freely generated by the elements $g_1,\cdots,g_{r+1}$. (Here, we again used the compatibility between (3.47) with localization indicated in §3.5.11). In other words, by changing the presentation, we may assume $\mathrm{A} \cong \mathrm{R}[x_1,\cdots,x_n]/(f_1,\cdots,f_m)$ and $\check{\mathcal{N}}_{\mathrm{X}/\mathbb{A}^n_{\mathrm{R}}}$ is freely generated by $f_1,\cdots,f_m$. The complex (3.47) is thus the following complex of free A-modules:

$$\bigoplus_{j=1}^{m} \mathrm{A}df_j \to \bigoplus_{i=1}^{n} \mathrm{A}dx_i, \quad df_j \mapsto \sum_{i=1}^{n} \frac{\partial f_j}{\partial x_i} dx_i. \tag{3.49}$$

In particular, we have $m \le n$.

Next, we consider the standard opens of X defined by the m-by-m minors of the matrix (3.49). They form an open cover of X because (3.49) is a summand. *Claim:* these standard opens are the spectra of standard smooth R-algebras. Without loss of generality, we consider the standard open $\mathrm{Spec}(\mathrm{A}_b) \to \mathrm{X}$ where b is the determinant of the first m-by-m minor. Thus we have a presentation:

$$\mathrm{A}_b \xrightarrow{\simeq} \mathrm{R}[x_1,\cdots,x_m,y,x_{m+1},\cdots,x_n]/(f_1,\cdots,f_{m+1})$$

where $f_{m+1} := b\cdot y - 1$, which realizes A_b as a standard smooth R-algebra. □

Corollary 3.6.4. *Let $f : \mathrm{X}\to\mathrm{S}$ be a smooth morphism of schemes of relative dimension r. Then there exists an open cover $\mathrm{X}_i \to \mathrm{X}$ $(i \in \mathrm{I})$ and a factorization of the induced morphism $f_i : \mathrm{X}_i \to \mathrm{S}$ as:*

$$\begin{array}{ccc} \mathrm{X}_i & \xrightarrow{\tilde{f}_i} & \mathbb{A}^r_{\mathrm{S}} \\ & \searrow^{f_i} & \downarrow p \\ & & \mathrm{S} \end{array}$$

where $\tilde{f}_i$ is étale and p is the projection map.

Proof. Since open immersions are étale, it suffices to prove this when f is the morphism induced from a standard smooth ring morphism $\mathrm{R} \to \mathrm{A}$ (Proposition 3.6.3). In this case, it follows from the factorization indicated in §3.6.2. ☐

3.6.5. Let $f : \mathrm{X} \to \mathrm{S}$ be a smooth morphism. We define the *canonical line bundle* on X (relative to S) to be $\omega_{\mathrm{X/S}} := \det(\Omega_{\mathrm{X/S}})$. This is the line bundle $\bigwedge^r \Omega_{\mathrm{X/S}}$ if f is smooth of relative dimension r. We also write ω_{X} if the base scheme S is clear from the context.

Given a closed immersion $\mathrm{Y} \to \mathrm{X}$ of smooth schemes over S, we have an isomorphism of line bundles:

$$\omega_{\mathrm{X/S}}|_{\mathrm{Y}} \xrightarrow{\simeq} \omega_{\mathrm{Y/S}} \otimes \det(\check{\mathcal{N}}_{\mathrm{Y/X}}). \tag{3.50}$$

Indeed, this follows from Corollary 3.5.21(2) by taking determinants.

Example 3.6.6 (Euler sequence)**.** For any $n \in \mathbf{Z}_{\geq 0}$, the projective space $\mathbb{P}^n_{\mathbf{Z}}$ is smooth over $\mathrm{Spec}(\mathbf{Z})$. Indeed, this follows from the smoothness of $\mathbb{A}^n_{\mathbf{Z}} \to \mathrm{Spec}(\mathbf{Z})$ and the fact that $\mathbb{P}^n_{\mathbf{Z}}$ admits an open cover by $(n+1)$ copies of $\mathbb{A}^n_{\mathbf{Z}}$ (*cf.* the proof of Proposition 3.1.4). The sheaf of differential forms $\Omega_{\mathbb{P}^n_{\mathbf{Z}}}$ fits into a short exact sequence:

$$0 \to \Omega_{\mathbb{P}^n_{\mathbf{Z}}} \xrightarrow{\alpha} \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^n_{\mathbf{Z}}}(-1) \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^n_{\mathbf{Z}}} \to 0, \tag{3.51}$$

called the *Euler sequence*. Here, β is the sum of maps $\mathrm{X}_i : \mathcal{O}_{\mathbb{P}^n_{\mathbf{Z}}}(-1) \to \mathcal{O}_{\mathbb{P}^n_{\mathbf{Z}}}$, where X_i is the ith homogeneous coordinate of $\mathbb{P}^n_{\mathbf{Z}}$ $(0 \leq i \leq n)$. Over the chart of $\mathbb{P}^n_{\mathbf{Z}}$ where X_i is invertible, the morphism α carries the basis element $d(\mathrm{X}_j/\mathrm{X}_i)$ $(j \neq i)$ to the section with $1/\mathrm{X}_i$ placed at the jth coordinate and $-\mathrm{X}_j/\mathrm{X}_i^2$ placed at the ith coordinate.[16]

It follows from (3.51) that $\omega_{\mathbb{P}^n_{\mathbf{Z}}}$ is identified with $\mathcal{O}_{\mathbb{P}^n_{\mathbf{Z}}}(-n-1)$, by taking determinants.

Example 3.6.7 (Smooth hypersurfaces)**.** Let R be a ring and consider the projective space $\mathbb{P}^n_{\mathrm{R}}$ over $\mathrm{Spec}(\mathrm{R})$. Let f be a section of $\mathcal{O}_{\mathbb{P}^2_{\mathrm{R}}}(d)$ such that its vanishing locus $i : \mathrm{X} \to \mathbb{P}^n_{\mathrm{R}}$ is smooth over $\mathrm{Spec}(\mathrm{R})$. (For explicit f, smoothness of X can be checked using the Jacobian criterion.) Then we have an isomorphism:

$$\omega_{\mathrm{X}} \xrightarrow{\simeq} \mathcal{O}_{\mathbb{P}^n_{\mathrm{R}}}(d-n-1)|_{\mathrm{X}}. \tag{3.52}$$

Indeed, we have a short exact sequence:

$$0 \to \mathcal{O}_{\mathbb{P}^2_{\mathrm{R}}}(-d) \xrightarrow{f} \mathcal{O}_{\mathbb{P}^2_{\mathrm{R}}} \to i_* \mathcal{O}_{\mathrm{X}} \to 0.$$

Comparing with (2.29), we find that the ideal sheaf defining the closed subscheme X of $\mathbb{P}^n_{\mathrm{R}}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^n_{\mathrm{R}}}(-d)$. Thus, the conormal sheaf of X is isomorphic to $\mathcal{O}_{\mathbb{P}^n_{\mathrm{R}}}(-d)|_{\mathrm{X}}$. The isomorphism (3.50) then reads as follows:

$$\mathcal{O}_{\mathbb{P}^n_{\mathrm{R}}}(-n-1)|_{\mathrm{X}} \xrightarrow{\simeq} \omega_{\mathrm{X}} \otimes \mathcal{O}_{\mathbb{P}^n_{\mathrm{R}}}(-d)|_{\mathrm{X}},$$

where we used the Euler sequence to identify $\omega_{\mathbb{P}^n_{\mathrm{R}}}$ with $\mathcal{O}_{\mathbb{P}^n_{\mathrm{R}}}(-n-1)$ (*cf.* Example 3.6.6). This yields the isomorphism (3.52).

Example 3.6.8 (Hodge classes of line bundles)**.** Let R be a ring and X be a smooth scheme over $\mathrm{Spec}(\mathrm{R})$. Let $\mathcal{L}$ be a line bundle over X. We shall attach to $\mathcal{L}$ an Ω_{X}-torsor.

[16] Note that the pullback of (3.51) along $\pi : \mathbb{A}^{n+1}_{\mathbf{Z}} \smallsetminus 0 \to \mathbb{P}^n_{\mathbf{Z}}$ is identified with the short exact sequence:

$$0 \to \pi^* \Omega_{\mathbb{P}^n_{\mathbf{Z}}} \to \bigoplus_{i=0}^{n} \Omega_{\mathbb{A}^{n+1}_{\mathbf{Z}} \smallsetminus 0} \to \Omega_{\pi} \to 0.$$

obtained in Corollary 3.5.21(1).

Namely, we consider the sheaf of abelian groups on $(\mathsf{Sch}^{\mathrm{aff}})_{/\mathrm{X}}$ underlying Ω_{X}: it assigns to $\mathrm{Spec}(\mathrm{A}) \to \mathrm{X}$ the abelian group $\Omega_{\mathrm{A}/\mathrm{R}}$. Recall the sheaf of abelian groups $\mathbb{G}_m$ on $(\mathsf{Sch}^{\mathrm{aff}})_{/\mathrm{X}}$ which assigns to $\mathrm{Spec}(\mathrm{A}) \to \mathrm{X}$ the abelian group $\mathrm{A}^\times$ of units in A. Consider the morphism of sheaves:

$$d\log : \mathbb{G}_m \to \Omega_{\mathrm{X}}, \quad f \mapsto f^{-1}df.$$

Note that a line bundle $\mathcal{L}$ over X is equivalent to a $\mathbb{G}_m$-torsor (Proposition 3.2.19), so we may change the structure group along $d\log$ to obtain an Ω_{X}-torsor. The isomorphism class of this Ω_{X}-torsor is called the *Hodge class* of $\mathcal{L}$.

4. Points of schemes

In this section, we introduce the underlying topological space $|\mathrm{X}|$ of a scheme X and discuss several notions associated to them. The topological space $|\mathrm{X}|$ is a strange beast. Its role is very different from, say, the underlying topological space of a differentiable manifold. For example, it is (quasi-)compact for the complex plane $\mathrm{X} = \mathbb{A}^1_{\mathbf{C}}$ and is practically never Hausdorff. For X a projective, smooth curve over $\mathbf{C}$ (the algebro-geometric model for a compact Riemann surface), $|\mathrm{X}|$ also has no idea how many "holes" X has.

However, $|\mathrm{X}|$ keeps track of a different host of information about X, such as its "algebraic cycles" and the closure relations among them.

4.1. The topological space $|\mathrm{X}|$.

4.1.1. Let X be a scheme.

We define an equivalence relation on the set of field-valued points of X: Two morphisms $\mathrm{Spec}(\mathrm{K}_1) \to \mathrm{X}$, $\mathrm{Spec}(\mathrm{K}_2) \to \mathrm{X}$ are *equivalent* if there is another field K_{12} containing K_1, K_2 such that the diagram below commutes:

$$\begin{array}{ccc} \mathrm{Spec}(\mathrm{K}_{12}) & \to & \mathrm{Spec}(\mathrm{K}_1) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathrm{K}_2) & \longrightarrow & \mathrm{X} \end{array}$$

To see that this relation is transitive, we observe that given morphism of spectra of fields $\mathrm{Spec}(\mathrm{L}_1) \to \mathrm{Spec}(\mathrm{K}) \leftarrow \mathrm{Spec}(\mathrm{L}_2)$, the tensor product $\mathrm{L}_1 \otimes_{\mathrm{K}} \mathrm{L}_2$ is not the zero ring, so there is a morphism $\mathrm{L}_1 \otimes_{\mathrm{K}} \mathrm{L}_2 \to \mathrm{L}$ for some field L.

We define $|\mathrm{X}|$ to be the set of equivalence classes of field-valued points of X. The set $|\mathrm{X}|$ is called the *underlying set* of the scheme X. Elements of $|\mathrm{X}|$ are called *points* of X (to be distinguished from R-points for $\mathrm{R} \in \mathsf{Ring}$.)

Remark 4.1.2. If $\mathrm{X} = \mathrm{Spec}(\mathrm{A})$ is an affine scheme, then $|\mathrm{X}|$ is in bijection with the set of prime ideals of A via the map sending a field-valued point $\mathrm{Spec}\,\mathrm{K} \to \mathrm{X}$ to the kernel of the corresponding ring map $\mathrm{A} \to \mathrm{K}$.

The inverse of this map sends a prime ideal $\mathfrak{p}$ to the class of the field-valued point $\mathrm{A} \to \kappa(\mathfrak{p})$, where $\kappa(\mathfrak{p}) \coloneqq \mathrm{A}_{\mathfrak{p}}/\mathfrak{p}\mathrm{A}_{\mathfrak{p}}$ is the residue field of the localization $\mathrm{A}_{\mathfrak{p}}$.

4.1.3. Given a morphism $f : \mathrm{Y} \to \mathrm{X}$ of schemes, we obtain a morphism of sets $|f| : |\mathrm{Y}| \to |\mathrm{X}|$ sending the class of a field-valued point of Y to the class of its image. Thus, the association $\mathrm{X} \mapsto |\mathrm{X}|$ defines a functor:

$$\mathsf{Sch} \to \mathsf{Set}, \quad \mathrm{X} \mapsto |\mathrm{X}|. \tag{4.1}$$

If f is a monomorphism (respectively, epimorphism) of Zariski sheaves, then $|f|$ is injective (respectively, surjective).

Remark 4.1.4. Given morphisms of schemes $X_1 \to X \leftarrow X_2$, the natural map:

$$|X_1 \times_X X_2| \to |X_1| \times_{|X|} |X_2| \tag{4.2}$$

is surjective but not injective in general. For an example of non-injectivity, take $X := \operatorname{Spec} k$ for a field k and $X_1 = X_2 = \mathbb{A}^1_k$. The fiber product $X_1 \otimes_X X_2 \cong \mathbb{A}^2_k \cong \operatorname{Spec} k[x, y]$ has points defined by prime ideals (0) and $(x - y)$ which have the same images in $|\mathbb{A}^1_k| \times |\mathbb{A}^1_k|$.

However, if $X_1 \to X$ is a monomorphism, then (4.2) is injective (hence bijective): This is because $X_1 \times_X X_2 \to X_2$ is also a monomorphism (*cf.* Lemma 1.4.8), so the composition $|X_1 \times_X X_2| \to |X_1| \times_{|X|} |X_2| \to |X_2|$ is already injective.

4.1.5. Let X be a scheme and $x \in |X|$. Consider the filtered colimit of rings:

$$\mathcal{O}_{X,x} := \operatorname*{colim}_{x \in |\operatorname{Spec} R|} R,$$

indexed by open affine subschemes $\operatorname{Spec} R \to X$ whose underlying set contains x. The ring $\mathcal{O}_{X,x}$ is called the *local ring* of X at x.

By construction, there is a morphism $\operatorname{Spec}(\mathcal{O}_{X,x}) \to X$ which factors through every open affine subscheme of X whose underlying set contains x. If we fix such an open affine subscheme $\operatorname{Spec} A \to X$, then x corresponds to a prime ideal $\mathfrak{p}$ of A and $\mathcal{O}_{X,x}$ is identified with the localization $A_{\mathfrak{p}}$. In particular, $\mathcal{O}_{X,x}$ is a local ring in the sense of commutative algebra: It has a unique maximal ideal which we denote by $\mathfrak{m}_{X,x}$.

Let $\kappa(x)$ denote the residue field of $\mathcal{O}_{X,x}$, which we refer to simply as the *residue field* of x. Denote by $\hat{\mathcal{O}}_{X,x}$ the completion of $\mathcal{O}_{X,x}$ along $\mathfrak{m}_{X,x}$ and call it the *completed local ring* of X at x. In summary, every point x of X gives rise to morphisms of schemes:

$$\operatorname{Spec}(\kappa(x)) \to \operatorname{Spec}(\hat{\mathcal{O}}_{X,x}) \to \operatorname{Spec}(\mathcal{O}_{X,x}) \to X.$$

Proposition 4.1.6. *Let* X *be a scheme. There is a topology on* $|X|$ *such that a subset of* $|X|$ *is open if and only if it is the image of* $|f|$ *for some open immersion* $f : U \to X$.

Proof. Let us call a subset of $|X|$ *open* if it is the image of $|f|$ for some open immersion $f : U \to X$. Since $\mathrm{id} : X \to X$ and $\varnothing \to X$ are open immersions, open subsets of $|X|$ include the empty set and $|X|$.

Given a finite collection of open immersions $f_i : U_i \to X$ $(i = 1, \cdots, n)$, we obtain an open immersion $f : U_1 \times_X U_2 \times_X \cdots \times_X U_n \to X$, whose image is $\bigcap_{i=1}^n |U_i|$, where each $|U_i|$ is viewed as a subset of $|X|$.

Given an arbitrary collection of open immersions $f_i : U_i \to X$ $(i \in I)$, we let U denote the sheaf image of the induced morphism $\bigsqcup_{i \in I} U_i \to X$, *i.e.* the sheafification of the presheaf image. We thus obtain morphisms of Zariski sheaves:

$$\bigsqcup_{i \in I} U_i \to U \to X, \tag{4.3}$$

where the first morphism is an epimorphism and the second one is a monomorphism. The formation of (4.3) is furthermore compatible with base change in X.

We claim that U is a scheme and the natural morphism $U \to X$ is an open immersion. This will be enough, since $|U|$ equals $\bigcup_{i \in I} |U_i|$ as a subset of $|X|$. To prove that U is a scheme, it suffices to prove that each $U_i \to U$ is an open immersion, because $U_i \to U$ $(i \in I)$ will then be an open cover of U by schemes. However, since $U \to X$ is a monomorphism and each $U_i \to X$ is an open immersion, so is the morphism $U_i \to U$ (*cf.* Remark 1.5.4). To prove that $U \to X$ is an open immersion, we reduce to the case where $X = \operatorname{Spec} A$ is affine, where it follows from the definition of open immersions. ☐

Remark 4.1.7. In fact, the poset of open immersions into X is identified with the poset of open subsets of $|X|$. Indeed, given two open immersions $U \to X \leftarrow V$ such that $|U| = |V| \subset |X|$, we argue that $U \cong V$ as open subschemes of X.

By taking the fiber product $U \times_X V$, this reduces to the following assertion: Any open immersion $U \to X$ which induces a surjection $|U| \to |X|$ is an isomorphism. This reduces to the case where X is affine, where the assertion is clear.

4.1.8. Let us now upgrade (4.1) to a functor:

$$\mathsf{Sch} \to \mathsf{Top}, \quad X \mapsto |X|, \tag{4.4}$$

by equipping the set $|X|$ with the topology defined by Proposition 4.1.6.

Given a morphism of schemes $f : Y \to X$, the induced map $|f| : |Y| \to |X|$ is indeed continuous because given an open immersion $U \to X$, the base change $f^{-1}U := U \times_X Y$ satisfies $|f^{-1}U| = f^{-1}|U|$ as subsets of $|Y|$ (*cf.* Remark 4.1.4).

We call $|X|$ the *underlying topological space* of the scheme X.

Corollary 4.1.9. *A scheme* X *is quasi-compact if and only if* $|X|$ *is quasi-compact.*[17]

Proof. This follows from Proposition 4.1.6. □

4.1.10. Recall that a topological space T is *connected* if it is not the union of two disjoint closed subsets.

A topological space T is *irreducible* if $T \neq \varnothing$ and whenever $T = T_1 \cup T_2$ for two closed subsets $T_1, T_2 \subset T$, we have $T = T_1$ or $T = T_2$. This is equivalent to saying that $T \neq \varnothing$ and every two nonempty open subsets of T have nonempty intersection. A maximal closed irreducible subset of T is called an *irreducible component*.

Clearly, being irreducible is stronger than being connected. If a topological space T is irreducible, then so is any of its nonempty open subsets.

Lemma 4.1.11. *Let* $T \subset T'$ *be a dense subset of a topological space. Then* T *is irreducible if and only if* T' *is irreducible.*

Proof. Suppose that T is irreducible. Any two nonempty open subsets $U_1, U_2 \subset T'$ satisfy $U_1 \cap T \neq \varnothing$, $U_2 \cap T \neq \varnothing$ because T is dense in T', so $(U_1 \cap U_2) \cap T$ is nonempty. In particular, $U_1 \cap U_2$ is nonempty, so T' is irreducible.

Suppose that T' is irreducible. Any two nonempty open subset $V_1, V_2 \subset T$ can be written as $V_1 = U_1 \cap T$, $V_2 = U_2 \cap T$ for nonempty open subsets $U_1, U_2 \subset T'$. Then $U_1 \cap U_2$ is nonempty. But because T is dense in T', $V_1 \cap V_2 = (U_1 \cap U_2) \cap T$ is also nonempty, so T is irreducible. □

4.1.12. Let T be a topological space. A point $x \in T$ is called *closed* if $\{x\}$ is closed.

It follows from Lemma 4.1.11 that the closure of any singleton $\overline{\{x\}}$, for $x \in T$, is irreducible. If a closed irreducible subset $Z \subset T$ is of the form $\overline{\{x\}}$ for some $x \in T$, then x is called a *generic point* of Z. The topological space T is called *sober* if every irreducible closed subset of T has a unique generic point.

If $y \in \overline{\{x\}}$, we say that y is a *specialization* of x, or that x is a *generalization* of y. This relationship is sometimes denoted by $x \rightsquigarrow y$. Note that if T is sober, then specialization defines a partial order on the points of T.

[17] A topological space is called *quasi-compact* if every open cover admits a finite subcover. This notion is sometimes called "compact".

The *Krull dimension* $\dim(T) \in \mathbf{Z} \cup \{\infty\}$ of T is the supremum of the length n of a chain of irreducible closed subsets of T:

$$Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n.$$

Lemma 4.1.13. *Let* X *be a scheme. Then* $|X|$ *is sober.*

Proof. We first prove this for $X = \mathrm{Spec}(A)$ an affine scheme. In this case, the association $\mathfrak{p} \mapsto V(\mathfrak{p})$ defines a bijection between primes of A and irreducible closed subsets of $|\mathrm{Spec}(A)|$. Furthermore, $\mathfrak{p}$ is the unique generic point of $V(\mathfrak{p})$.

For any scheme X, let $Z \subset |X|$ be an irreducible closed subset. Let $\mathrm{Spec}(A) \to X$ be an open immersion such that $Z \cap |\mathrm{Spec}(A)|$ is nonempty. Since Z is irreducible, so is $Z \cap |\mathrm{Spec}(A)|$ as it is a nonempty open subset. The affine case shows that we can find a point $x \in |\mathrm{Spec}(A)|$ such that $Z \cap |\mathrm{Spec}(A)|$ coincides with the closure of $\{x\}$ in $|\mathrm{Spec}(A)|$. *Claim*: Z coincides with $\overline{\{x\}}$. Indeed, $Z = \overline{\{x\}} \cup (Z \smallsetminus |\mathrm{Spec}(A)|)$, but $Z \neq Z \smallsetminus |\mathrm{Spec}(A)|$, so $Z = \overline{\{x\}}$. We have now found a generic point of Z.

Suppose that Z has two generic points $x, y \in |X|$. Since $y \in \overline{\{x\}}$, any open subset of $|X|$ containing y must also contain x. Thus we may choose an open affine subscheme containing both and the uniqueness in the affine case shows $x = y$. □

Remark 4.1.14. Let X be a scheme. For every point $x \in |X|$, we have a canonical morphism $\mathrm{Spec}(\mathcal{O}_{X,x}) \to X$ constructed in §4.1.5. The image of $|\mathrm{Spec}(\mathcal{O}_{X,x})|$ in $|X|$ consists precisely of generalizations of x.

4.1.15. Let X be a scheme. We say:

(1) X is *connected* (respectively, *irreducible*) if $|X|$ is;
(2) the *Krull dimension* $\dim(X)$ is the Krull dimension of $|X|$:

$$\dim(X) := \dim(|X|).$$

Note that by Lemma 4.1.13, $\dim(X)$ coincides with the supremum of the length n of a chain of specializations $x_n \rightsquigarrow x_{n-1} \rightsquigarrow \cdots \rightsquigarrow x_0$ of distinct points. In particular, for an affine scheme $X = \mathrm{Spec}(A)$, $\dim(X)$ equals the Krull dimension $\dim(A)$ of the ring A.

By taking the local ring at x_0 for all such chains, we also see that $\dim(X)$ also equals the supremum of $\dim(\mathcal{O}_{X,x})$ over all $x \in |X|$.

Remark 4.1.16. Let X be a scheme. For $x \in |X|$, the number $\dim(\mathcal{O}_{X,x}) \in \mathbf{Z} \cup \{\infty\}$ should be thought of as the *codimension* of the point x, so we also write it as $\mathrm{codim}(x)$.

If $X = \mathrm{Spec}(A)$ and x corresponds to a prime $\mathfrak{p}$, then $\dim(\mathcal{O}_{X,x})$ is called *height* of $\mathfrak{p}$ in commutative algebra. In summary, we have equalities by definition:

$$\mathrm{ht}(\mathfrak{p}) := \mathrm{codim}(x) := \dim(\mathcal{O}_{X,x}).$$

Let us also write $X^{(n)}$ for the set of codimension-n points of X ($n \geq 0$). By Lemma 4.1.13, $X^{(0)}$ is in bijection with the irreducible components of $|X|$. Elements of $X^{(0)}$ are also called the *generic points* of the scheme X.

Example 4.1.17 (Effective Cartier divisors have codimension 1)**.** Let X be a locally Noetherian scheme and $Z \to X$ be a nonempty effective Cartier divisor. Let ξ be the generic point of any irreducible component of $|Z|$. Then ξ is a codimension-1 point of X.

Indeed, this translates to the following ring-theoretic statement: If A is a ring and $\mathfrak{p}$ is a minimal prime containing a non-zero-divisor $a \in A$, then $\mathrm{ht}(\mathfrak{p}) = 1$. Indeed, we have $\mathrm{ht}(\mathfrak{p}) \leq 1$ by Krull's Hauptidealsatz (*cf.* [Sta18, 00KV]), but $\mathrm{ht}(\mathfrak{p}) \geq 1$ because every minimal prime consists of zero-divisors (Proof: Every element in a minimal prime $\mathfrak{p} \subset A$ becomes nilpotent in $A_{\mathfrak{p}}$).

4.1.18. Finally, we classify all locally Noetherian schemes (*cf.* Lemma 1.8.4) of Krull dimension 0.

Recall that given a local ring R, the following conditions are equivalent:[18]

(1) R is Artinian, *i.e.* ideals of R satisfy the descending chain condition;
(2) R is Noetherian and $\dim(R) = 0$;
(3) R has finite length as an object of Mod_R.

Indeed, (3) implies both Artinian and Noetherian properties; it also implies $\dim(R) = 0$ as the maximal ideal $\mathfrak{m} \subset R$ is nilpotent, *i.e.* $\mathfrak{m}^n = 0$ for some $n \in \mathbf{Z}$ (Nakayama), so it is contained in any prime. Hence (3) ⇒ (1) & (2). On the other hand, (1) ⇒ (3) again by nilpotence of $\mathfrak{m}$. To see (2) ⇒ (1), note that $\mathfrak{m}$ coincides with the radical of R, but because it is finitely generated, it is nilpotent.

4.1.19. A scheme X is called *Noetherian* if it is locally Noetherian and quasi-compact.

Lemma 4.1.20. *Let* X *be a Noetherian scheme. Then* $|X|$ *has finitely many irreducible components.*

Proof. For convenience, we call a topological space T *Noetherian* if its closed subsets satisfy the descending chain condition. Thus, if A is a Noetherian ring, then $|\mathrm{Spec}(A)|$ is a Noetherian topological space.

Claim: a Noetherian topological space T has finitely many irreducible components. Assume otherwise, we consider the set $\mathcal{Z}$ of closed subsets of T which are *not* the union of finitely many irreducible subsets. Then $\mathcal{Z} \neq \emptyset$, so the descending chain condition produces a minimal element $Z \in \mathcal{Z}$. This Z cannot be irreducible, so $Z = Z_1 \cup Z_2$ for closed subsets $Z_1, Z_2 \neq Z$. Then both Z_1 and Z_2 are unions of finitely many irreducible subsets, so the same holds for Z; contradiction.

Claim: if a topological space T is a finite union of Noetherian subspaces T_i ($i \in I$), then T is Noetherian. Indeed, any chain of closed subsets of T which stablizes in each T_i stabilizes in T, because I is finite.

Finally, we note that $|X|$ can be expressed as a finite union of $|X_i|$ ($i \in I$), where each $X_i = \mathrm{Spec}(R_i) \to X$ is an open immersion with R_i Noetherian. Hence $|X|$ is a Noetherian topological space, so it has finitely many irreducible components. □

Proposition 4.1.21. *Let* X *be a locally Noetherian scheme with* $\dim(X) = 0$*. Then* X *is a disjoint union of spectra of local Artinian rings.*

Proof. It suffices to prove that $|X|$ has the discrete topology. Indeed, then every $x \in |X|$ is open, so the unique open immersion $U \to X$ with $|U| = \{x\}$ must agree with $\mathrm{Spec}(\mathcal{O}_{X,x})$. The open cover of X they define yields an isomorphism in Sch:

$$\bigsqcup_{x \in |X|} \mathrm{Spec}(\mathcal{O}_{X,x}) \xrightarrow{\simeq} X.$$

On the other hand, each $\mathcal{O}_{X,x}$ is Noetherian and of dimension 0, hence Artinian.

To prove that $|X|$ has the discrete topology, it suffices to treat the case $X = \mathrm{Spec}(A)$ is affine. We may further assume that A is reduced because $|\mathrm{Spec}(A)|$ is homeomorphic to $|\mathrm{Spec}(A/\sqrt{0})|$. Since A is Noetherian, it has finitely many minimal primes by Lemma

[18] These conditions are also equivalent for any $R \in \mathsf{Ring}$ not necessarily local, but this is partly subsumed by Proposition 4.1.21 below, so we will not recall it here. (See [Sta18, 00KH, 00JB] for references.)

4.1.20.[19] Since $\dim(\mathrm{A}) = 0$, these minimal primes are all maximal. Let us denote them by $\mathfrak{m}_i$ ($i \in \mathrm{I}$). Because $\bigcap_{i\in\mathrm{I}} \mathfrak{m}_i = \sqrt{0} = (0)$, the Chinese remainder theorem yields an isomorphism:

$$\mathrm{A} \xrightarrow{\simeq} \prod_{i\in\mathrm{I}} \mathrm{A}/\mathfrak{m}_i.$$

In particular, the topological space $|\mathrm{Spec}(\mathrm{A})|$ is discrete. □

4.2. Normalization.

4.2.1. Proposition 4.1.21 tells us that locally Noetherian schemes of dimension 0 are simply discrete collections of fat points. In dimension 1, we do not have such a simple description. Indeed, Noetherian schemes of dimension 1 are supposed to model compact Riemann surfaces as well as integers in a number field, both of which are rich mathematical objects.

In this subsection, we will make some baby steps towards understanding locally Noetherian schemes of dimension 1. Namely, we will show that they can be "desingularized".

4.2.2. Given a nonzero ring R, we can attach to it a sequence of modifications:

$$\mathrm{R} \to \mathrm{R}_{\mathrm{red}} \to \prod_{\mathfrak{p} \text{ minimal}} \mathrm{R}/\mathfrak{p} \to \prod_{\mathfrak{p} \text{ minimal}} (\mathrm{R}/\mathfrak{p})_\nu =: \mathrm{R}_\nu. \tag{4.5}$$

The first step is the quotient of R by its nilradical $\sqrt{0}$, giving a reduced ring $\mathrm{R}_{\mathrm{red}} := \mathrm{R}/\sqrt{0}$. The second step is the quotient of R at its minimal primes, so each $\mathrm{R}/\mathfrak{p}$ is a domain. The third step replaces each $\mathrm{R}/\mathfrak{p}$ by its normalization $(\mathrm{R}/\mathfrak{p})_\nu$, *i.e.* the integral closure in its field of fractions, so each $(\mathrm{R}/\mathfrak{p})_\nu$ is a normal domain.

The process (4.5) can be seen as making the ring R increasingly "regular". At the opposite end, we have the notion of a "regular" ring.

Recall that a local ring R, with maximal ideal $\mathfrak{m}$ and residue field κ, is *regular* if it is Noetherian and $\dim(\mathrm{R})$ equals the dimension of $\mathfrak{m}/\mathfrak{m}^2$ as a κ-vector space:

$$\dim(\mathrm{R}) = \dim_\kappa(\mathfrak{m}/\mathfrak{m}^2).$$

(By Nakayama lemma, this implies that $\mathfrak{m}$ can be generated by $\dim(\mathrm{R})$ elements.)

4.2.3. The key feature of dimension-1 Noetherian local rings is that the notions of normality and regularity coincide. They also turn out to be equivalent to a third notion: a "discrete valuation ring".

Recall that a *discrete valuation ring* is a principal ideal domain R with exactly one nonzero maximal ideal $\mathfrak{m}$. Upon choosing a generator $\pi \in \mathfrak{m}$, we can define a map out of its field of fractions f.f.(R), called *order of vanishing*:

$$\mathrm{ord} : \mathrm{f.f.}(\mathrm{R}) \to \mathbf{Z} \cup \{\infty\}, \tag{4.6}$$

sending $u\pi^n$ ($u \in \mathrm{R}^\times$) to the integer n and $0 \in \mathrm{R}$ to ∞.

Proposition 4.2.4. *Let* R *be a local ring with maximal ideal* $\mathfrak{m}$ *and residue field* κ*. Then the following statements are equivalent:*

(1) R *is a Noetherian normal domain and* $\dim(\mathrm{R}) = 1$*;*
(2) R *is regular and* $\dim(\mathrm{R}) = 1$*;*
(3) R *is a discrete valuation ring.*

Proof. This is [Sta18, 00PD]. □

[19] Here is an alternative, ring-theoretic, proof of this fact. First, one shows that any finite A-module M has an associated prime (*e.g.* a maximal element among the annihilators of elements of M). Using the ascending chain condition, we find a finite filtration $0 = \mathfrak{a}_0 \subsetneq \mathfrak{a}_1 \subsetneq \cdots \subsetneq \mathfrak{a}_n = \mathrm{A}$ where each $\mathfrak{a}_i/\mathfrak{a}_{i-1}$ is isomorphic to $\mathrm{A}/\mathfrak{p}_i$ for some prime $\mathfrak{p}_i$ ($1 \le i \le n$). Any prime $\mathfrak{p}$ of A contains one of the $\mathfrak{p}_i$'s because $\mathrm{A}_\mathfrak{p} \neq 0$.

Remark 4.2.5. Let R be a discrete valuation ring with maximal ideal $\mathfrak{m}$. Then $|\mathrm{Spec}(\mathrm{R})|$ has two points: a *special point* x corresponding to the prime $\mathfrak{m}$ and a *generic point* η corresponding to the prime (0) (*cf.* Remark 4.1.16). They are related by a specialization $\eta \rightsquigarrow x$. The function (4.6), as its name suggests, returns the "order of vanishing" of $f \in \mathrm{f.f.}(\mathrm{R})$ at the special point x, where negative values indicate "poles".

For example, consider $\mathrm{R} \coloneqq \mathrm{K}[x]_{(x)}$ where K is a field. Its field of fractions is identified with the field $\mathrm{K}(x)$ and $\mathrm{ord}(f)$ is precisely the order of vanishing of $f \in \mathrm{K}(x)$ at the origin.

Example 4.2.6 (Nodal curve)**.** Let K be a field and consider the subring $\mathrm{A} \subset \mathrm{K}[x]$ consisting of elements $f \in \mathrm{K}[x]$ such that $f(0) = f(1)$. Note that A is a finitely generated K-algebra because any $f \in \mathrm{K}[x]$ is of the form $(x^2 - x) \cdot g + \lambda$ for some $g \in \mathrm{K}[x]$ and $\lambda \in \mathrm{K}$, so A is generated by $x^2 - x$ and $x^3 - x^2$ as a K-algebra.

Claim: $\mathrm{K}[x]$ is the normalization of A.

Indeed, $\mathrm{K}[x] \subset \mathrm{K}(x)$ is integrally closed, so it suffices to show that any element of $\mathrm{K}[x]$ is integral over A. In fact, it suffices to show that x is integral over A, but this holds because $x^2 - x \in \mathrm{A}$. (This also implies that $\mathrm{A} \to \mathrm{K}[x]$ is finite, since it is of finite type.)

Note that $\mathfrak{a} \coloneqq (x^2 - x)$ is a maximal ideal of A. Indeed, we first observe that it is the intersection $(x) \cap \mathrm{A}$: This is because any $f \in \mathrm{A}$ which is divisible by x must also be divisible by $x - 1$. Hence $\mathfrak{a}$ is a prime ideal of A. It is maximal because $\mathrm{A} \subset \mathrm{K}[x]$ is integral, so it satisfies the going-up property. Let us write 0 for the corresponding closed point of Spec A. Its residue field is K since the residue field of $(x) \subset \mathrm{K}[x]$ is K.

Consider the short exact sequence of finite A-modules:

$$0 \to \mathrm{A} \to \mathrm{K}[x] \to \mathrm{K} \to 0, \tag{4.7}$$

where the second map sends f to $f(0) - f(1)$. Localizing at the element $x^2 - x \in \mathrm{A}$, the last term vanishes, so we obtain an isomorphism of affine schemes:

$$\mathbb{A}^1_{\mathrm{K}} \smallsetminus \{0, 1\} \to \mathrm{Spec}\,\mathrm{A} \smallsetminus 0.$$

On the other hand, if we complete (4.7) along the ideal $\mathfrak{a}$, we obtain a short exact sequence (using the exactness of completion of finite modules over Noetherian rings, *cf.* [Sta18, 00MA]):

$$0 \to \hat{\mathrm{A}}_{\mathfrak{a}} \to \mathrm{K}[[x]] \times \mathrm{K}[[y]] \to \mathrm{K} \to 0.$$

where the second map sends (f, g) to $f(0) - g(0)$. This shows that $\hat{\mathrm{A}}_{\mathfrak{a}}$ is isomorphic to the completion of $\mathrm{K}[x, y]/(xy)$ along (x, y), *i.e.* formally locally at 0, Spec A looks like the union of the two coordinate axes in $\mathbb{A}^2_{\mathrm{K}}$. Such singularities are called *nodes*.

4.2.7. Let X be a scheme.

We say that X is *normal* (respectively, *regular*) if for every $x \in |\mathrm{X}|$, the local ring $\mathcal{O}_{\mathrm{X},x}$ is a normal domain (respectively, a regular local ring). In particular, normality and regularity are local properties (*cf.* §1.8.2).

We say that X is *integral* if $\mathrm{X} \neq \varnothing$ and for every nonempty open immersion $\mathrm{Spec}(\mathrm{R}) \to \mathrm{X}$ (*i.e.* $\mathrm{Spec}(\mathrm{R}) \neq \varnothing$), the ring R is a domain. Note that integrality is *not* a local property: The disjoint union of two integral schemes is *not* integral.

Lemma 4.2.8. *A scheme* X *is integral and normal if and only if for every nonempty open immersion* $\mathrm{Spec}(\mathrm{R}) \to \mathrm{X}$*, the ring* R *is a normal domain.*

Proof. This is equivalent to the assertion that an integral domain R is a normal domain if and only if $\mathrm{R}_{\mathfrak{p}}$ is a normal domain for every prime $\mathfrak{p}$ of R.

To prove the "⇒" direction, we note a more general fact. Namely, given a morphism $\mathrm{R} \to \mathrm{B}$ in Ring and a multiplicatively closed subset $\mathrm{S} \subset \mathrm{R}$, if $\mathrm{R} \to \mathrm{A} \subset \mathrm{B}$ is the integral closure

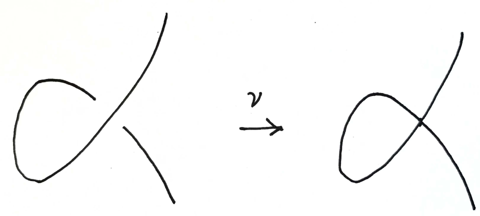

FIGURE 5. The morphism $\nu : \mathbb{A}^1_K \to \mathrm{Spec}(A)$ of affine schemes induced by the subring $A \subset K[x]$ (*cf.* Example 4.2.6) is the normalization of a node. Geometrically, we think of Spec(A) as obtained from $\mathbb{A}^1_K = \mathrm{Spec}(K[x])$ by "pinching" the K-points 0 and 1.

of R in B, then $S^{-1}R \to S^{-1}A \subset S^{-1}B$ is the integral closure of $S^{-1}R$ in $S^{-1}B$. The proof of this fact is by "clearing the denominators" (see [Sta18, 0307] for details).

The desired statement follows from applying this observation to $B := \mathrm{f.f.}(R)$, the field of fractions of R, and $S := R \setminus \mathfrak{p}$.

To prove the "⇐" direction, we note that being a domain, R equals the intersection:

$$R = \bigcap_{\mathfrak{p}} R_{\mathfrak{p}} \subset \mathrm{f.f.}(R).$$

Indeed, for any $f \in \bigcap_{\mathfrak{p}} R_{\mathfrak{p}}$, consider the ideal $\mathfrak{a} \subset R$ consisting of elements $a \in R$ such that $af \in R$. The for every prime $\mathfrak{p}$, there exists $a \in \mathfrak{a}$, $a \notin \mathfrak{p}$. Hence $\mathfrak{a} = R$.

The desired statement follows, because the intersection of integrally closed subrings of f.f.(R) is integrally closed. □

4.2.9. Let X be a scheme. Let $T \subset |X|$ be a closed subset. Then the partial order of closed immersions $Z \to X$ such that $|Z| \to |X|$ has image T has an initial object.

We construct the initial object $Z \to X$ by specifying its ideal sheaf $\mathcal{I}_Z$: for every open immersion $\mathrm{Spec}(R) \to X$, $T \cap |\mathrm{Spec}(R)|$ is a closed subset of $|\mathrm{Spec}(R)|$, so it corresponds to a radical ideal $\mathfrak{a} \subset R$, and we set $\Gamma(\mathrm{Spec}(R), \mathcal{I}_Z) := \mathfrak{a}$. By Corollary 2.2.5, it remains to prove that for an open immersion $\mathrm{Spec}(R_f) \to \mathrm{Spec}(R)$, we have:

$$\Gamma(\mathrm{Spec}(R), \mathcal{I}_Z) \otimes_R R_f \xrightarrow{\simeq} \Gamma(\mathrm{Spec}(R_f), \mathcal{I}_Z).$$

This follows because $\mathfrak{a}_f \subset R_f$ is the radical ideal corresponding to $T \cap |\mathrm{Spec}(R_f)|$.

Note that the inital object $Z \to X$ has the property that Z is reduced. We call it the *induced reduced subscheme* of the closed subset $T \subset |X|$.

In particular, we may apply this construction to the set |X| itself to obtain a closed subscheme $X_{\mathrm{red}} \to X$. By construction, for every open immersion $\mathrm{Spec}(R) \to X$, we have a Cartesian square:

$$\begin{array}{ccc} \mathrm{Spec}(R_{\mathrm{red}}) & \to & X_{\mathrm{red}} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(R) & \longrightarrow & X \end{array}$$

So every morphism $f : Y \to X$ in Sch, with Y reduced, factors through X_{red}.

Proposition 4.2.10. *A scheme* X *is integral if and only if it is irreducible and reduced.*

Proof. Suppose that X is irreducible and reduced. Then for every open immersion $\mathrm{Spec}(\mathrm{R}) \to \mathrm{X}$ with $\mathrm{Spec}(\mathrm{R}) \neq \varnothing$, $\mathrm{Spec}(\mathrm{R})$ is irreducible and R is reduced. Hence R is an integral domain.

Suppose that X is integral. In particular, X is reduced. It remains to prove that X is irreducible. By our hypothesis, for every nonempty open immersion $\mathrm{Spec}(\mathrm{R}) \to \mathrm{X}$, $\mathrm{Spec}(\mathrm{R})$ is irreducible. We need to prove that any two nonempty open immersions $\mathrm{Spec}(\mathrm{R}_1), \mathrm{Spec}(\mathrm{R}_2) \to \mathrm{X}$ has a nonempty intersection. Suppose that $\mathrm{Spec}(\mathrm{R}_1) \cap \mathrm{Spec}(\mathrm{R}_2) \cong \varnothing$, then we obtain an open immersion:

$$\mathrm{Spec}(\mathrm{R}_1 \times \mathrm{R}_2) \xrightarrow{\simeq} \mathrm{Spec}(\mathrm{R}_1) \sqcup \mathrm{Spec}(\mathrm{R}_2) \to \mathrm{X}.$$

But our hypothesis implies that $\mathrm{Spec}(\mathrm{R}_1 \times \mathrm{R}_2)$ is irreducible; contradiction. □

4.2.11. Let X be an integral scheme. Then X is irreducible (*cf.* Proposition 4.2.10), so it has a unique generic point $\eta \in \mathrm{X}^{(0)}$ (*cf.* Lemma 4.1.13). Define the *function field* of X to be:

$$\mathrm{f.f.}(\mathrm{X}) := \mathcal{O}_{\mathrm{X},\eta}.$$

This is indeed a field, because $\mathcal{O}_{\mathrm{X},\eta}$ is reduced and $\dim(\mathcal{O}_{\mathrm{X},\eta}) = 0$. In particular, $\mathcal{O}_{\mathrm{X},\eta}$ is identified with the residue field $\kappa(\eta)$.

Note that given any open immersion $\mathrm{Spec}(\mathrm{R}) \to \mathrm{X}$, the map $\mathrm{Spec}(\mathcal{O}_{\mathrm{X},\eta}) \to \mathrm{X}$ factors through $\mathrm{Spec}(\mathrm{R})$ and its image in $|\mathrm{Spec}(\mathrm{R})|$ is the prime (0). Hence, f.f.(X) is identified with the field of fractions of R. We may thus view R as a subring:

$$\mathrm{R} \subset \mathrm{f.f.}(\mathrm{X}).$$

4.2.12. Next, we construct normalizations. We shall do this in a more general setting, which amounts to globalizing the construction of integral closures for all morphisms in Ring.

Let $f : \mathrm{Y} \to \mathrm{X}$ be an affine morphism of schemes. Recall that f corresponds to some quasi-coherent sheaf of $\mathcal{O}_{\mathrm{X}}$-algebras $\mathcal{B}$ (*cf.* Corollary 2.4.13).

Let us define a quasi-coherent sheaf of $\mathcal{O}_{\mathrm{X}}$-subalgebras $\mathcal{A} \subset \mathcal{B}$, *i.e.* a morphism $\mathcal{A} \to \mathcal{B}$ in $\mathsf{CAlg}(\mathsf{QCoh}(\mathrm{X}))$ which is injective on the underlying objects of $\mathsf{QCoh}(\mathrm{X})$. Indeed, for every open immersion $\mathrm{Spec}(\mathrm{R}) \to \mathrm{X}$, we define:

$$\mathrm{R} \to \Gamma(\mathrm{Spec}(\mathrm{R}), \mathcal{A}) \subset \Gamma(\mathrm{Spec}(\mathrm{R}), \mathcal{B})$$

to be the integral closure of R in $\Gamma(\mathrm{Spec}(\mathrm{R}), \mathcal{B})$. To define an object of $\mathsf{QCoh}(\mathrm{X})$, we need to show that the natural map:

$$\Gamma(\mathrm{Spec}(\mathrm{R}), \mathcal{A}) \otimes_{\mathrm{R}} \mathrm{R}_f \to \Gamma(\mathrm{Spec}(\mathrm{R}_f), \mathcal{A})$$

is an isomorphism for every $f \in \mathrm{R}$ (*cf.* Corollary 2.2.5), but this has been observed in the proof of Lemma 4.2.8.

We call the affine morphism $\mathrm{Spec}_{\mathrm{X}}(\mathcal{A}) \to \mathrm{X}$ the *relative normalization* of X in Y. The morphism $f : \mathrm{Y} \to \mathrm{X}$ factors canonically through $\mathrm{Spec}_{\mathrm{X}}(\mathcal{A})$.

4.2.13. Let X be an integral scheme. We define the *normalization* of X to be the relative normalization of X in Spec(f.f.(X)), so we have a commutative diagram:

$$\begin{array}{ccc} \mathrm{Spec}(\mathrm{f.f.}(\mathrm{X})) & \xrightarrow{\tilde{\eta}} & \widetilde{\mathrm{X}} \\ & \searrow^{\eta} & \downarrow{\scriptstyle \nu} \\ & & \mathrm{X} \end{array}$$

where η is the generic point of $|\mathrm{X}|$, thought of as a morphism from its residue field f.f.(X) to X. By construction, $\widetilde{\mathrm{X}}$ is also an integral scheme and ν induces an isomorphism on function fields $\mathrm{f.f.}(\mathrm{X}) \xrightarrow{\simeq} \mathrm{f.f.}(\widetilde{\mathrm{X}})$. By Lemma 4.2.8, $\widetilde{\mathrm{X}}$ is normal.

4.2.14. By construction, normalization is only functorial with respect to morphisms $f : \mathrm{Y} \to \mathrm{X}$ of integral schemes carrying the generic point of $|\mathrm{Y}|$ to the generic point of $|\mathrm{X}|$. (One cannot expect it to be functorial with respect to all morphisms. Consider, for example, the inclusion of the nodal point in a nodal curve, *cf.* Example 4.2.6.)

More generally, a morphism $f : \mathrm{Y} \to \mathrm{X}$ in Sch is called *dominant* if $|f| : |\mathrm{Y}| \to |\mathrm{X}|$ has dense image. For integral schemes, this coincides with the condition above.

Lemma 4.2.15. *Let $f : \mathrm{Y} \to \mathrm{X}$ be a morphism of integral schemes. Then the following are equivalent:*

1. *(1) f is dominant;*
2. *(2) $|f|$ maps the generic point of $|\mathrm{Y}|$ to the generic point of $|\mathrm{X}|$.*

Proof. Let x (respectively, y) denote the generic point of $|\mathrm{X}|$ (respectively, $|\mathrm{Y}|$). Since $\{x\}$ is dense in $|\mathrm{X}|$, we have (2) ⇒ (1). To prove (1) ⇒ (2), we assume that f is dominant. Then the image of $|\mathrm{Y}| = \overline{\{y\}}$ under $|f|$ is contained in $\overline{\{f(y)\}}$. But if $f(y) \neq x$, then $\overline{\{f(y)\}}$ would be a proper closed subset of $|\mathrm{X}|$, hence not dense. □

Remark 4.2.16. In summary, we have globalized the construction (4.5) to schemes. Namely, to each scheme X, we may attach the following sequence of morphisms:

$$\coprod_{\eta \in \mathrm{X}^{(0)}} \widetilde{\mathrm{X}}_\eta \to \coprod_{\eta \in \mathrm{X}^{(0)}} \mathrm{X}_\eta \to \mathrm{X}_{\mathrm{red}} \to \mathrm{X}. \tag{4.8}$$

Here, X_i is the induced reduced subscheme of X (hence of $\mathrm{X}_{\mathrm{red}}$) defined by the closed subset $\overline{\{\eta\}} \subset |\mathrm{X}|$. Thus, each X_η is integral and each $\widetilde{\mathrm{X}}_\eta$ is integral and normal. Note that by construction, the morphisms in (4.8) induce bijections on generic points.

Furthermore, when X is locally Noetherian and $\dim(\mathrm{X}) = 1$, each of the schemes $\widetilde{\mathrm{X}}_\eta$ is regular by Proposition 4.2.4.

Remark 4.2.17. Let X be a locally Noetherian scheme. Then maps from spectra of discrete valuation rings to X control all specializations in $|\mathrm{X}|$ in the following way.

Claim: For any specialization $y \rightsquigarrow x$ in $|\mathrm{X}|$, there is a discrete valuation ring R equipped with a morphism of schemes:

$$\mathrm{Spec}(\mathrm{R}) \to \mathrm{X}, \tag{4.9}$$

sending the generic point to y and the special point to x.

Indeed, let us equip $\mathrm{Y} := \overline{\{y\}}$ with the induced reduced subscheme structure and view x as a point of $|\mathrm{Y}|$. Then the local ring $\mathrm{A} := \mathcal{O}_{\mathrm{Y},x}$ is a Noetherian local domain whose maximal ideal corresponds to x. It suffices to find a discrete valuation ring R with a morphism:

$$\mathrm{Spec}\,\mathrm{R} \to \mathrm{Spec}\,\mathrm{A} \tag{4.10}$$

which induces an isomorphism of fraction fields and sends the closed point to x.

To construct the morphism (4.10), we consider the blow-up $\mathrm{Bl}_x \mathrm{Spec}\,\mathrm{A}$ of $\mathrm{Spec}\,\mathrm{A}$ along the closed point x (*cf.* Proposition 3.4.7). Let ξ be the generic point of any irreducible component of the exceptional divisor E in $\mathrm{Bl}_x \mathrm{Spec}\,\mathrm{A}$. Then the local ring B of $\mathrm{Bl}_x \mathrm{Spec}\,\mathrm{A}$ at ξ is a Noetherian local ring of Krull dimension 1 (*cf.* Example 4.1.17). Taking R to be the normalization of B suffices.

4.2.18. *Valuation rings.* To control specializations on an arbitrary scheme, we need a non-Noetherian version of discrete valuation rings.

Let A, B be local domains contained in a field K. We say that B *dominates* A if $\mathrm{A} \subset \mathrm{B}$ and $\mathfrak{m}_{\mathrm{B}} \cap \mathrm{A} = \mathfrak{m}_{\mathrm{A}}$. Here, $\mathfrak{m}_{\mathrm{A}}$ (respectively $\mathfrak{m}_{\mathrm{B}}$) is the maximal ideal of A (respectively B). (Note that (4.10) is an example of a domination relation.)

A local domain A is a *valuation ring* if it is maximal among local domains contained in its fraction field f.f.(A).

Given a valuation ring A, the *special point* of $|\mathrm{Spec}(A)|$ is the maximal ideal $\mathfrak{m}_A$. Clearly, it is a specialization of the generic point, corresponding to the prime ideal (0).

Remark 4.2.19. By definition, a field is a valuation ring. Furthermore, any discrete valuation ring is a valuation ring. In fact, a valuation ring is Noetherian if and only if it is a field or a discrete valuation ring [Sta18, 00II].

Lemma 4.2.20. *Let* A *be a local domain contained in a field* K*. Then there exists a valuation ring* $B \subset K$ *with fraction field* K *dominating* A*.*

Proof. We shall apply Zorn's lemma to the set of local domains in K which dominate A, ordered by the dominance relation. Indeed, given any chain A_i $(i \in I)$ of local domains in K dominating A, their union $\bigcup_{i\in I} A_i$ is also a local domain in K dominating A. Thus this set has a maximal element $B \subset K$. To fininsh the proof, we need to prove that $\mathrm{f.f.}(B) = K$.

Suppose that $\mathrm{f.f.}(B) \subsetneq K$. Take $t \in K \smallsetminus \mathrm{f.f.}(B)$. If t is transcendental over B, then $B[t]_{(t,\mathfrak{m}_B)} \subset K$ is a local domain dominating B. (Here, $\mathfrak{m}_B$ stands for the maximal ideal of B.) If t is algebraic over B, then for some $f \in B$, $ft \in K$ is integral over B, so the subring B' of K generated by B and ft is finite over B. The finite extension $B \subset B'$ then defines a surjective morphism $\mathrm{Spec}(B') \to \mathrm{Spec}(B)$ (Nakayama), so $\mathfrak{m}_B$ lifts to a prime ideal $\mathfrak{p}$ of B' and $(B')_{\mathfrak{p}}$ dominates B. □

4.2.21. Let X be a scheme and $y \rightsquigarrow x$ be a specialization in $|X|$. Let us construct a morphism for some valuation ring R:

$$f : \mathrm{Spec}(R) \to X, \tag{4.11}$$

such that $|f|$ carries the generic point to y and the special point to x.

Indeed, we equip $Y := \overline{\{y\}}$ with the induced reduced subscheme structure and apply Lemma 4.2.20 to the local domain $A := \mathcal{O}_{Y,x}$ contained in its fraction field $K := \kappa(y)$. By construction, the fraction field of R is identified with $\kappa(y)$.

4.3. Chevalley's theorem.

4.3.1. Given a morphism of schemes $f : Y \to X$, what does the image of $|f|$ look like? In general, it can be arbitrary: for any subset T of $|X|$, taking Y to be the disjoint union of points in T yields a morphism $f : Y \to X$ such that $|f|$ has image T.

After one imposes some finiteness conditions on f, however, the answer becomes much more interesting. In this subsection, we will prove Chevalley's theorem (*cf.* Theorem 4.3.8), which characterizes the images of morphisms of finite presentation.

4.3.2. Let T be a topological space. We say that T is *spectral* if:

1. T is sober (*cf.* §4.1.12);
2. T is quasi-compact;
3. the intersection of two quasi-compact open subsets of T is quasi-compact;
4. T admits a basis consisting of quasi-compact open subsets.

Lemma 4.3.3. *Let* X *be a quasi-compact, quasi-separated scheme. Then* $|X|$ *is spectral.*

Proof. Sobriety (1) follows from Lemma 4.1.13. Quasi-compactness (2) follows from Corollary 4.1.9. The intersection property (3) is equivalent to quasi-separatedness. To find a basis of quasi-compact open subsets (4), we may take the collection of open subsets $|U| \subset |X|$ with U an affine scheme. □

Remark 4.3.4. Hochster [Hoc69, Theorem 6] proved that every spectral topological space is of the form $|\mathrm{Spec}(\mathrm{A})|$ for some $\mathrm{A} \in \mathsf{Ring}$.

4.3.5. Let T be a spectral topological space.

A subset Z of T is called *constructible* if it belongs to the Boolean algebra generated by quasi-compact open subsets, *i.e.* the smallest subset of the power set of T closed under binary intersection, binary union, and complement.[20]

Note that Z is constructible if and only if it is of the form $\bigcup_{i\in \mathrm{I}}(\mathrm{U}_i \smallsetminus \mathrm{V}_i)$ ($i \in \mathrm{I}$ finite) where $\mathrm{U}_i, \mathrm{V}_i \subset \mathrm{T}$ are quasi-compact open subsets. (The key step is showing that sets of this form are closed under binary intersection, and this makes use of axiom (3).)

Here is the nice thing about constructible sets: Checking their closedness (respectively, openness) amounts to checking their closure under specializations (respectively, generalizations).

Lemma 4.3.6. *Let* X *be a quasi-compact, quasi-separated scheme. Let* $\mathrm{Z} \subset |\mathrm{X}|$ *be a constructible subset. Then:*

(1) Z *is closed if and only if it is closed under specialization;*
(2) Z *is open if and only if it is closed under generalization.*

Proof. [21] (2) follows from (1) by taking complements, so we will only prove (1). The "⇒" direction is clear, so we prove the "⇐" direction. Since closedness can be checked on an open cover, we may assume that $\mathrm{X} = \mathrm{Spec}(\mathrm{A})$ is affine.

Write $\mathrm{Z} = \bigcup_{i\in\mathrm{I}}(\mathrm{U}_i \smallsetminus \mathrm{V}_i)$ ($i \in \mathrm{I}$ finite). *Claim*: Z is the image of an affine scheme. Indeed, by taking a disjoint union, it suffices to show that $\mathrm{U} \smallsetminus \mathrm{V}$ is the image of an affine scheme, for any quasi-compact opens $\mathrm{U}, \mathrm{V} \subset |\mathrm{Spec}(\mathrm{A})|$. Writing U as a finite union of images of standard opens $\mathrm{Spec}(\mathrm{A}_j) \to \mathrm{Spec}(\mathrm{A})$ ($j \in \mathrm{J}$), we see that each closed subset $|\mathrm{Spec}(\mathrm{A}_j)| \smallsetminus \mathrm{V}$ is the image of an affine scheme, so the same holds for $\mathrm{U} \smallsetminus \mathrm{V}$.

It remains to prove that given a morphism of affine schemes $f : \mathrm{Spec}(\mathrm{B}) \to \mathrm{Spec}(\mathrm{A})$, the image of $|f|$ is closed if it is closed under specialization. Take a point $\mathfrak{p} \in |\mathrm{Spec}(\mathrm{A})|$ off the image of $|f|$. Then the image of $|\mathrm{Spec}(\mathrm{A}_\mathfrak{p})| \to |\mathrm{Spec}(\mathrm{A})|$ is also off the image of $|f|$. Hence:

$$\operatorname*{colim}_{a\notin\mathfrak{p}} \mathrm{B} \otimes_\mathrm{A} \mathrm{A}_a \xrightarrow{\simeq} \mathrm{B} \otimes_\mathrm{A} \mathrm{A}_\mathfrak{p} \xrightarrow{\simeq} 0.$$

This is a filtered colimit, so $1 \in \mathrm{B}$ vanishes in $\mathrm{B} \otimes_\mathrm{A} \mathrm{A}_a$ for some $a \notin \mathfrak{p}$, showing that $|\mathrm{Spec}(\mathrm{A}_a)|$ is an open neighborhood of $\mathfrak{p}$ off the image of $|f|$. □

Remark 4.3.7. The proof of Lemma 4.3.6 establishes the following fact: If $f : \mathrm{Y} \to \mathrm{X}$ is a quasi-compact morphism of schemes, then the image of $|f|$ is closed if it is closed under specialization.

Indeed, by taking an affine cover of X, we reduce to the case where $\mathrm{X} = \mathrm{Spec}(\mathrm{A})$ is affine. Then by taking a *finite* affine cover $\mathrm{Y}_j \to \mathrm{Y}$ ($j \in \mathrm{J}$) and replacing Y by the disjoint union $\bigsqcup_{j\in\mathrm{J}} \mathrm{Y}_j$, we reduce to the case where $\mathrm{Y} = \mathrm{Spec}(\mathrm{B})$ is affine. Then the assertion established in the proof of Lemma 4.3.6.

Theorem 4.3.8 (Chevalley)**.** *Let* $f : \mathrm{Y} \to \mathrm{X}$ *be a morphism in* Sch*. Suppose that:*

[20] We only define "constructible subsets" in a spectral topological space. For purposes of scheme theory, not much generality is lost: For any scheme X, a subset $\mathrm{Z} \subset |\mathrm{X}|$ is "locally constructible" (as defined in [Sta18, 005G] via retro-compactness) if and only if $\mathrm{Z} \cap |\mathrm{U}|$ is constructible for any open affine subscheme $\mathrm{U} \to \mathrm{X}$ [Sta18, 054C]. Of course, $|\mathrm{U}|$ is spectral.

[21] The proof given here is ring-theoretic. See [Sta18, 0903] for a topological proof which applies directly to any spectral topological space (using compactness of the "constructible topology"). By Remark 4.3.4, these concern the same class of topological spaces.

(1) f is of finite presentation;
(2) X is quasi-compact and quasi-separated (so $|X|$ and $|Y|$ are both spectral).

Then $|f| : |Y| \to |X|$ preserves constructible subsets.

Remark 4.3.9. Let X be a quasi-compact, quasi-separated scheme. In the proof of Lemma 4.3.6, we have expressed any constructible subset of $|X|$ as the image of $|f|$, for some morphism $f : Y \to X$ of finite presentation. Hence, Theorem 4.3.8 characterizes precisely the subsets of $|X|$ which can arise as the image of such morphisms.

4.3.10. Let us reduce Theorem 4.3.8 to two (very) special cases. Indeed, let $Z \subset |Y|$ be a constructible subset. To check that $|f|(Z) \subset |X|$ is constructible, we may do so for its intersection with $|\mathrm{Spec}(A)|$ for any open immersion $\mathrm{Spec}(A) \subset X$. Thus we may assume that $X = \mathrm{Spec}(A)$ is affine. Moreover, by covering Y with open affine subschemes, we also reduce to the case where $Y = \mathrm{Spec}(B)$ is affine.

Now, the morphism f factors as $\mathrm{Spec}(B) \xrightarrow{i} \mathbb{A}^n_A \xrightarrow{p} \mathrm{Spec}(A)$, where i is a finitely presented closed immersion, *i.e.* defined by a finitely generated ideal. Thus, by factoring i and p, we reduce to the following cases:

(1) the closed immersion $i : \mathrm{Spec}(A/a) \to \mathrm{Spec}(A)$ for some $a \in A$;
(2) the projection $p : \mathbb{A}^1_A \to \mathrm{Spec}(A)$.

In other words, Theorem 4.3.8 follows from Lemma 4.3.11 and Lemma 4.3.12 below.

Lemma 4.3.11. *Let A be a ring with $f \in A$. Then the closed embedding $i : |\mathrm{Spec}(A/f)| \to |\mathrm{Spec}(A)|$ preserves constructible subsets.*

Proof. Let U, V be quasi-compact open subsets of $|\mathrm{Spec}(A/f)|$. We need to show that $U \setminus V$ is a constructible subset of $|\mathrm{Spec}(A)|$.

Observe that U (respectively, V) is the intersection of a quasi-compact open subset $\widetilde{U} \subset |\mathrm{Spec}(A)|$ (respectively, $\widetilde{V} \subset |\mathrm{Spec}(A)|$) with $|\mathrm{Spec}(A/f)|$. (The complement of U is defined by a finitely generated ideal in A/f; lift the generators to A and use them to define the complement of $\widetilde{U}$.) Then $U \setminus V = (\widetilde{U} \setminus \widetilde{V}) \cap |\mathrm{Spec}(A/f)|$ is constructible. □

Lemma 4.3.12. *Let A be a ring. Then the projection map $p : |\mathbb{A}^1_A| \to |\mathrm{Spec}(A)|$ preserves constructible subsets.*

Proof. Let U, V be quasi-compact open subsets of $|\mathbb{A}^1_A|$. We need to show that the image of $p(U \setminus V)$ is constructible. Writing U as a finite union of standard opens in $\mathbb{A}^1_A$, we reduce to the case where $U = D(f)$, the complement of the closed subscheme of $\mathbb{A}^1_A$ defined by a single element $f \in A[T]$. Let us write V as complement to the closed subscheme of $\mathbb{A}^1_A$ defined by $g_1, \cdots, g_n \in A[T]$ ($n \in \mathbf{Z}_{\ge 0}$), so we have:

$$U \setminus V = D(f) \cap |\mathrm{Spec}(A[x]/(g_1, \cdots, g_n))|.$$

We shall arrange $g_1, \cdots, g_n$ so that:

$$\deg(g_1) \le \cdots \le \deg(g_n).$$

Let us reduce the result for $g_1, \cdots, g_n$ to one for $g'_1, \cdots, g'_m \in A[T]$ ($m \in \mathbf{Z}_{\ge 0}$) where either $m < n$, or $m = n$ and all $\deg(g'_i) \le \deg(g_i)$ where the inequality is strict for some i.

Indeed, we let $a \in A$ be the leading coefficient of g_1. The constructibility of $p(U \setminus V)$ can be verified after pulling back along $|\mathrm{Spec}(A/a)| \sqcup |\mathrm{Spec}(A_a)| \to |\mathrm{Spec}(A)|$. The base change to $\mathrm{Spec}(A/a)$ makes g_1 vanish, so we reduce the number n. The base change to $\mathrm{Spec}(A_a)$ makes a invertible, so we may replace g_2 by:

$$g'_2 := g_2 - \frac{a_2}{a_1} \cdot T^{\deg(g_2) - \deg(g_1)} \cdot g_1.$$

to reduce the degree of g_2 without affecting the ideal $(g_1, \cdots, g_n)$.

This process terminates in two ways: either $n = 0$, or $n = 1$ and the leading coefficient of $g := g_1$ is invertible. Let us now treat these two cases.

Suppose that $n = 0$. We want to show that $p(\mathrm{D}(f))$ is constructible. Let us write $f = r_m \mathrm{T}^m + \cdots + r_0$ with $r_i \in \mathrm{A}$ $(0 \le i \le m)$. *Claim*:

$$p(\mathrm{D}(f)) = \bigcup_{1 \le i \le m} \mathrm{D}(r_i).$$

Indeed, a point $x \in |\mathrm{Spec}(\mathrm{A})|$ falls outside $p(\mathrm{D}(f))$ if and only if the base change below is empty:

$$\begin{array}{ccccc} \mathbb{A}^1_{\kappa(x)} \cap \mathrm{D}(f) & \longrightarrow & \mathrm{D}(f) & \subset & \mathbb{A}^1_{\mathrm{A}} \\ \downarrow & & \downarrow & & \\ \mathrm{Spec}(\kappa(x)) & \longrightarrow & \mathrm{Spec}(\mathrm{A}) & & \end{array}$$

(We use the same notation $\mathrm{D}(f)$ for the open subscheme of $\mathbb{A}^1_{\mathrm{A}}$ defined by $\mathrm{D}(f) \subset |\mathbb{A}^1_{\mathrm{A}}|$.) This happens if and only if f is nilpotent in $\kappa(x)[\mathrm{T}]$, *i.e.* all r_i vanish in $\kappa(x)[\mathrm{T}]$.

Suppose that $n = 1$ and the leading coefficient of $g \in \mathrm{A}[\mathrm{T}]$ is invertible. Write $\mathrm{Z} := \mathrm{Spec}(\mathrm{A}[\mathrm{T}]/g)$ for the subscheme of $\mathbb{A}^1_{\mathrm{A}}$ defined by g. We want to show that $p(\mathrm{D}(f) \cap |\mathrm{Z}|)$ is constructible. In this case, $\mathrm{A}[\mathrm{T}]/g$ is a finite free A-module. Multiplication by f defines an A-linear endomorphism of $\mathrm{A}[\mathrm{T}]/g$, so it has a characteristic polynomial $\mathrm{P}(f) = \mathrm{T}^m + r_{m-1}\mathrm{T}^{m-1} + \cdots + r_0$ with $r_i \in \mathrm{A}$ $(0 \le i \le m-1)$. *Claim:*

$$p(\mathrm{D}(f) \cap |\mathrm{Z}|) = \bigcup_{1 \le i \le m-1} \mathrm{D}(r_i).$$

Indeed, a point $x \in |\mathrm{Spec}(\mathrm{A})|$ falls outside the left-hand-side if and only if the base change below is empty:

$$\begin{array}{ccccc} \mathrm{Z}_{\kappa(x)} \cap \mathrm{D}(f) & \to & \mathrm{Z} \cap \mathrm{D}(f) & \subset & \mathrm{Z} \\ \downarrow & & \downarrow & & \\ \mathrm{Spec}(\kappa(x)) & \longrightarrow & \mathrm{Spec}(\mathrm{A}) & & \end{array}$$

where $\mathrm{Z}_{\kappa(x)} := \mathrm{Z} \times_{\mathrm{Spec}(\mathrm{A})} \mathrm{Spec}(\kappa(x))$. This happens if and only if f is nilpotent in $\kappa(x)[\mathrm{T}]/g$, if and only if $\mathrm{P}(f) = \mathrm{T}^m$ in $\kappa(x)[\mathrm{T}]$, *i.e.* all r_i vanish in $\kappa(x)[\mathrm{T}]$. □

Corollary 4.3.13 (Zariski's lemma)**.** *Let* $\mathrm{K} \subset \mathrm{L}$ *be a field extension such that* L *is a finite type* K*-algebra. Then* L *is a finite extension of* K*.*

Proof. We begin with an observation: The subset $\{\eta\} \subset |\mathbb{A}^1_{\mathrm{K}}|$, where η is the generic point, is *not* constructible. Indeed, this is because any open subset of $|\mathbb{A}^1_{\mathrm{K}}|$ is the complement of finitely many closed points, but $|\mathbb{A}^1_{\mathrm{K}}|$ has infinitely many closed points.

Let $\mathrm{K} \subset \mathrm{L}$ be a field extension with a surjective ring map $\varphi : \mathrm{K}[x_1, \cdots, x_n] \to \mathrm{L}$. It suffices to prove that each $\varphi(x_i) \in \mathrm{L}$ $(1 \le i \le n)$ is algebraic over K. Applying Chevalley's theorem (*cf.* Theorem 4.3.8) to the projection map $\mathbb{A}^n_{\mathrm{K}} \to \mathbb{A}^1_{\mathrm{K}}$ onto the ith factor, we see that the intersection of $\ker(\varphi) \cap \mathrm{K}[x_i]$ is a maximal ideal of $\mathrm{K}[x_i]$, so it contains a nonzero element $f \in \mathrm{K}[x_i]$. This shows that $\varphi(x_i)$ is a zero of the polynomial f. □

Corollary 4.3.14 (Hilbert's Nullstellensatz)**.** *Let k be an algebraically closed field and* A *be a finite type k-algebra. If* $\mathrm{A} \ne 0$*, then* Spec A *has a k-point.*

Proof. By Corollary 4.3.13, every closed point of $|\mathrm{Spec}\,\mathrm{A}|$ has residue field k. □

Corollary 4.3.15. *Let $f : \mathrm{Y} \to \mathrm{X}$ be a morphism of schemes. Suppose that f is flat and locally of finite presentation. Then $|f|$ sends open subsets of $|\mathrm{Y}|$ to open subsets of $|\mathrm{X}|$.*

Proof. Since open immersions are flat and locally of finite presentation, it suffices to prove that the image of $|f|$ is open. This allows us to replace X by an affine scheme Spec A. Since image commutes with union, we may also replace Y be an affine scheme Spec B.

We now appeal to the fact that a flat ring map $\mathrm{A} \to \mathrm{B}$ satisfies the going-down property (*cf.* [Sta18, 00HS]), *i.e.* the image of $|\operatorname{Spec} \mathrm{B}| \to |\operatorname{Spec} \mathrm{A}|$ is closed under generalization. By Chevalley's Theorem (*cf.* Theorem 4.3.8) and Lemma 4.3.6, it must be open. ☐

4.4. Properness.

4.4.1. In this subsection, we introduce the notion of "properness" for a morphism of schemes. It is the algebro-geometric analogue of a compact manifold.

We shall prove that "properness" can be detected by morphisms out of valuation rings, *cf.* §4.2.18. Then we use this criterion to prove that the projective space is "proper".

4.4.2. Let $f : \mathrm{Y} \to \mathrm{X}$ be a morphism of schemes.

We say that f is *closed* if the induced map on topological spaces $|f| : |\mathrm{Y}| \to |\mathrm{X}|$ is closed, *i.e.* the image of a closed subset is closed. We say that f is *universally closed* if for any morphism of schemes $\mathrm{X}' \to \mathrm{X}$, its base change $f' : \mathrm{Y}' := \mathrm{Y} \times_{\mathrm{X}} \mathrm{X}' \to \mathrm{X}'$ is closed.

We say that f is *proper* if it is of finite type, separated, and universally closed. In particular, proper morphisms are quasi-compact and quasi-separated. Clearly, a composition of proper morphisms is still proper.

We also have the following implications:

$$\text{closed immersions} \;\Rightarrow\; \text{proper} \;\Rightarrow\; \text{separated.}$$

Remark 4.4.3. The property of being proper is stable under base change and Zariski local on the target. Indeed, these statements hold for "being locally of finite type", quasi-compactness and quasi-separatedness, and universal closedness.

4.4.4. Given a morphism $f : \mathrm{Y} \to \mathrm{X}$ of schemes, we may consider commutative diagrams (depicted with the solid arrows):

$$\begin{array}{ccc} \operatorname{Spec}(\mathrm{K}) & \longrightarrow & \mathrm{Y} \\ \downarrow & \nearrow & \downarrow f \\ \operatorname{Spec}(\mathrm{R}) & \longrightarrow & \mathrm{X} \end{array} \tag{4.12}$$

where R is a valuation ring with fraction field K, and the left vertical morphism in (4.12) is induced from the inclusion $\mathrm{R} \subset \mathrm{K}$.

The morphism f is said to satisfy the *existence part* (respectively, *uniqueness part*) *of the valuative criterion* if for every commutative diagram (4.12), there exists (respectively, exists at most one) a morphism $\operatorname{Spec}(\mathrm{R}) \to \mathrm{Y}$ rendering both triangles commute. (The morphism $\operatorname{Spec}(\mathrm{R}) \to \mathrm{Y}$ is depicted by the dotted arrow in (4.12).)

Proposition 4.4.5 (Valuative criterion of universal closedness)**.** *Let $f : \mathrm{Y} \to \mathrm{X}$ be a quasi-compact morphism of schemes. Then the following are equivalent:*

(1) f is universally closed;
(2) f satisfies the existence part of the valuative criterion.

Proof. (1) ⇒ (2). Suppose that f is universally closed. To construct a lift in a commutative diagram (4.12), we may replace X by Spec(R) upon taking a base change. Let $y \in |\mathrm{Y}|$ be the image of $|\mathrm{Spec(K)}|$. Since $|f|$ carries $\overline{\{y\}}$ to a closed subset of $|\mathrm{Spec(R)}|$, there exists some $x \in \overline{\{y\}}$ whose image is the special point of $|\mathrm{Spec(R)}|$. Thus the inclusion $\mathrm{R} \subset \mathrm{K}$ factors as $\mathrm{R} \subset \mathcal{O}_{\mathrm{Y},x} \subset \mathrm{K}$, where $\mathcal{O}_{\mathrm{Y},x}$ dominates R. This implies that $\mathrm{R} = \mathcal{O}_{\mathrm{Y},x}$ as subrings of K.

(2) ⇒ (1). Since the existence part of the valuative criterion is stable under base change, it suffices to prove that (2) implies that f is closed. By equipping a closed subset of $|\mathrm{Y}|$ with the induced reduced subscheme structure, we reduce to proving that the image of $|f|$ is closed. By Remark 4.3.7, it suffices to prove that the image of $|f|$ is stable under specialization. Namely, given any specialization $x' \rightsquigarrow x$ in $|\mathrm{X}|$ and $y' \in |\mathrm{Y}|$ mapping to x', we need to find $y \in |\mathrm{Y}|$ mapping to x.

Equip $\mathrm{X}' := \overline{\{x'\}}$ with the induced reduced closed subscheme structure and consider the local domain $\mathcal{O}_{\mathrm{X}',x}$ with fraction field $\kappa(x')$. The fact that y' maps to x' gives rise to an inclusion $\mathcal{O}_{\mathrm{X}',x} \subset \kappa(x') \subset \kappa(y')$. Apply Lemma 4.2.20 to $\mathrm{A} := \mathcal{O}_{\mathrm{X}',x}$ and $\mathrm{K} := \kappa(y')$, we find a valuation ring R with fraction field K dominating $\mathcal{O}_{\mathrm{X}',x}$. Applying the valuative criterion to the diagram:

$$\begin{array}{ccccc} \mathrm{Spec(K)} & \xrightarrow{\simeq} & \mathrm{Spec}(\kappa(y')) & \to & \mathrm{Y} \\ \downarrow & & & \nearrow & \downarrow \\ \mathrm{Spec(R)} & \to & \mathrm{Spec}(\mathcal{O}_{\mathrm{X}',x}) & \to & \mathrm{X} \end{array}$$

we find $y \in |\mathrm{Y}|$ as the image of the closed point of $|\mathrm{Spec(R)}|$ under the dotted arrow. □

4.4.6. Recall that a morphism $f : \mathrm{Y} \to \mathrm{X}$ in Sch is a locally closed immersion if it can be factored as $f = j \cdot i$, where i is a closed immersion and j is an open immersion, *cf.* §2.5.1. Note that given a locally closed immersion f, the following are equivalent:

(1) f is a closed immersion;
(2) f is universally closed;
(3) f is closed.

The implications (1) ⇒ (2) ⇒ (3) are clear. To prove (3) ⇒ (1), we factor f as $\mathrm{Y} \xrightarrow{i} \mathrm{U} \xrightarrow{j} \mathrm{X}$ and setting $\mathrm{V} := |\mathrm{X}| \smallsetminus |\mathrm{Y}|$ equipped with the open subscheme structure, we find that U, V form an open cover of X over which f restricts to a closed immersion. Since being a closed immersion is a Zariski local property, this implies that f is a closed immersion.

Lemma 4.4.7. *Let $f : \mathrm{Y} \to \mathrm{X}$ be a morphism of schemes. Then its diagonal $\Delta_f : \mathrm{Y} \to \mathrm{Y} \times_{\mathrm{X}} \mathrm{Y}$ is a locally closed immersion.*

Proof. Take open affine covers $\mathrm{Spec}(\mathrm{B}_i) \to \mathrm{Y}$ ($i \in \mathrm{I}$), $\mathrm{Spec}(\mathrm{A}_i) \to \mathrm{X}$ ($i \in \mathrm{I}$) such that f restricts to morphisms $f_i : \mathrm{Spec}(\mathrm{B}_i) \to \mathrm{Spec}(\mathrm{A}_i)$ for each $i \in \mathrm{I}$.

Then each $\mathrm{Spec}(\mathrm{B}_i \otimes_{\mathrm{A}_i} \mathrm{B}_i) \to \mathrm{Y} \times_{\mathrm{X}} \mathrm{Y}$ is an open immersion, and Δ_f factors through their union U, viewed as an open subscheme of $\mathrm{Y} \times_{\mathrm{X}} \mathrm{Y}$. Furthermore, over each $\mathrm{Spec}(\mathrm{B}_i \otimes_{\mathrm{A}_i} \mathrm{B}_i) \subset \mathrm{U}$, Δ_f is given by the spectrum of the multiplication map $\mathrm{B}_i \otimes_{\mathrm{A}_i} \mathrm{B}_i \to \mathrm{B}_i$, which is surjective. Thus, the factorization:

$$\Delta_f : \mathrm{Y} \to \mathrm{U} \to \mathrm{Y} \times_{\mathrm{X}} \mathrm{Y}$$

exhibits Δ_f as a locally closed immersion. □

Proposition 4.4.8 (Valuative criterion of separatedness)**.** *Let $f : \mathrm{Y} \to \mathrm{X}$ be a quasi-separated morphism of schemes. Then the following are equivalent:*

(1) f is separated;
(2) f satisfies the uniqueness part of the valuative criterion.

Proof. The hypothesis means that $\Delta_f : \mathrm{Y} \to \mathrm{Y} \times_{\mathrm{X}} \mathrm{Y}$ is quasi-compact. On the other hand, it is a locally closed immersion (*cf.* Lemma 4.4.7), so the property of being a closed immersion is equivalent to universal closedness. According to Proposition 4.4.5, the following are then equivalent:

(1) Δ_f is a closed immersion (*i.e.* f is separated);
(2) Δ_f satisfies the existence part of the valuative criterion.

Finally, note that the existence part of the valuative criterion for Δ_f is equivalent to the uniqueness part of the valuative criterion for f. □

Proposition 4.4.9 (Valuative criterion of properness)**.** *Let $f : \mathrm{Y} \to \mathrm{X}$ be a morphism in* Sch*, assumed quasi-separated and of finite type. Then the following are equivalent:*

(1) f is proper;
(2) f satisfies both the existence and uniqueness parts of the valuative criterion.

Proof. Since f is of finite type, it is in particular quasi-compact. Combining Proposition 4.4.5 and Proposition 4.4.8, we see that the following are equivalent for f:

(1) f is universally closed and separated;
(2) f satisfies both the existence and uniqueness parts of the valuative criterion.

With the given conditions on f, (1) is equivalent to the properness of f. □

4.4.10. As an immediate consequence of the valuative criteria, we prove the following permanence property of separated and proper morphisms.

Lemma 4.4.11. *Given a diagram in* Sch*:*

$$\begin{array}{ccc} \mathrm{Y} & \xrightarrow{f'} & \mathrm{X}' \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle g} \\ & & \mathrm{X} \end{array}$$

the following statements hold:

(1) if f is separated, then so is f';
(2) if f is proper and g is separated, then f' is proper.

Proof. For (1), we know from Lemma 1.7.10(1) that f' is quasi-separated. To prove that f' is separated, it suffices to prove that f' satisfies the uniqueness part of the valuative criterion (*cf.* Proposition 4.4.8), but this follows from the same property of f.

For (2), we know from (1) that f' is separated. We also know from Lemma 1.7.10 that f' is quasi-compact. Furthermore, by Lemma 1.8.13, f' is also of finite type. It thus remains to prove that f' satisfies the existence part of the valuative criterion (*cf.* Proposition 4.4.9), but this follows from the valuative criteria applied to f and g. □

Example 4.4.12. The valulative criterion of properness implies the existence and uniqueness of "limit points" in the following sense. Given a proper morphism $f : \mathrm{Y} \to \mathrm{X}$ and a regular Noetherian scheme S of Krull dimension 1 with a closed point $s \in |\mathrm{S}|$, a lift in the following diagram exists uniquely:

$$\begin{array}{ccc} \mathrm{S} \smallsetminus \{s\} & \to & \mathrm{Y} \\ \downarrow & \nearrow & \downarrow{\scriptstyle f} \\ \mathrm{S} & \longrightarrow & \mathrm{X} \end{array} \tag{4.13}$$

Indeed, the local ring $\mathcal{O}_{S,s}$ is a discrete valuation ring, so the morphism $\operatorname{Spec}\mathcal{O}_{S,s} \smallsetminus \{s\} \to Y$ induced from (4.13) extends uniquely to a morphism $\operatorname{Spec}\mathcal{O}_{S,s} \to Y$ of schemes over X by the valuative criterion (*cf.* Proposition 4.4.9). Since f is locally of finite presentation, the morphism $\operatorname{Spec}\mathcal{O}_{S,s} \to Y$ extends to some open affine subscheme $\operatorname{Spec} R$ of S containing s (*cf.* Proposition 1.8.16). The situation is summarized in the diagram below:

$$\begin{array}{ccccccc} \operatorname{Spec}\mathcal{O}_{S,s} \smallsetminus \{s\} & \to & \operatorname{Spec} R \smallsetminus \{s\} & \to & S \smallsetminus \{s\} & \to & Y \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \operatorname{Spec}\mathcal{O}_{S,s} & \longrightarrow & \operatorname{Spec} R & \longrightarrow & S & \longrightarrow & X \end{array}$$

Replacing $\operatorname{Spec} R$ by a smaller open affine subscheme, we may assume that the extension $\operatorname{Spec} R \to Y$ coincides with the given morphism $S \smallsetminus \{s\} \to Y$ over $\operatorname{Spec} R \smallsetminus \{s\}$. Thus the two glue to the desired morphism $S \to Y$ in (4.13).

4.4.13. In the remainder of this subsection, we study the relation between properness and closed immersions into projective spaces.

Theorem 4.4.14. *For each $n \geq 0$, the morphism $\mathbb{P}^n_{\mathbf{Z}} \to \operatorname{Spec}\mathbf{Z}$ is proper.*

Proof. The fact that $\mathbb{P}^n_{\mathbf{Z}} \to \operatorname{Spec}\mathbf{Z}$ is separated can be checked directly using the standard open cover by affine space (*cf.* Proposition 3.1.4): For any $0 \leq i, j \leq n$, the intersection $U_i \cap U_j$ is affine and the morphism $U_i \cap U_j \to U_i \times U_j$ is a closed immersion.

It remains to prove that $\mathbb{P}^n_{\mathbf{Z}} \to \operatorname{Spec}\mathbf{Z}$ is universally closed. By Proposition 4.4.5, it suffices to observe that given a valuation ring R with fraction field K, any 1-dimensional K-subspace of $K^{\oplus n+1}$ extends to a line subbundle of $R^{\oplus n+1}$ over $\operatorname{Spec} R$. By applying an automorphism of $K^{\oplus n+1}$, we may assume that the 1-dimensional K-subspace is Ke_1, where e_1 is the first basis vector. This subspace extends to Re_1. □

4.4.15. Let $f : X \to S$ be a morphism of schemes.

If f factors as $X \xrightarrow{i} \mathbb{P}^n_S \xrightarrow{p} S$ ($n \in \mathbf{Z}_{\geq 0}$) where i is a closed immersion and p denotes the base change of $\mathbb{P}^n_{\mathbf{Z}} \to \operatorname{Spec}(\mathbf{Z})$ to S, then f is proper.

Conversely, if f is proper and S is Noetherian, then we can dominate X by a closed subscheme of a projective space over S, as the following theorem shows (in conjunction with Lemma 4.4.11).

Theorem 4.4.16 (Chow's lemma)**.** *Let* S *be a Noetherian affine scheme and* $f : X \to S$ *be a separated morphism of finite type. Then there exists a commutative diagram in* Sch*:*

$$\begin{array}{ccc} Y & \xrightarrow{\iota} & \mathbb{P}^n_S \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{f} & S \end{array}$$

satisfying the following properties:

(1) p is the base change of $\mathbb{P}^n_{\mathbf{Z}} \to \operatorname{Spec}(\mathbf{Z})$ to S *for some $n \in \mathbf{Z}_{\geq 0}$;*
(2) ι is a locally closed immersion;
(3) π is proper and there exists a dense open subscheme $U \subset X$ such that the induced map $\pi_U : Y \times_X U \to U$ is an isomorphism.

4.4.17. In the proof of Theorem 4.4.16, we shall repeatedly use the fact that given a quasi-compact locally closed immersion $f : Y \to X$ with scheme theoretic image $\overline{Y}$, the morphism $Y \to \overline{Y}$ is an open immersion (*cf.* Lemma 2.5.7) and $|\overline{Y}|$ is the closure of $|Y| \subset |X|$. Indeed, the proof of Lemma 4.3.6 shows that any point of $|\overline{Y}|$ is a specialization of a point of $|Y|$.

Proof of Theorem 4.4.16. We first reduce to the case where X is irreducible. Indeed, $|\mathrm{X}|$ has finitely many irreducible components $|\mathrm{Z}_i|$ ($i \in \mathrm{I}$) by the Noetherian hypothesis (*cf.* Lemma 4.1.20). We equip each $|\mathrm{Z}_i|$ with the structure of a closed subscheme $\mathrm{Z}_i \to \mathrm{X}$ as follows: Take the open subset $\mathring{\mathrm{Z}}_i := |\mathrm{X}| \smallsetminus \bigcup_{j \neq i} |\mathrm{Z}_j|$, equip it with the induced open subscheme structure, and set Z_i to be the scheme-theoretic image of $\mathring{\mathrm{Z}}_i \to \mathrm{X}$. Then Z_i has underlying topological space $|\mathrm{Z}_i|$, and $\bigsqcup_{i \in \mathrm{I}} \mathrm{Z}_i \to \mathrm{X}$ is proper and restricts to an isomorphism over $\bigsqcup_{i \in \mathrm{I}} \mathring{\mathrm{Z}}_i$. The problem for X is thus reduced to the problem for each Z_i.

Let us now assume that X is irreducible, with an affine open cover $\mathrm{X}_j \to \mathrm{X}$ ($j \in \mathrm{J}$ finite). For each $j \in \mathrm{J}$, we choose a locally closed immersion $\mathrm{X}_j \hookrightarrow \mathbb{P}^{n_j}_{\mathrm{S}}$ for some $n_j \in \mathbf{Z}_{\geq 0}$ and let $\overline{\mathrm{X}}_j$ denote its scheme-theoretic image. Then $\overline{\mathrm{X}}_j$ is a closed subscheme of $\mathbb{P}^{n_j}_{\mathrm{S}}$ and $\prod_{j \in \mathrm{J}} \overline{\mathrm{X}}_j$ is a closed subscheme of some $\mathbb{P}^n_{\mathrm{S}}$ by the Segre embedding (*cf.* Example 3.1.13).

Consider the locally closed immersion:

$$\mathrm{U} := \bigcap_{j \in \mathrm{J}} \mathrm{X}_j \xrightarrow{\Delta} \prod_{j \in \mathrm{J}} \mathrm{X}_j \to \prod_{j \in \mathrm{J}} \overline{\mathrm{X}}_j, \tag{4.14}$$

where U is a dense open subscheme of X because X is irreducible. Let $\overline{\mathrm{Y}}$ denote the scheme-theoretic image of the morphism (4.14), so $\overline{\mathrm{Y}}$ is a closed subscheme of $\prod_{j \in \mathrm{J}} \overline{\mathrm{X}}_j$, hence of $\mathbb{P}^n_{\mathrm{S}}$. Define open subschemes $\mathrm{Y}_j \subset \overline{\mathrm{Y}}$ by the Cartesian square:

$$\begin{array}{ccc} \mathrm{Y}_j & \longrightarrow & \overline{\mathrm{Y}} \\ \downarrow & & \downarrow{\scriptstyle \pi_j} \\ \mathrm{X}_j & \longrightarrow & \overline{\mathrm{X}}_j \end{array}$$

where π_j is the projection onto the jth factor. We then obtain a collection of morphisms indexed by $j \in \mathrm{J}$:

$$\mathrm{Y}_j \to \mathrm{X}_j \hookrightarrow \mathrm{X}. \tag{4.15}$$

Set $\mathrm{Y} := \bigcup_{j \in \mathrm{J}} \mathrm{Y}_j$ as an open subscheme of $\overline{\mathrm{Y}}$. We claim that the collection (4.15) glues into a morphism of schemes:

$$\pi : \mathrm{Y} \to \mathrm{X}.$$

By the sheaf property, it suffices to prove that $\pi_j|_{\mathrm{Y}_j \cap \mathrm{Y}_{j'}} = \pi_{j'}|_{\mathrm{Y}_j \cap \mathrm{Y}_{j'}}$ for $j, j' \in \mathrm{J}$. Note that each Y_j contains U as an open dense subscheme and the restriction of (4.15) to U coincides with the open immersion $\mathrm{U} \hookrightarrow \mathrm{X}$. Since $\mathrm{X} \to \mathrm{S}$ is separated, $\pi_j|_{\mathrm{Y}_j \cap \mathrm{Y}_{j'}}$ and $\pi_{j'}|_{\mathrm{Y}_j \cap \mathrm{Y}_{j'}}$ coincide on a closed subscheme of $\mathrm{Y}_j \cap \mathrm{Y}_{j'}$ containing U, which must be $\mathrm{Y}_j \cap \mathrm{Y}_{j'}$ itself.

Note that we have a locally closed immersion $\iota : \mathrm{Y} \hookrightarrow \overline{\mathrm{Y}} \hookrightarrow \mathbb{P}^n_{\mathrm{S}}$ by construction. It remains to prove that π is proper and induced an isomorphism over $\mathrm{U} \hookrightarrow \mathrm{X}$. Both statements will follow once we identify its base change $\pi^{-1}\mathrm{X}_j \to \mathrm{X}_j$ with the projection $\pi_j : \mathrm{Y}_j \to \mathrm{X}_j$, *i.e.* the open subschemes Y_j and $\pi^{-1}\mathrm{X}_j$ of Y coincide. We have a commutative diagram:

$$\begin{array}{ccccccc} \mathrm{Y}_j & \longrightarrow & \pi^{-1}\mathrm{X}_j & \hookrightarrow & \mathrm{Y} & \xhookrightarrow{\iota} & \mathbb{P}^n_{\mathrm{S}} \\ & {\scriptstyle \pi_j}\searrow & \downarrow & & \downarrow{\scriptstyle \pi} & & \downarrow \\ & & \mathrm{X}_j & \hookrightarrow & \mathrm{X} & \longrightarrow & \mathrm{S} \end{array}$$

Since $\mathrm{Y} \to \mathrm{S}$ is separated, so is $\pi : \mathrm{Y} \to \mathrm{X}$ (*cf.* Lemma 4.4.11). This implies that $\pi^{-1}\mathrm{X}_j \to \mathrm{X}_j$ is separated. On the other hand, π_j is proper, so the open immersion $\mathrm{Y}_j \to \pi^{-1}\mathrm{X}_j$ is also proper (*cf.* Lemma 4.4.11). Since $\pi^{-1}\mathrm{X}_j$ contains the dense open subscheme U, it is irreducible, so we conclude that $\mathrm{Y}_j \cong \pi^{-1}\mathrm{X}_j$. □

Remark 4.4.18. The statement of Theorem 4.4.16 holds when S is any Noetherian scheme (*i.e.* not necessarily affine). Change in the proof: To construct the locally closed immersion $X_j \hookrightarrow \mathbb{P}^{n_j}_S$ for each affine open subscheme X_j of X, we invoke [Sta18, 01VS].

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