ÉTALE METAPLECTIC COVERS OF REDUCTIVE GROUP SCHEMES

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Abstract. Given a reductive group scheme G, we give a linear algebraic description of reduced étale 4-cocycles on its classifying stack B(G). These cocycles form a 2-groupoid, which we interpret as parameters of metaplectic covers of G. We use our linear algebraic description to define the Langlands dual of a metaplectic cover.

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Introduction

The goal of this article is to build a theory of covering groups of reductive group schemes G over an arbitrary base scheme S, relying on étale cohomology. The idea for such a theory is due to Deligne [Del96], who also worked out the case where G is semisimple and simply connected. The generalization to the reductive case, carried out in the present article, is motivated by the geometrization of the Langlands program [BD91] [AG15] [FS24], as well as the possibility of casting covering groups under the same framework, see [FL10] [McN12] [Wei18] [GG18] [GL18].

There are already two existing foundations for the Langlands program for covering groups of reductive groups:

(1) Weissman [Wei18] has defined the L-groups of covering groups using Brylinski and Deligne's parametrization of them by algebraic K-theory [BD01];
(2) Gaitsgory–Lysenko [GL18] has defined the "metaplectic dual data" of factorization gerbes on the affine Grassmannian associated to G and a smooth curve X, interpreted as parameters of the metaplectic geometric Langlands program.

The advantage of Weissman's foundation is that it works over number fields and function fields alike, whereas the advantage of Gaitsgory and Lysenko's theory is that it is intrinsically geometric, allowing for example, a straightforward formulation of the geometric Satake equivalence.

The foundation we construct is their "least common multiple": it is essentially equivalent to that of [GL18] but remains valid in the number field context. It includes all covering groups which arise from algebraic K-theory, as well as an extra class coming from abelianized cohomology of G, which has applications in Langlands functoriality by Kaletha's work [Kal22]. Furthermore, we prove a number of structural results which lift classical computations to the geometric level, including for example the behavior of certain covering groups when restricted to parabolic subgroups.

The main intended applications of the present article are to extend V. Lafforgue's spectral decomposition [Laf18] to covering groups over function fields and Fargues and Scholze's spectral action [FS24] to covering groups over a p-adic field (jointly with Gaisin, Imai, and Koshikawa). The present article partially serves as a collection of the group-theoretic inputs they require. I beg the reader's forgiveness for not including concrete applications herein.

0.1. Étale metaplectic covers.

0.1.1. Let us explain the core definition of this article. Let S be a scheme, $G \to S$ be a group scheme of finite type, and A be a locally constant étale sheaf of finite abelian groups whose order is invertible on S. We define an A-*valued étale metaplectic cover* of G to be a rigidified section of $B^4 A(1)$ over the classifying stack BG.[1]

Here, A(1) is the Tate twist of A introduced for convention, $B^4 A(1)$ is its fourth iterated classifying stack, and "rigidified" means being equipped with a trivialization over the netural point $e : S \to BG$.

We use the language of Higher Algebra, as developed by Lurie [Lur09] [Lur17], in making this definition, although it can be avoided by working with chain complexes on simplicial schemes as in [Del96]. However, for the definition of the L-group, as well as the intended applications, we will need to consider algebraic structures on higher groupoids, which can be cumbersome to formulate without Higher Algebra.

Let us explain how this definition is related to other notions of covering groups.

0.1.2. *Topological covers.* To connect this definition to classical covering groups, we assume that S is the spectrum of a nonarchimedean local field F and A is a constant sheaf (for instance, the abelian group $\mu(F)$ of roots of unity in F). Then any étale metaplectic cover of G induces a central extension of topological groups:

$$1 \to A \to \tilde{G} \to G(F) \to 1. \tag{0.1}$$

The idea is simply to consider the monoidal morphism $G \to B^3 A(1)$ associated to an étale metaplectic cover, evaluate it on Spec(F), and use the vanishing of the étale cohomology group $H^3(F, A(1))$ and the Tate duality isomorphism $H^2(F, A(1)) \cong H^0(F, A)$. The archimedean case is addressed separately since H^3 may not vanish.

If F is a global field with topological ring of adèles $\mathbb{A}_F$, we obtain a central extension of $G(\mathbb{A}_F)$ (or possibly a subgroup thereof when F contains real places) equipped with a canonical splitting over G(F). When A appears as a subgroup of the multiplicative group of a coefficient field, we arrive at the notion of "genuine" automorphic representations.

The most classical example of (0.1) is Kubota's double cover of $SL_2(F)$, for F of characteristic $\neq 2$ (see [Kub67]), which is induced from the étale metaplectic cover represented by the mod 2 universal second Chern class $[c_2] \in H^4(B(SL_2), \{\pm 1\}^{\otimes 2})$.

0.1.3. *Central extensions by* $\underline{K}_2$. Let us relate étale metaplectic covers to central extensions of G by the Zariski sheafified second algebraic K-group $\underline{K}_2$, considered by Brylinski–Deligne

[1] We use the adjective "metaplectic" since "étale cover" has an established meaning, although the covering groups we study include far more instances than those having to do with the symplectic group.

[BD01]. The relevant étale metaplectic covers have coefficient group $\mathrm{A} = \mu_{\mathrm{N}}$, for $\mathrm{N} \geq 1$ being an integer invertible on S.

The essential point is that over a smooth scheme over a field, there is a canonical functor from central extensions of G by $\underline{\mathrm{K}}_2$ to étale metaplectic covers. It is a version of the étale realization of motivic cohomology of BG. The construction of this functor is essentially due to Gaitsgory [Gai20, §6] and involves strengthening some results of [EKLV98, §6] proving the equivalence between K-cohomology of BG and integral motivic cohomology in both the Zariski and étale topologies, see Theorem 2.3.8 in the main body of the text.

If S is the spectrum of a nonarchimedean local field F containing a primitive Nth root of unity, then we have a commutative diagram:

$$\begin{array}{ccc} \left\{\begin{matrix}\text{central extensions}\\ \text{of G by } \underline{\mathrm{K}}_2\end{matrix}\right\} & & \\ \Big\downarrow & \searrow & \\ & & \left\{\begin{matrix}\text{étale metaplectic covers}\\ \text{of G valued in } \mu_{\mathrm{N}}\end{matrix}\right\} \\ & \swarrow & \\ \left\{\begin{matrix}\text{central extensions}\\ \text{of G(F) by } \mu_{\mathrm{N}}(\mathrm{F})\end{matrix}\right\} & & \end{array}$$

where the vertical functor is that of [BD01, §10]. There is an analogous commutative diagram for global fields. They show that étale metaplectic covers *refine* the parametrization of covering groups by algebraic K-theory.

For example, the fact that the restriction of the metaplectic double cover of Sp_{2n} to the Siegel parabolic is induced along the determinant of the Levi quotient GL_n is witnessed by the corresponding étale metaplectic cover but not by the K-theoretic one.

0.1.4. *Geometrization.* Perhaps most importantly, A-valued étale metaplectic covers define A-gerbes on the geometric objects relevant to the Langlands program. When A is contained in the multiplicative group of a coefficient field, say $\overline{\mathbb{Q}}_\ell$ for a prime ℓ invertible on S, the induced $\overline{\mathbb{Q}}_\ell^\times$-gerbes allow us to form twisted categories of constructible sheaves.

Let us first discuss the global function field context: the base scheme is a smooth, proper, geometrically connected curve X over a finite field with field of fractions F. Let $\mathrm{Bun}_{\mathrm{G}}$ denote the moduli stack of G-bundles on X. Given an étale metaplectic cover of G, its pullback along the universal G-bundle $\mathrm{Bun}_{\mathrm{G}} \times \mathrm{X} \to \mathrm{BG}$ defines a section of the complex $\mathrm{A}(1)[4]$ over $\mathrm{Bun}_{\mathrm{G}} \times \mathrm{X}$. Pairing with the fundamental class of X defines a map:

$$\Gamma(\mathrm{Bun}_{\mathrm{G}} \times \mathrm{X}, \mathrm{A}(1)[4]) \to \Gamma(\mathrm{Bun}_{\mathrm{G}}, \mathrm{A}[2]),$$

and a section of the target is precisely an A-gerbe on $\mathrm{Bun}_{\mathrm{G}}$. Constructible sheaves twisted by the induced $\overline{\mathbb{Q}}_\ell^\times$-gerbe define unramified genuine automorphic forms on the corresponding covering group of $\mathrm{G}(\mathbb{A}_{\mathrm{F}})$ by the trace of Frobenius. The construction has a variant in the ramified situation as well.

Besides $\mathrm{Bun}_{\mathrm{G}}$, étale metaplectic covers naturally define $\overline{\mathbb{Q}}_\ell^\times$-gerbes on the Hecke stack compatible with the convolution structure. The variant for the local Hecke stack is furthermore compatible with its factorization structure. This observation allows a significant amount of the geometric Langlands program to be transported to the metaplectic context, as explained in [GL18].

When the base is a p-adic field F, étale metaplectic covers also define A-gerbes on the Fargues–Scholze $\mathrm{Bun}_{\mathrm{G}}$: the v-stack assigning the groupoid of G-bundles on the Fargues–Fontaine curve X_{S} to any affinoid perfectoid space S in characteristic p, see [FS24]. The construction is a variant of the global function field case, using the properness and cohomological smoothness of the "mirror curve". Over the open locus $[*/\underline{\mathrm{G(F)}}]$ of $\mathrm{Bun}_{\mathrm{G}}$, the resulting A-gerbe is canonically rigidified and recovers the topological central extension $\tilde{\mathrm{G}}$ of G(F).[2] The category of lisse sheaves twisted by its induced $\overline{\mathbb{Q}}_\ell^\times$-gerbe is a geometric incarnation of the category of genuine smooth representations of $\tilde{\mathrm{G}}$.

In geometric applictions, étale metaplectic covers offer a technical advantage over their K-theoretic counterparts, because the theory requires no regularity assumption on the base scheme, which is needed for [BD01].

0.2. Classification.

0.2.1. We hope to have conveyed some sense of the utility of having a theory of covering groups based on étale cohomology. We now turn to the main theorem of the article, which is a classification of étale metaplectic covers of any reductive group scheme $\mathrm{G} \to \mathrm{S}$: by definition, they form the space of rigidified sections $\Gamma_e(\mathrm{BG}, \mathrm{B}^4\mathrm{A}(1))$. Our classification is in complete parallel with [BD01, Theorem 7.2], the only difference being that $\Gamma_e(\mathrm{BG}, \mathrm{B}^4\mathrm{A}(1))$ is a 2-groupoid rather than a 1-groupoid.

Let $\underline{\Gamma}_e(\mathrm{BG}, \mathrm{B}^4\mathrm{A}(1))$ denote the étale sheaf on S, assigning the space of rigidified sections of $\mathrm{B}^4\mathrm{A}(1)$ over $\mathrm{BG} \times_{\mathrm{S}} \mathrm{S}_1$ for any S-scheme S_1. By abstract nonsense, each homotopy sheaf $\pi_i\underline{\Gamma}(\mathrm{BG}, \mathrm{B}^4\mathrm{A}(1))$ is isomorphic to $\mathrm{R}^{4-i}p_*\mathrm{A}(1)$, where $p : \mathrm{BG} \to \mathrm{S}$ denotes the projection map. The higher direct images $\mathrm{R}^{4-i}p_*\mathrm{A}(1)$ are computed by the étale cohomology of BG, which is standard. The part of the problem which we resolve is how these homotopy sheaves "fit together" in $\underline{\Gamma}_e(\mathrm{BG}, \mathrm{B}^4\mathrm{A}(1))$.

Sections 4-5 describe $\underline{\Gamma}_e(\mathrm{BG}, \mathrm{B}^4\mathrm{A}(1))$ in two stages.

0.2.2. *Tori.* Suppose that $\mathrm{T} \to \mathrm{S}$ is a torus with sheaf of cocharacters Λ. Then Theorem 4.3.2 expresses $\underline{\Gamma}_e(\mathrm{BT}, \mathrm{B}^4\mathrm{A}(1))$ as both a pushout and a pullback (in the ∞-categorical sense) of constructions of linear algebraic nature:

$$\begin{array}{ccccc} \underline{\mathrm{Maps}}_{\mathbb{Z}}(\wedge^2\Lambda, \mathrm{A}(-1)) & \to & \underline{\mathrm{Maps}}_{\mathbb{Z}}(\Lambda^{\otimes 2}, \mathrm{A}(-1)) & & \\ \downarrow{\scriptstyle \Psi(-1)} & & \downarrow & & \\ \underline{\mathrm{Maps}}_{\mathbb{Z}}(\Lambda, \mathrm{B}^2\mathrm{A}) & \longrightarrow & \underline{\Gamma}_e(\mathrm{BT}, \mathrm{B}^4\mathrm{A}(1)) & \longrightarrow & \underline{\mathrm{Maps}}_{\mathbb{Z}}(\Gamma^2\Lambda, \mathrm{A}(-1)) \\ & & \downarrow & & \downarrow{\scriptstyle \Psi(-1)} \\ & & \underline{\mathrm{Maps}}_{\mathbb{Z}}(\mathrm{H}^{(2)}(\Lambda), \mathrm{B}^2\mathrm{A}) & \to & \underline{\mathrm{Maps}}_{\mathbb{Z}}(\wedge^2\Lambda, \mathrm{BA}) \end{array} \tag{0.2}$$

Here, $\underline{\mathrm{Maps}}_{\mathbb{Z}}(-,-)$ denotes the sheaf of $\mathbb{Z}$-linear morphisms and the arrows labeled $\Psi(-1)$ are constructed using the "Kummer torsor" of roots of $(-1) \in \mathbb{G}_m$. The sheaf $\mathrm{H}^{(2)}(\Lambda)$ is an extension:

$$\mathrm{B}(\wedge^2\Lambda) \to \mathrm{H}^{(2)}(\Lambda) \to \Lambda,$$

and will be introduced in §4; here we only mention informally that maps out of $\mathrm{H}^{(2)}(\Lambda)$ encode "central extensions of Λ with prescribed commutators". The horizontal map out of

[2]This observation is first made by Arthur-César Le Bras.

$\underline{\Gamma}_e(\mathrm{BT}, \mathrm{B}^4\mathrm{A}(1))$ in (0.2) attaches an $\mathrm{A}(-1)$-valued quadratic form Q to every étale metaplectic cover of T: this is its only discrete invariant.

The diagram (0.2) has concrete analogues in the K-theoretic context. The two maps received by $\underline{\Gamma}_e(\mathrm{BT}, \mathrm{B}^4\mathrm{A}(1))$ correspond to the constructions of central extensions by cocycles, respectively by exact sequences of abelian sheaves. The pullback square, on the other hand, is an analogue of Brylinski and Deligne's description of central extensions of T by $\underline{\mathrm{K}}_2$ as those of Λ by $\mathbb{G}_m$ with prescribed commutators, see [BD01, Theorem 3.16].

0.2.3. *Reductive group schemes.* Suppose that $\mathrm{G} \to \mathrm{S}$ is a reductive group scheme. There is an étale sheaf of based root data $(\Delta \subset \Phi \subset \Lambda, \check{\Delta} \subset \check{\Phi} \subset \check{\Lambda})$, $\Phi \cong \check{\Phi}$ over S associated to G. For instance, once a Borel subgroup $\mathrm{B} \subset \mathrm{G}$ is chosen, then Λ is canonically identified with the sheaf of cocharacters of its maximal quotient torus.

Our description of $\underline{\Gamma}_e(\mathrm{BG}, \mathrm{B}^4\mathrm{A}(1))$ is supposed to be an extension of the association of based root data to reductive group schemes. First of all, the calculation of étale cohomology of BG yields a canonical triangle:

$$\underline{\mathrm{Maps}}_{\mathbb{Z}}(\pi_1\mathrm{G}, \mathrm{B}^2\mathrm{A}) \to \underline{\Gamma}_e(\mathrm{BG}, \mathrm{B}^4\mathrm{A}(1)) \to \mathrm{Quad}(\Lambda, \mathrm{A}(-1))_{\mathrm{st}}, \tag{0.3}$$

where $\pi_1\mathrm{G}$ is the algebraic fundamental group, and $\mathrm{Quad}(\Lambda, \mathrm{A}(-1))_{\mathrm{st}}$ is a subsheaf of the sheaf of $\mathrm{A}(-1)$-valued quadratic forms on Λ characterized by the equality:

$$b(\alpha, \lambda) = \mathrm{Q}(\alpha)\langle\check{\alpha}, \lambda\rangle, \quad \alpha \in \Delta, \lambda \in \Lambda,$$

for b the symmetric form associated to Q. (It sends λ_1, λ_2 to $\mathrm{Q}(\lambda_1 + \lambda_2) - \mathrm{Q}(\lambda_1) - \mathrm{Q}(\lambda_2)$.) We call such quadratic forms *strictly Weyl-invariant*, or simply *strict*. The triangle (0.3) reduces to the middle triangle in (0.2) for $\mathrm{G} = \mathrm{T}$ a torus, and yields an equivalence:

$$\underline{\Gamma}_e(\mathrm{BG}, \mathrm{B}^4\mathrm{A}(1)) \cong \mathrm{Quad}(\Lambda, \mathrm{A}(-1))_{\mathrm{st}} \tag{0.4}$$

when G is simply connected.

To describe $\underline{\Gamma}_e(\mathrm{BG}, \mathrm{B}^4\mathrm{A}(1))$, we will need to fix a Borel subgroup $\mathrm{B} \subset \mathrm{G}$. The restriction of an étale metaplectic cover along $\mathrm{B} \subset \mathrm{G}$ canonically descends to one of the "universal Cartan" $\mathrm{T} := \Lambda \otimes \mathbb{G}_m$. It follows rather easily from the fiber sequence (0.3) that the following commutative diagram is Cartesian:

$$\begin{array}{ccc} \underline{\Gamma}_e(\mathrm{BG}, \mathrm{B}^4\mathrm{A}(1)) & \xrightarrow{\mathrm{res}_{\mathrm{B}}} & \underline{\Gamma}_e(\mathrm{BT}, \mathrm{B}^4\mathrm{A}(1))_{\mathrm{st}} \\ \downarrow & & \downarrow \\ \underline{\Gamma}_e(\mathrm{BG}_{\mathrm{sc}}, \mathrm{B}^4\mathrm{A}(1)) & \xrightarrow{\mathrm{res}_{\mathrm{B}_{\mathrm{sc}}}} & \underline{\Gamma}_e(\mathrm{BT}_{\mathrm{sc}}, \mathrm{B}^4\mathrm{A}(1))_{\mathrm{st}} \end{array} \tag{0.5}$$

where G_{sc} denotes the simply connected form with its own universal Cartan T_{sc}, and the horizontal functors are the restrictions along B and the induced Borel subgroup $\mathrm{B}_{\mathrm{sc}} \subset \mathrm{G}_{\mathrm{sc}}$, see Theorem 5.1.13.

Combining the description of étale metaplectic covers of tori and that of simply connected groups, the Cartesian diagram (0.5) classifies étale metaplectic covers of reductive group schemes. We state it in parallel with [BD01, Theorem 7.2].

Theorem A. *Suppose that* $\mathrm{G} \to \mathrm{S}$ *is a reductive group scheme equipped with a Borel subgroup* $\mathrm{B} \subset \mathrm{G}$*. Then* $\underline{\Gamma}_e(\mathrm{BG}, \mathrm{B}^4\mathrm{A}(1))$ *is equivalent to the sheaf of quadruples* $(\mathrm{Q}, \mathrm{F}, h, \varphi)$ *where:*

(1) Q *is a strictly Weyl-invariant quadratic form on* Λ*;*
(2) $\mathrm{F} : \mathrm{H}^{(2)}(\Lambda) \to \mathrm{B}^2\mathrm{A}$ *is a* $\mathbb{Z}$*-linear morphism;*

(3) h is an isomorphism between the restriction of F *to* $\mathrm{B}(\wedge^2\Lambda)$ *and a* $\mathbb{Z}$*-linear morphism* $\wedge^2(\Lambda) \to \mathrm{BA}$ *canonically attached to* Q*;*

(4) φ is an isomorphism between the restriction of the pair (F, h) *to* Λ_{sc} *(i.e. the span of* Δ*) and a pair defined by the restriction of* Q *to* Λ_{sc}*.*

There is, however, an important caveat in this description: the necessity of fixing a Borel subgroup. If two Borel subgroups $\mathrm{B}_1, \mathrm{B}_2 \subset \mathrm{G}$ are conjugate under a section $g \in \mathrm{G}$, then we do find an isomorphism of functors $\mathrm{res}_{\mathrm{B}_1} \cong \mathrm{res}_{\mathrm{B}_2}$. However, this isomorphism *depends* on the section g and not on the Borel subgroups $\mathrm{B}_1, \mathrm{B}_2$ alone.

If the existence of Borel subgroups seems too restrictive, one can reformulate the classification Theorem A using a maximal torus as in [BD01, Theorem 7.2], but the issue persists: for two maximal tori $\mathrm{T}_1, \mathrm{T}_2$, there is no canonical way to relate the functors defined by restrictions along $\mathrm{T}_1, \mathrm{T}_2 \subset \mathrm{G}$.

0.3. The L-group.

0.3.1. Our main motivation for classifying étale metaplectic covers is to define their L-groups, following Gaitsgory and Lysenko [GL18]. It appears at first sight that *op.cit.* makes critical use of the geometry of the affine Grassmannian. However, once étale metaplectic covers are classified, the construction of "metaplectic dual data" carries over and an L-group in the style of Langlands can also be extracted formally. (In fact, only a small part of the classification is needed for this construction, and the reader who is only interested in the L-group can safely skip Section 3 and most of Sections 4-5.)

In this construction, we fix a field of coefficients, say $\overline{\mathbb{Q}}_\ell$ for a prime ℓ invertible over the base scheme S, and assume that A is a subsheaf of the constant sheaf $\overline{\mathbb{Q}}_\ell^\times$. From a reductive group scheme $\mathrm{G} \to \mathrm{S}$ equipped with an étale metaplectic cover μ, we shall construct a triple $(\mathrm{H}, \mathscr{G}_{\mathrm{Z}_{\mathrm{H}}(\overline{\mathbb{Q}}_\ell)}, \epsilon)$ called *metaplectic dual data*, where:

(1) H is an étale local system over S of pinned split reductive groups over $\overline{\mathbb{Q}}_\ell$;

(2) $\mathscr{G}_{\mathrm{Z}_{\mathrm{H}}(\overline{\mathbb{Q}}_\ell)}$ is an étale $\mathrm{Z}_{\mathrm{H}}(\overline{\mathbb{Q}}_\ell)$-gerbe over S. Here, Z_{H} denotes the center of H, so $\mathrm{Z}_{\mathrm{H}}(\overline{\mathbb{Q}}_\ell)$ is naturally an étale sheaf of abelian groups over S;

(3) ϵ is a homomorphism $\{\pm 1\} \to \mathrm{Z}_{\mathrm{H}}(\overline{\mathbb{Q}}_\ell)$.

The construction of H has first appeared in the metaplectic Satake isomorphism [FL10] [McN12] and uses only the quadratic form Q associated to an étale metaplectic cover under the second map of (0.3). For example, the sheaf of characters of its maximal torus T_{H} is precisely the kernel $\Lambda^\sharp \subset \Lambda$ of the associated symmetric form b. The subsheaf of roots $\Lambda^{\sharp, r} \subset \Lambda^\sharp$ is spanned by $\mathrm{ord}(\mathrm{Q}(\alpha))\alpha$ for $\alpha \in \Delta$, etc. In particular, the character group of its center $\hat{\mathrm{Z}}_{\mathrm{H}} \cong \Lambda^\sharp / \Lambda^{\sharp, r}$ is a subquotient of Λ. The map ϵ is simply defined by the restriction of Q to $\Lambda^\sharp$, which is 2-torsion valued.

The construction of $\mathscr{G}_{\mathrm{Z}_{\mathrm{H}}(\overline{\mathbb{Q}}_\ell)}$ poses the biggest challenge in both [GL18] and Weissman [Wei18] (where it is termed the "second twist"). Let us sketch its construction in our context, leaving the details to §6.

0.3.2. *Construction of* $\mathscr{G}_{\mathrm{Z}_{\mathrm{H}}(\overline{\mathbb{Q}}_\ell)}$. We need an observation concerning étale metaplectic covers of tori, see §0.2.2. Namely, if the symmetric form b associated to an étale metaplectic cover μ vanishes, then μ acquires the structure of an $\mathbb{E}_\infty$-*monoidal* morphism $\mathrm{BT} \to \mathrm{B}^4\mathrm{A}(1)$. Taking loop spaces yields an $\mathbb{E}_\infty$-monoidal morphism $\mathrm{T} \to \mathrm{B}^3\mathrm{A}(1)$, and upon evaluation at $\mathbb{G}_m$ viewed as a constant group scheme over S, we obtain an $\mathbb{E}_\infty$-monoidal morphism $\Lambda \to \mathrm{B}^2\mathrm{A}$.

We shall temporarily assume that the reductive group scheme G has a Borel subgroup $B \subset G$. Given an étale metaplectic cover μ of G, we apply the above observation to the torus $T^\sharp := \Lambda^\sharp \otimes \mathbb{G}_m$. From the commutative diagram (0.5), we shall obtain an $\mathbb{E}_\infty$-monoidal $\Lambda^\sharp \to B^2A$ trivialized over $\Lambda^{\sharp,r}$. This gives us an $\mathbb{E}_\infty$-monoidal morphism:

$$\nu : \hat{Z}_H \to B^2A.$$

Finally, we observe that the following inclusion:

$$\underline{\mathrm{Maps}}_{\mathbb{Z}}(\hat{Z}_H, B^2A) \to \underline{\mathrm{Maps}}_{\mathbb{E}_\infty}(\hat{Z}_H, B^2A)$$

admits a retraction, so ν defines a $\mathbb{Z}$-linear morphism $\nu^0 : \hat{Z}_H \to B^2A$. The induced $\mathbb{Z}$-linear morphism $\hat{Z}_H \to B^2(\overline{\mathbb{Q}}_\ell^\times)$ is equivalent to a $Z_H(\overline{\mathbb{Q}}_\ell)$-gerbe: this is $\mathscr{G}_{Z_H(\overline{\mathbb{Q}}_\ell)}$.

0.3.3. *Independence of* B. In §0.2.3, we have mentioned the dependence of the functor res_B on the Borel subgroup $B \subset G$. This fact haunts us again in the definition of $\mathscr{G}_{Z_H(\overline{\mathbb{Q}}_\ell)}$.

However, we shall prove that $\mathscr{G}_{Z_H(\overline{\mathbb{Q}}_\ell)}$ does *not* depend on B, i.e. it is functorially associated to the pair (G, μ). This fact allows us to remove the hypothesis on the existence of a Borel subgroup by étale descent.

Remark 0.3.4. If the base scheme is a smooth curve X over a field k, our $Z_H(\overline{\mathbb{Q}}_\ell)$-gerbe differs slightly from the one constructed in [GL18].[3] Namely, the $\{\pm 1\}$-gerbe of ϑ-*characteristics*, i.e. square roots of the canonical bundle $\omega_{X/k}$, induces a $Z_H(\overline{\mathbb{Q}}_\ell)$-gerbe ω_X^ϵ under the homomorphism ϵ, and the gerbe of *op.cit.* is given by $\mathscr{G}_{Z_H(\overline{\mathbb{Q}}_\ell)} \otimes \omega_X^\epsilon$.

0.3.5. *The* L-*group.* When S is the spectrum of a field F with a fixed algebraic closure $\bar{F}$, the metaplectic dual data $(H, \mathscr{G}_{Z_H(\overline{\mathbb{Q}}_\ell)}, \epsilon)$ defines an extension of topological groups:

$$1 \to H(\overline{\mathbb{Q}}_\ell) \to {}^L H_F \to \mathrm{Gal}(\bar{F}/F) \to 1, \tag{0.6}$$

upon rigidifying $\mathscr{G}_{Z_H(\overline{\mathbb{Q}}_\ell)}$ along $\mathrm{Spec}(\bar{F})$. Here, we confuse H with its fiber at $\mathrm{Spec}(\bar{F})$.

Roughly speaking, this is because $\mathscr{G}_{Z_H(\overline{\mathbb{Q}}_\ell)}$, equipped with its rigidification, defines a Galois 2-cocycle with values in $Z_H(\overline{\mathbb{Q}}_\ell)$. Such a 2-cocycle defines an extension of $\mathrm{Gal}(\bar{F}/F)$ by $Z_H(\overline{\mathbb{Q}}_\ell)$ compatible with the Galois action, which then induces (0.6) along the Galois-equivariant inclusion $Z_H(\overline{\mathbb{Q}}) \subset H(\overline{\mathbb{Q}}_\ell)$.

Remark 0.3.6. If F is a local field of characteristic $\neq 2$, our L-group (0.6) differs slightly from the one constructed in [Wei18]. Namely, ours is missing the twist by the meta-Galois group $\widetilde{\mathrm{Gal}}(\bar{F}/F)$, induced along the homomorphism ϵ. There is an analogous discrepancy for a global field F.

This difference is in fact the same as the one in Remark 0.3.4: the meta-Galois group *is* the central extension associated to the $\{\pm 1\}$-gerbe of ϑ-characteristics. The reason that we do not incorprate these twists in our definition of the metaplectic dual data, or the L-groups, is that they have to do with special features of the base scheme but we prefer to give a purely group-theoretic definition. Incorporating them poses no difficulty in any case.

[3] Since *op.cit.* is limited to the split case and does not address the dependence on auxiliary data, our construction is slightly more precise.

0.4. Metaplectic vs. quantum.

0.4.1. In the final §7, we give a group-theoretic interpretation of the relationship between étale metaplectic covers and "quantum parameters" of the geometric Langlands program, taking place over a field k of characteristic zero.

The notion of quantum parameters originated in the study of affine Kac–Moody Lie algebras, where it is given by the "level" of the central extension of a loop Lie algebra $\mathfrak{g}((t))$, see [Kac90]. When $\mathfrak{g}$ arises as the Lie algebra of a split reductive group scheme G, these levels are classified by G-invariant symmetric forms κ on $\mathfrak{g}$. They correspond bijectively to the fourth de Rham cohomology classes of BG.

In the geometric Langlands program of Beilinson and Drinfeld [BD91], the base scheme is a smooth, geometrically connected curve X over k. The study of Eisenstein series indicates that it is natural to include an extension of coherent sheaves:

$$0 \to \Omega^1_{\mathrm{X}/k} \to \mathrm{E} \to \mathfrak{g}_{\mathrm{ab}} \otimes \mathscr{O}_{\mathrm{X}} \to 0$$

in the space of quantum parameters, where $\mathfrak{g}_{\mathrm{ab}}$ denotes the abelianization of $\mathfrak{g}$, see [Zha23].

The space of pairs (κ, E) has the following group-theoretic intepretation: they are the rigidified fourth de Rham cochains on BG living in the second Hodge filtrant, i.e. the connective part of $\underline{\Gamma}_e(\mathrm{BG}, \Omega^{\geq 2}[4])$. This definition can be stated over any smooth k-scheme S and any reductive group scheme over S, not necessarily split.

0.4.2. We have thus arrived at a number of notions of "metaplectic covers" of a reductive group scheme $\mathrm{G} \to \mathrm{S}$, all having to do with the fourth cohomology of BG. Let us summarize their relationship in the following diagram:

$$\begin{array}{ccccc}
\left\{\begin{matrix}\text{central extensions} \\ \text{of G by } \underline{\mathrm{K}}_2\end{matrix}\right\} & \cong & \tau^{\leq 0}\underline{\Gamma}_e(\mathrm{BG}, \mathbb{Z}(2)[4]) & & \\
& & \downarrow & & \\
& & \tau^{\leq 0}\underline{\Gamma}_e(\mathrm{BG}, \mathbb{Q}(2)[4]) & \to & \left\{\begin{matrix}\text{quantum paramters} \\ \text{associated to G}\end{matrix}\right\} \\
& & \downarrow & & \\
\left\{\begin{matrix}\mathbb{Q}/\mathbb{Z}(1)\text{-valued étale} \\ \text{metaplectic covers of G}\end{matrix}\right\} & \cong & \tau^{\leq 0}\underline{\Gamma}_e(\mathrm{BG}, \mathbb{Q}/\mathbb{Z}(2)[4]) & &
\end{array} \tag{0.7}$$

Here, the column is given by the motivic cohomology groups of BG. Quantum parameters and étale metaplectic covers (generalized to infinite torsion abelian groups by taking colimits) arise from them by the de Rham, respectively étale realization functors. Diagram (0.7) is meant to provide some guidance in formulating the compatibility between the metaplectic and quantum geometric Langlands correspondences.

When the base field is $\mathbb{C}$, one has in addition a Betti realization functor of integral motivic cocycles on BG, whose natural target is the space of rigidified cocycles of the Deligne–Beilinson complex $\mathbb{Z}_{\mathrm{D}}(2)[4]$ (or $\mathbb{Q}_{\mathrm{D}}(2)[4]$) on the analytification $\mathrm{BG}_{\mathrm{an}}$. These Deligne-Beilinson cocycles may be interpreted as classifying "Chern–Simons theories" with gauge group the compact real form of G_{an} and have been studied by Brylinski and McLaughlin [BM94] [BM96].

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1. General topology

The notion of étale metaplectic covers we develop is 2-categorical in an essential way. The 2-isomorphisms we shall encounter fall into two kinds: ones which exist for *general* reasons and ones which exist for *particular* reasons. While the bulk of the paper deals with the latter, we also need an effective language to handle the former. This language is provided by Lurie's theory of Higher Algebra.

The goal of this section is to record facts from Higher Algebra that we will use in the main body of the text, mostly without explicit mention.

1.1. Structured spaces.

1.1.1. Let us begin with Grothendieck's dictionary between chain complexes in cohomological degrees $[-1,0]$ and small, strictly commutative Picard groupoids ([AGV73, Exposé XVIII, §1.4]). Recall: a symmetric monoidal groupoid $\mathscr{A}$ is a *Picard groupoid* if the monoidal product $-\otimes a : \mathscr{A} \to \mathscr{A}$ is an equivalence for all $a \in \mathscr{A}$. It is *strictly commutative* if the commutativity constraint $a_1 \otimes a_2 \cong a_2 \otimes a_1$ is the identity map whenever $a_1 = a_2$.

To a chain complex $\mathrm{K}^{-1} \xrightarrow{d} \mathrm{K}^0$, we attach a small, strictly commutative Picard groupoid whose objects are elements $a \in \mathrm{K}^0$ and there is an isomorphism $a_1 \xrightarrow{\sim} a_2$ for each $f \in \mathrm{K}^{-1}$ with $df = a_2 - a_1$. This association defines an equivalence of categories between chain complexes in cohomological degrees $[-1,0]$ and small, strictly commutative Picard groupoids ([AGV73, Exposé XVIII, Proposition 1.4.15]).

The central objects of this paper—étale metaplectic covers—form the 2-categorical version of a small, strictly commutative Picard groupoid. Homotopical algebra provides us with the tools to concisely express the data defining them.

1.1.2. Let Spc (resp. Spc_*) denote the ∞-category of (resp. pointed) spaces. The stable ∞-category of spectra Sptr is the stabilization of Spc_* (c.f. [Lur17, Proposition 1.4.2.24]).

It is equipped with a canonical functor $\Omega^\infty : \mathrm{Sptr} \to \mathrm{Spc}_*$ and a t-structure such that Ω^∞ factors through the ∞-category $\mathrm{Sptr}^{\le 0}$ of connective spectra ([Lur17, Proposition 1.4.3.4]). The resulting functor $\mathrm{Sptr}^{\le 0} \to \mathrm{Spc}_*$, still denoted by Ω^∞, has left adjoint Σ^∞_+.

1.1.3. Let us put the functor Ω^∞ in a different context. For an integer $0 \le n \le \infty$, write $\mathbb{E}_n(\mathrm{Spc})$ for the ∞-category of $\mathbb{E}_n$-monoid objects in Spc. (See [Lur17, §5.1] for the definition of the $\mathbb{E}_n$-operad.) Equivalently, this is the ∞-category of $\mathbb{E}_n$-algebras in Spc with respect to the Cartesian symmetric monoidal structure ([Lur17, Proposition 2.4.2.5]).

We refer to an object of $\mathbb{E}_n(\mathrm{Spc})$ simply as an $\mathbb{E}_n$-*space*. The ∞-category $\mathbb{E}_0(\mathrm{Spc})$ is equivalent to Spc_*.

For $n \ge 1$, an $\mathbb{E}_n$-space $\mathscr{A}$ is grouplike if and only if $\pi_0(\mathscr{A})$ is a group with respect to the induced monoid structure ([Lur17, Definition 5.2.6.2, Example 5.2.6.4]). We denote the

∞-category of grouplike $\mathbb{E}_n$-spaces by $\mathbb{E}_n^{\mathrm{grp}}(\mathrm{Spc})$. A version of May's recognition theorem ([Lur17, Remark 5.2.6.26]) states that there is a canonical equivalence of ∞-categories:

$$\mathrm{Sptr}^{\leq 0} \cong \mathbb{E}_\infty^{\mathrm{grp}}(\mathrm{Spc}). \tag{1.1}$$

Under (1.1), the forgetful functor $\mathbb{E}_\infty^{\mathrm{grp}}(\mathrm{Spc}) \to \mathbb{E}_0(\mathrm{Spc})$ corresponds to Ω^∞. This allows us to view connective spectra as spaces equipped with an "algebraic" structure.

1.1.4. Every commutative ring R may be viewed as an $\mathbb{E}_\infty$-algebra of Sptr with respect to the smash product ([Lur17, Remark 7.1.0.3]). The ∞-categorical version of the Schwede–Shipley theorem ([Lur17, Theorem 7.1.2.13]) compares the ∞-derived category D(R) and the ∞-category $\mathrm{Mod}_{\mathrm{R}}$ of R-module spectra. It states that there is a canonical equivalence of symmetric monoidal ∞-categories:

$$\mathrm{D}(\mathrm{R}) \cong \mathrm{Mod}_{\mathrm{R}}. \tag{1.2}$$

Under (1.2), the t-structure on R-module spectra corresponds to the natural t-structure on D(R). In particular, the ∞-category of connective R-module spectra $\mathrm{Mod}_{\mathrm{R}}^{\leq 0}$ is equivalent to that of nonpositively graded complexes of R-modules $\mathrm{D}^{\leq 0}(\mathrm{R})$.

This last equivalence can be viewed as a generalization of Grothendieck's dictionary, where the role of small, strictly commutative Picard groupoids is played by connective $\mathbb{Z}$-module spectra. The equivalence in §1.1.1 is the special case for $\mathrm{R} = \mathbb{Z}$, when both sides are restricted to 1-coconnective objects.

1.1.5. The following chain of forgetful functors relates all "structured spaces" which we shall consider in this paper:

$$\begin{aligned}\mathrm{Mod}_{\mathrm{R}}^{\leq 0} \to \mathrm{Mod}_{\mathbb{Z}}^{\leq 0} &\to \mathrm{Sptr}^{\leq 0} \\ &\cong \mathbb{E}_\infty^{\mathrm{grp}}(\mathrm{Spc}) \to \mathbb{E}_1^{\mathrm{grp}}(\mathrm{Spc}) \to \mathrm{Spc}_* \to \mathrm{Spc}.\end{aligned} \tag{1.3}$$

Lemma 1.1.6. *All functors in* (1.3) *are conservative. They preserve sifted colimits and arbitrary limits.*

Proof. The first two functors preserve limits by [Lur17, Proposition 4.6.2.17]. Since the $\mathbb{E}_\infty$-operad is coherent, we may apply [Lur17, Corollary 3.4.4.6] to conclude that they preserve arbitrary colimits and deduce from [Lur17, Corollary 3.4.3.3] that they are conservative.

For the functors on the bottom row, conservativity is a consequence of [Lur17, Lemma 3.2.2.6]. For any integer $0 \leq n \leq \infty$, the functor $\mathbb{E}_n(\mathrm{Spc}) \to \mathrm{Spc}$ preserves arbitrary limits ([Lur17, Corollary 3.2.2.4]) and *sifted* colimits ([Lur17, Proposition 3.2.3.1]). Together with conservativity, this implies the same for the functors:

$$\mathbb{E}_\infty(\mathrm{Spc}) \to \mathbb{E}_1(\mathrm{Spc}) \to \mathrm{Spc}_* \to \mathrm{Spc}.$$

Finally, the full subcategory of grouplike objects in $\mathbb{E}_n(\mathrm{Spc})$ ($n \geq 1$) is closed under arbitrary limits (by definition) and arbitrary colimits ([Lur17, Remark 5.2.6.9]). ☐

1.2. Sheaves.

1.2.1. Let $\mathscr{C}$ be a site. Denote by $\mathrm{PShv}(\mathscr{C})$ the ∞-category of presheaves of spaces on $\mathscr{C}$. It contains the full subcategory $\mathrm{Shv}(\mathscr{C}) \subset \mathrm{PShv}(\mathscr{C})$ of sheaves of spaces, characterized by the property that for any covering sieve $\mathscr{S} \subset \mathrm{Hom}_{\mathscr{C}}(-, c)$, the canonical map:

$$\mathscr{F}(c) \to \lim_{(f : c_1 \to c) \in \mathscr{S}} \mathrm{F}(c_1)$$

is an equivalence. The ∞-category $\mathrm{Shv}(\mathscr{C})$ is an ∞-topos in the sense of [Lur09, §6].

1.2.2. For each ∞-category in (1.3), we may consider the ∞-category of (pre)sheaves valued in it. Since the functors connecting them preserve limits (Lemma 1.1.6), the corresponding functors on presheaves preserve the full subcategories of sheaves.

Lemma 1.1.6 also implies that sheaves valued in $\mathbb{E}_\infty^{\mathrm{grp}}(\mathrm{Spc})$ (resp. $\mathbb{E}_1^{\mathrm{grp}}(\mathrm{Spc})$ or Spc_*) are canonically equivalent to grouplike $\mathbb{E}_\infty$-monoids (resp. grouplike $\mathbb{E}_1$-monoids or pointed objects) in $\mathrm{Shv}(\mathscr{C})$. In particular, (1.3) gives rise to a chain of ∞-categories:

$$\mathrm{Shv}(\mathscr{C}, \mathrm{Mod}_{\mathrm{R}}^{\le 0}) \to \mathrm{Shv}(\mathscr{C}, \mathrm{Mod}_{\mathbb{Z}}^{\le 0}) \to \mathrm{Shv}(\mathscr{C}, \mathrm{Sptr}^{\le 0}) \cong \mathbb{E}_\infty^{\mathrm{grp}}(\mathrm{Shv}(\mathscr{C})) \to \mathbb{E}_1^{\mathrm{grp}}(\mathrm{Shv}(\mathscr{C})) \to \mathrm{Shv}_*(\mathscr{C}) \to \mathrm{Shv}(\mathscr{C}). \tag{1.4}$$

The functors in (1.4) are conservative and limit-preserving. We could add to (1.4) a limit-preserving (but evidently not conservative) functor $\mathrm{Shv}(\mathscr{C}, \mathrm{Mod}_{\mathrm{R}}) \to \mathrm{Shv}(\mathscr{C}, \mathrm{Mod}_{\mathrm{R}}^{\le 0})$ defined by $\tau^{\le 0}$ on the underlying presheaves.

1.2.3. Let $\mathscr{A}$ be a symmetric monoidal ∞-category with arbitrary limits and colimits and such that the unit $\mathbf{1}$ is both final and initial. Then for $n \ge 0$, the ∞-category of $\mathbb{E}_n$-algebras of $\mathscr{A}$ and the ∞-category of $\mathbb{E}_n$-algebras of $\mathscr{A}^{\mathrm{op}}$ (i.e. $\mathbb{E}_n$-coalgebras of $\mathscr{A}$) are related by a pair of adjoint functors ([Lur17, Remark 5.2.3.6]):

$$\mathrm{Bar}^{(n)} : \mathbb{E}_n(\mathscr{A}) \rightleftarrows \mathbb{E}_n(\mathscr{A}^{\mathrm{op}})^{\mathrm{op}} : \mathrm{Cobar}^{(n)}.$$

We shall apply this construction to $\mathscr{A} = \mathrm{Shv}(\mathscr{C}, \mathrm{Mod}_{\mathrm{R}}^{\le 0})$, $\mathrm{Shv}(\mathscr{C}, \mathrm{Sptr}^{\le 0})$, $\mathrm{Shv}_*(\mathscr{C})$, equipped with the *Cartesian* symmetric monoidal structure.

In the first two cases, the Cartesian symmetric monoidal structure coincides with the co-Cartesian one, so the forgetful functor $\mathbb{E}_n(\mathscr{A}) \to \mathscr{A}$ is an equivalence ([Lur17, Proposition 2.4.3.9]). Under this equivalence, $\mathrm{Bar}^{(n)}$ corresponds to n-fold suspension $[n]$ ([Lur17, Example 5.2.2.4]). Since the functor $\Omega^\infty : \mathrm{Shv}(\mathscr{C}, \mathrm{Sptr}^{\le 0}) \to \mathrm{Shv}_*(\mathscr{C})$ is symmetric monoidal, the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{Shv}(\mathscr{C}, \mathrm{Sptr}^{\le 0}) & \xrightarrow{[n]} & \mathrm{Shv}(\mathscr{C}, \mathrm{Sptr}^{\le 0}) \\ \big\downarrow {\scriptstyle \mathbb{E}_n(\Omega^\infty)} & & \big\downarrow {\scriptstyle \Omega^\infty} \\ \mathbb{E}_n^{\mathrm{grp}}(\mathrm{Shv}(\mathscr{C})) & \xrightarrow{\mathrm{B}^n} & \mathrm{Shv}_*(\mathscr{C}) \end{array} \tag{1.5}$$

Here, the functor B^n is obtained from $\mathrm{Bar}^{(n)}$ by composing with the functor forgetting the $\mathbb{E}_n$-coalgebra structure. We have an analogous commutative diagram when we replace $\mathrm{Sptr}^{\le 0}$ in the top row by $\mathrm{Mod}_{\mathrm{R}}^{\le 0}$.

1.2.4. Given $\mathscr{F} \in \mathrm{Shv}(\mathscr{C})$ and integer $n \ge 0$, we let $\pi_n \mathscr{F}$ denote the sheafification of the presheaf $c \mapsto \pi_n(\mathscr{F}(c))$. Write $\mathrm{Shv}(\mathscr{C})_{\ge n}$ for the full subcategory of $\mathrm{Shv}(\mathscr{C})$ consisting of objects $\mathscr{F}$ with $\pi_k \mathscr{F} = 0$ for all $0 \le k < n$. Namely, this is the ∞-category of n-connective objects of $\mathrm{Shv}(\mathscr{C})$. We also use the notation $\mathrm{Shv}_*(\mathscr{C})_{\ge n}$ for the pointed version.

Because $\mathrm{Shv}(\mathscr{C})$ is an ∞-topos, B^n in (1.5) defines an equivalence of ∞-categories onto $\mathrm{Shv}_*(\mathscr{C})_{\ge n}$ ([Lur17, Theorem 5.2.6.15]).

1.2.5. Let us mention a concrete description of $\mathrm{B} := \mathrm{B}^1$. Every $\mathbb{E}_1$-monoid in $\mathrm{Shv}(\mathscr{C})$ defines a simplicial object $\mathscr{F}^{[n]}$ ($[n] \in \Delta^{\mathrm{op}}$) of $\mathrm{Shv}(\mathscr{C})$ with $\mathscr{F}^{[0]} \cong \mathrm{pt}$. The functor B sends it to the geometric realization $\mathrm{colim}_{[n]} \mathscr{F}^{[n]}$, pointed by $\mathscr{F}^{[0]}$ ([Lur17, Example 5.2.6.13]).

In particular, given a sheaf of groups H on $\mathscr{C}$, B(H) is the classifying stack of H in the classical sense. For a sheaf of abelian groups (resp. R-modules) A on $\mathscr{C}$ and $n \geq 1$, we view $B^n(A)$ as the n-fold classifying stack of A. The commutative diagram (1.5) tells us that $B^n(A)$ is canonically identified with the image of $A[n]$ under Ω^∞. In particular, $B^n(A)$ inherits the structure of a sheaf valued in $\mathrm{Sptr}^{\leq 0}$ (resp. $\mathrm{Mod}_R^{\leq 0}$).

1.3. Complexes.

1.3.1. In light of (1.2), an object of the ∞-category $\mathrm{Shv}(\mathscr{C}, \mathrm{Mod}_R)$ can be viewed as a sheaf of complexes of R-modules. In the classical context, we are more accustomed to complexes of sheaves of R-modules. Let us explain how these two points of views are related.

We continue to fix a site $\mathscr{C}$ and a commutative ring R. The t-structure on Mod_R induces a t-structure on $\mathrm{Shv}(\mathscr{C}, \mathrm{Mod}_R)$. Its heart is identified with the category of sheaves of R-modules on $\mathscr{C}$. According to [Lur18, Corollary 2.1.2.4], we have a fully faithful, t-exact functor:

$$D^+(\mathrm{Shv}(\mathscr{C}, \mathrm{Mod}_R)^\heartsuit) \to \mathrm{Shv}(\mathscr{C}, \mathrm{Mod}_R). \tag{1.6}$$

The essential image of (1.6) is the full subcategory $\mathrm{Shv}(\mathscr{C}, \mathrm{Mod}_R)^{>-\infty}$ of left-bounded objects.

1.3.2. Let $\mathrm{ob} : \mathrm{Shv}(\mathscr{C}, \mathrm{Mod}_R) \to \mathrm{PShv}(\mathscr{C}, \mathrm{Mod}_R)$ denote the forgetful functor. Its left adjoint is the sheafification functor, which is t-exact by [Lur18, Proposition 1.3.4.7]. In particular, ob is left t-exact. We denote by $R(\mathrm{ob}^\heartsuit)$ the right derived functor of its truncation. Explicitly, given a left-bounded complex $\mathscr{F}$ of sheaves of R-modules, $R(\mathrm{ob}^\heartsuit)(\mathscr{F})$ is the presheaf whose value at $c \in \mathscr{C}$ is the complex of R-modules $R\Gamma(c, \mathscr{F})$.

The following commutative diagram arises from the analogous diagram for the sheafification functor by passing to the right adjoint:

$$\begin{array}{ccc} D^+(\mathrm{Shv}(\mathscr{C}, \mathrm{Mod}_R)^\heartsuit) & \xrightarrow{\cong} & \mathrm{Shv}(\mathscr{C}, \mathrm{Mod}_R)^{>-\infty} \\ \big\downarrow {\scriptstyle R(\mathrm{ob}^\heartsuit)} & & \big\downarrow {\scriptstyle \mathrm{ob}} \\ D^+(\mathrm{PShv}(\mathscr{C}, \mathrm{Mod}_R)^\heartsuit) & \xrightarrow{\cong} & \mathrm{PShv}(\mathscr{C}, \mathrm{Mod}_R)^{>-\infty} \end{array} \tag{1.7}$$

It implies that the image of $\mathscr{F}$ under (1.6) has an explicit description: its value at $c \in \mathscr{C}$ is the complex of R-modules $R\Gamma(c, \mathscr{F})$.

Lemma 1.3.3. *Suppose that* A *is a sheaf of* R*-modules on* $\mathscr{C}$. *For integers* $0 \leq k \leq n$ *and* $c \in \mathscr{C}$, *there is a canonical isomorphism of* R*-modules:*

$$\pi_k \Gamma(c, B^n(A)) \cong H^{n-k}(c, A).$$

Proof. This follows from (1.7) and the discussion in §1.2.5. ☐

2. Étale metaplectic covers

Let S be a scheme and $G \to S$ be a group scheme of finite type. Suppose that A is a locally constant étale sheaf of finite abelian groups on S whose order is invertible on S.

This is the general context in which we will define metaplectic covers.

Definition 2.0.1. An *étale metaplectic cover* of G with values in A is a morphism of pointed étale stacks $BG \to B^4A(1)$.

Thus, A-valued étale metaplectic covers of G form the groupoid $\Gamma_e(\mathrm{BG}, \mathrm{B}^4\mathrm{A}(1))$, viewed as sections of $\mathrm{B}^4\mathrm{A}(1)$ over BG rigidified along $e : \mathrm{S} \to \mathrm{BG}$ (i.e. equipped with a trivialization after pulling back by e).

Here, the Tate twist A(1) is defined as $\mathrm{A}(1) := \mathrm{A} \otimes \mathbb{G}_m[-1]$ or equivalently $\lim_{\mathrm{N}}(\mathrm{A} \otimes \mu_{\mathrm{N}})$, where the transition maps are given by $\mu_{\mathrm{N}_1} \twoheadrightarrow \mu_{\mathrm{N}}$, $a \mapsto a^{\mathrm{N}_1/\mathrm{N}}$ whenever $\mathrm{N} \mid \mathrm{N}_1$. Then A(1) is again a locally constant étale sheaf of finite abelian groups on S. Likewise, we write A(−1) for $\mathrm{colim}_{\mathrm{N}}(\mathrm{A} \otimes \mu_{\mathrm{N}}^{\otimes -1})$.

In the remainder of this section, we first explain how to obtain covering groups in the classical sense from étale metaplectic covers. This construction is a reformulation of Deligne [Del96, §2, §5-6], although the language of higher categories allows us to avoid direct contact with resolutions. Then we explain in §2.3 the relationship between étale metaplectic covers and central extensions of G by $\underline{\mathrm{K}}_2$, which may be viewed as an analogue of our theory with "integral coefficients".

Remark 2.0.2. It is sometimes natural to forget the distinguished point of the étale stack BG. Indeed, a morphism of étale stacks $\mu : \mathrm{BG} \to \mathrm{B}^4\mathrm{A}(1)$ automatically induces an étale metaplectic cover of $\mathrm{G}_{\mathrm{P}} := \underline{\mathrm{Aut}}(\mathrm{P})$ for any G-torsor $\mathrm{P} \to \mathrm{S}$: writing $\mathrm{S} \xrightarrow{e_{\mathrm{P}}} \mathrm{BG} \xrightarrow{p} \mathrm{S}$ where e_{P} corresponds to P, we form $\mu_{\mathrm{P}} := \mu \otimes (p^* e_{\mathrm{P}}^* \mu)^{\otimes -1}$.

We may view morphisms $\mathrm{BG} \to \mathrm{B}^4\mathrm{A}(1)$ as defining étale metaplectic covers on all pure inner forms of G at once. For a chosen G-torsor $\mathrm{P} \to \mathrm{S}$, we have a projection $\mu \mapsto \mu_{\mathrm{P}}$ to the groupoid of étale metaplectic covers of G_{P}, for which the functor of forgetting the pointing provides a section.

2.1. Local fields.

2.1.1. Suppose that S is the spectrum of a nonarchimedean local field F. Fix an algebraic closure $\mathrm{F} \subset \bar{\mathrm{F}}$ which defines the Galois group $\mathrm{Gal}(\bar{\mathrm{F}}/\mathrm{F})$. In particular, A may be viewed as a finite abelian group equipped with a $\mathrm{Gal}(\bar{\mathrm{F}}/\mathrm{F})$-action.

From a pointed morphism $\mathrm{BG} \to \mathrm{B}^4\mathrm{A}(1)$, we shall functorially attach a central extension of topological groups:

$$1 \to \mathrm{H}_0(\mathrm{F}, \mathrm{A}) \to \tilde{\mathrm{G}} \to \mathrm{G}(\mathrm{F}) \to 1, \tag{2.1}$$

where $\mathrm{H}_0(\mathrm{F}, \mathrm{A})$ denotes the $\mathrm{Gal}(\bar{\mathrm{F}}/\mathrm{F})$-coinvariants of A, i.e. the zeroth homology group, and is equipped with the discrete topology.

In the case $\mathrm{F} = \mathbb{R}$, we will in general only obtain a central extension of a subgroup of $\mathrm{G}(\mathbb{R})$ which contains the neutral component; this will be addressed in §2.1.6. The case $\mathrm{F} = \mathbb{C}$ will give canonically split extensions of $\mathrm{G}(\mathbb{C})$.

2.1.2. The construction of (2.1) uses local Tate duality [Tat63, Theorem 2.1]. Write $\mathrm{A}^* := \underline{\mathrm{Hom}}(\mathrm{A}, \mathbb{Q}/\mathbb{Z})$. Then A^* is in Cartier duality with A(1), so cup product defines a perfect pairing on Galois cohomology groups:

$$\mathrm{H}^0(\mathrm{F}, \mathrm{A}^*) \otimes \mathrm{H}^2(\mathrm{F}, \mathrm{A}(1)) \to \mathrm{H}^2(\mathrm{F}, \mathbb{G}_m) \cong \mathbb{Q}/\mathbb{Z}, \tag{2.2}$$

where the second map is given by Brauer invariants.

Therefore, $\mathrm{H}^2(\mathrm{F}, \mathrm{A}(1))$ is identified with the Pontryagin dual of $\mathrm{H}^0(\mathrm{F}, \mathrm{A}^*)$, which agrees with the homology group $\mathrm{H}_0(\mathrm{F}, \mathrm{A})$. We summarize this isomorphism together with the vanishing statements of [Tat63, Theorem 2.1]:

$$\mathrm{H}^i(\mathrm{F}, \mathrm{A}(1)) \cong \begin{cases} \mathrm{H}_0(\mathrm{F}, \mathrm{A}) & i = 2 \\ 0 & i \geq 3 \end{cases} \tag{2.3}$$

2.1.3. Given a pointed morphism $\mu : \mathrm{BG} \to \mathrm{B}^4\mathrm{A}(1)$, we may form an $\mathbb{E}_1$-monoidal morphism $\Omega(\mu) : \mathrm{G} \to \mathrm{B}^3\mathrm{A}(1)$ by taking loop stacks. Evaluation at $\mathrm{Spec}(\mathrm{F})$ then yields maps of $\mathbb{E}_1$-monoidal groupoids:

$$\begin{aligned} &\mathrm{G}(\mathrm{F}) \to \Gamma(\mathrm{F}, \mathrm{B}^3\mathrm{A}(1)) \\ &\qquad \cong \mathrm{B}(\Gamma(\mathrm{F}, \mathrm{B}^2\mathrm{A}(1))) \xrightarrow{\pi_0} \mathrm{B}(\mathrm{H}^2(\mathrm{F}, \mathrm{A}(1))) \cong \mathrm{B}(\mathrm{H}_0(\mathrm{F}, \mathrm{A})). \end{aligned} \tag{2.4}$$

where the isomorphisms owe to the calculations (2.3). This composition defines a central extension (2.1) as an abstract group.

2.1.4. In order to equip $\tilde{\mathrm{G}}$ with a topology making the maps in (2.1) continuous, it suffices to construct a family of local sections of $\tilde{\mathrm{G}} \to \mathrm{G}(\mathrm{F})$ satisfying the properties (a)–(c) of [Del96, §2.9].

Let us paraphrase the construction of *op.cit.* in our language. Consider the $\mathbb{E}_1$-monoidal morphism $\Omega(\mu) : \mathrm{G} \to \mathrm{B}^3\mathrm{A}(1)$ corresponding to μ and write $\mathrm{G}^\dagger$ for the fiber of $\Omega(\mu)$ at the base point. Then $\mathrm{G}^\dagger$ is an étale sheaf of 2-groupoids and the projection $\mathrm{G}^\dagger(\mathrm{F}) \to \mathrm{G}(\mathrm{F})$ factors through $\tilde{\mathrm{G}}$.

Pullback along any $g \in \mathrm{G}(\mathrm{F})$ defines a section:

$$g^*\Omega(\mu) \in \Gamma(\mathrm{F}, \mathrm{B}^3\mathrm{A}(1)) \cong \underset{\mathrm{Spec}(\mathrm{F}) \to \mathrm{U}}{\mathrm{colim}} \Gamma(\mathrm{U}, \mathrm{B}^3\mathrm{A}(1)),$$

the colimit being taken over étale neighborhoods U of g. The null-homotopy of $g^*\Omega(\mu)$ is thus defined over some such U. By construction, $\mathrm{U} \to \mathrm{G}$ lifts to $\mathrm{G}^\dagger$, giving a commutative diagram:

$$\begin{array}{ccc} & & \mathrm{G}^\dagger(\mathrm{F}) \\ & \nearrow & \downarrow \\ \mathrm{U}(\mathrm{F}) & & \tilde{\mathrm{G}} \\ & \searrow & \downarrow \\ & & \mathrm{G}(\mathrm{F}) \end{array}$$

Since $\mathrm{U} \to \mathrm{G}$ is étale and $g \in \mathrm{G}(\mathrm{F})$ lifts to $\mathrm{U}(\mathrm{F})$, we find local sections of $\mathrm{U}(\mathrm{F}) \to \mathrm{G}(\mathrm{F})$ at g by the implicit function theorem. (This step uses the finite type hypothesis on G.) They define sections for the projection $\tilde{\mathrm{G}} \to \mathrm{G}(\mathrm{F})$.

Remark 2.1.5. Let $\mathscr{O} \subset \mathrm{F}$ denote the ring of integers and suppose that A extends to a locally constant étale sheaf of finite abelian groups on $\mathrm{Spec}(\mathscr{O})$ with invertible order, i.e. indivisible by the residue characteristic.

If G is an affine group scheme over $\mathrm{Spec}(\mathscr{O})$, then the central extension (2.1) acquires a canonical splitting over the subgroup $\mathrm{G}(\mathscr{O}) \subset \mathrm{G}(\mathrm{F})$. Indeed, the vanishing of $\mathrm{H}^2(\mathscr{O}, \mathrm{A}(1))$ implies that (2.4) is null-homotopic when restricted to $\mathrm{G}(\mathscr{O})$.

2.1.6. When $\mathrm{F} = \mathbb{R}$, we replace (2.2) by the perfect pairing on Tate cohomology groups:

$$\hat{\mathrm{H}}^0(\mathbb{R}, \mathrm{A}^*) \otimes \mathrm{H}^2(\mathbb{R}, \mathrm{A}(1)) \to \mathrm{H}^2(\mathbb{R}, \mathbb{G}_m) \cong \frac{1}{2}\mathbb{Z}/\mathbb{Z}. \tag{2.5}$$

Recall that $\hat{\mathrm{H}}^0(\mathbb{R}, \mathrm{A})$ is defined by an exact sequence involving the norm map:

$$0 \to \hat{\mathrm{H}}^{-1}(\mathbb{R}, \mathrm{A}) \to \mathrm{H}_0(\mathbb{R}, \mathrm{A}) \xrightarrow{\mathrm{Nm}} \mathrm{H}^0(\mathbb{R}, \mathrm{A}) \to \hat{\mathrm{H}}^0(\mathbb{R}, \mathrm{A}) \to 0. \tag{2.6}$$

We obtain from (2.5) the following isomorphism:

$$\mathrm{H}^2(\mathbb{R}, \mathrm{A}(1)) \cong \hat{\mathrm{H}}^{-1}(\mathbb{R}, \mathrm{A}). \tag{2.7}$$

The other change in the construction has to do with possibly nonzero $\mathrm{H}^3(\mathbb{R}, \mathrm{A}(1))$. For a pointed morphism $\mu : \mathrm{BG} \to \mathrm{B}^4\mathrm{A}(1)$, the induced $\mathbb{E}_1$-monoidal map $\Omega(\mu) : \mathrm{G} \to \mathrm{B}^3\mathrm{A}(1)$ defines a homomorphism:

$$\mathrm{G}(\mathbb{R}) \to \mathrm{H}^3(\mathbb{R}, \mathrm{A}(1)). \tag{2.8}$$

We write $\mathrm{G}(\mathbb{R})^0$ for the kernel of (2.8). Then the same construction as above, using (2.7) instead of (2.2), gives a central extension of topological groups:

$$1 \to \hat{\mathrm{H}}^{-1}(\mathbb{R}, \mathrm{A}) \to \tilde{\mathrm{G}} \to \mathrm{G}(\mathbb{R})^0 \to 1. \tag{2.9}$$

Finally, we also obtain a central extension of $\mathrm{G}(\mathbb{R})^0$ by $\mathrm{H}_0(\mathbb{R}, \mathrm{A})$ by inducing along the first map in (2.6).

2.2. Global fields.

2.2.1. We now suppose that S is the spectrum of a global field F and G is affine. As before, we fix an algebraic closure $\mathrm{F} \subset \bar{\mathrm{F}}$. Let $\mathbb{A}_\mathrm{F}$ denote the topological ring of adèles of F.

Let $\mu : \mathrm{BG} \to \mathrm{B}^4\mathrm{A}(1)$ be a pointed morphism over Spec(F). Then restriction of μ to each place v of F defines a pointed morphism $\mu_v : \mathrm{BG}_{\mathrm{F}_v} \to \mathrm{B}^4\mathrm{A}(1)$ over $\mathrm{Spec}(\mathrm{F}_v)$. In particular, we find central extensions $\tilde{\mathrm{G}}_v$ of $\mathrm{G}(\mathrm{F}_v)^0$, where the notation $\mathrm{G}(\mathrm{F}_v)^0$ stands for $\mathrm{G}(\mathrm{F}_v)$ for v nonarchimedean or complex, and for the kernel of (2.8) for v real.

2.2.2. In this section, we shall combine these $\tilde{\mathrm{G}}_v$ into a central extension of the adèlic points of G. More precisely, we functorially attach to μ a central extension:

$$1 \to \mathrm{H}_0(\mathrm{F}, \mathrm{A}) \to \tilde{\mathrm{G}} \to \mathrm{G}(\mathbb{A}_\mathrm{F})^0 \to 1, \tag{2.10}$$

equipped with a canonical splitting over $\mathrm{G}(\mathrm{F})^0$.

Here, $\mathrm{G}(\mathbb{A}_\mathrm{F})^0 \subset \mathrm{G}(\mathbb{A}_\mathrm{F})$ (resp. $\mathrm{G}(\mathrm{F})^0 \subset \mathrm{G}(\mathrm{F})$) denotes the subgroup of elements whose restriction along each real place v lies in $\mathrm{G}(\mathrm{F}_v)^0$.

2.2.3. Write $\mathscr{O}_\mathrm{F} \subset \mathrm{F}$ for the ring of integers. Fix a finite nonempty collection of places $v \in \mathrm{F}$ including all the archimedean ones. Its complement may be viewed as an open subscheme $\mathrm{V} \subset \mathrm{Spec}(\mathscr{O}_\mathrm{F})$.

We shall assume that A extends to a locally constant étale sheaf over V, with order invertible on V. Such an extension is unique, as the morphism from $\mathrm{Gal}(\bar{\mathrm{F}}/\mathrm{F})$ to the étale fundamental group of V is surjective, see [Sta18, 0BSD].

There is an exact sequence:

$$\mathrm{H}^2(\mathrm{V}, \mathrm{A}(1)) \to \bigoplus_{v \notin \mathrm{V}} \mathrm{H}^2(\mathrm{F}_v, \mathrm{A}(1)) \to \mathrm{H}_0(\mathrm{F}, \mathrm{A}) \to 0, \tag{2.11}$$

where the first map is the restriction, and the second map is the sum of local duality maps, composed with the projections $\mathrm{H}_0(\mathrm{F}_v, \mathrm{A}) \twoheadrightarrow \mathrm{H}_0(\mathrm{F}, \mathrm{A})$. This follows from [Tat63, Theorem 3.1], in view of the identification $\mathrm{H}^0(\mathrm{V}, \mathrm{A}^*) \cong \mathrm{H}^0(\mathrm{F}, \mathrm{A}^*)$.

The same Theorem yields the calculation of $\mathrm{H}^3(\mathrm{V}, \mathrm{A}(1))$, i.e. along the restriction maps, we have an isomorphism:

$$\mathrm{H}^3(\mathrm{V}, \mathrm{A}(1)) \cong \bigoplus_{v \text{ real}} \mathrm{H}^3(\mathrm{F}_v, \mathrm{A}(1)). \tag{2.12}$$

2.2.4. To construct (2.10), we need to extend the affine group scheme G and the pointed morphism μ to an open subscheme of $\mathrm{Spec}(\mathscr{O}_{\mathrm{F}})$.

To address the canonicity of the construction, we consider the filtered category of triples $(\mathrm{U}, \mathrm{G}_{\mathrm{U}}, \gamma)$ where $\mathrm{U} \subset \mathrm{Spec}(\mathscr{O}_{\mathrm{F}})$ is an open subscheme where A extends as a locally constant sheaf, $\mathrm{G}_{\mathrm{U}} \to \mathrm{U}$ is a affine group scheme of finite type, and γ is an isomorphism $\mathrm{G} \cong (\mathrm{G}_{\mathrm{U}}) \times_{\mathrm{U}} \mathrm{Spec}(\mathrm{F})$ of affine group schemes over Spec(F).

Lemma 2.2.5. *The functor defined by restriction along* $\mathrm{Spec}(\mathrm{F}) \to \mathrm{U}$*:*

$$\operatorname*{colim}_{(\mathrm{U},\mathrm{G}_{\mathrm{U}},\gamma)} \Gamma_e(\mathrm{BG}_{\mathrm{U}}, \mathrm{B}^4\mathrm{A}(1)) \to \Gamma_e(\mathrm{BG}, \mathrm{B}^4\mathrm{A}(1))$$

is an equivalence.

Proof. The existence of an integral model implies that G is identified with $\lim_{(\mathrm{U},\mathrm{G}_{\mathrm{U}},\gamma)} \mathrm{G}_{\mathrm{U}}$. The assertion of the Lemma, when BG is replaced by G^n (for $[n] \in \Delta^{\mathrm{op}}$), follows from the local finite presentation of the stack of étale local systems ([Sta18, 0GL2]).

Since $\mathrm{B}^4\mathrm{A}(1)$ is 4-truncated, the space of sections $\Gamma_e(\mathrm{BG}, \mathrm{B}^4\mathrm{A}(1))$ is computed by a cosimplicial limit over a finite subcategory of Δ^{op}, so we conclude by the commutation of filtered colimits with finite limits. ☐

2.2.6. Using Lemma 2.2.5, it suffices to construct a functor from $\Gamma_e(\mathrm{BG}_{\mathrm{U}}, \mathrm{B}^4\mathrm{A}(1))$ to the groupoid of central extensions (2.10) equipped with a splitting over $\mathrm{G}(\mathrm{F})^0$, which is moreover natural in the triplet $(\mathrm{U}, \mathrm{G}_{\mathrm{U}}, \gamma)$.

Indeed, we consider a pointed morphism $\mu_{\mathrm{U}} : \mathrm{BG}_{\mathrm{U}} \to \mathrm{B}^4\mathrm{A}(1)$. For any open subscheme $\mathrm{V} \subset \mathrm{U}$, we obtain a central extension of the topological group $\prod_{v\notin \mathrm{V}} \mathrm{G}(\mathrm{F}_v)^0 \times \prod_{v\in\mathrm{V}} \mathrm{G}(\mathscr{O}_v)$ by taking the product of $\tilde{\mathrm{G}}_v$ over $v \notin \mathrm{V}$:

$$\begin{aligned} 0 \to \bigoplus_{v\notin\mathrm{V}} \mathrm{H}^2(\mathrm{F}_v, \mathrm{A}(1)) \to \prod_{v\notin\mathrm{V}} \tilde{\mathrm{G}}_v \times \prod_{v\in\mathrm{V}} \mathrm{G}_{\mathrm{U}}(\mathscr{O}_v) \\ \to \prod_{v\notin\mathrm{V}} \mathrm{G}(\mathrm{F}_v)^0 \times \prod_{v\in\mathrm{V}} \mathrm{G}_{\mathrm{U}}(\mathscr{O}_v) \to 1. \end{aligned}$$

Taking the push-out along the second map in (2.11), we obtain a central extension $\tilde{\mathrm{G}}_{\mathrm{U},\mathrm{V}}$ of $\prod_{v\notin\mathrm{V}} \mathrm{G}(\mathrm{F}_v)^0 \times \prod_{v\in\mathrm{V}} \mathrm{G}_{\mathrm{U}}(\mathscr{O}_v)$ by $\mathrm{H}_0(\mathrm{F}, \mathrm{A})$.

This central extension is canonically split over $\mathrm{G}_{\mathrm{U}}(\mathrm{V})^0$, the kernel of the map $\mathrm{G}_{\mathrm{U}}(\mathrm{V}) \to \mathrm{H}^3(\mathrm{V}, \mathrm{A}(1))$. Indeed, $\mathrm{G}_{\mathrm{U}}(\mathrm{V})^0$ equals the subgroup whose restriction to the real places belong to $\mathrm{G}(\mathrm{F}_v)^0$, using the commutative diagram:

$$\begin{array}{ccc} \mathrm{G}_{\mathrm{U}}(\mathrm{V}) & \longrightarrow & \mathrm{H}^3(\mathrm{V}, \mathrm{A}(1)) \\ \downarrow & & \downarrow \cong \\ \prod_{v \text{ real}} \mathrm{G}(\mathrm{F}_v) & \to & \oplus_{v \text{ real}} \mathrm{H}^3(\mathrm{F}_v, \mathrm{A}(1)) \end{array}$$

where the isomorphism is (2.12). Furthermore, the composition of the first two maps in (2.11) vanishes, so the central extension of $\mathrm{G}_{\mathrm{U}}(\mathrm{V})^0$ by $\mathrm{H}^2(\mathrm{V}, \mathrm{A}(1))$ constructed from the recipe of §2.1 induces the split extension by $\mathrm{H}_0(\mathrm{F}, \mathrm{A})$.

In summary, we obtain a diagram of topological groups:

$$
\begin{array}{ccccccc}
 & & & & \mathrm{G}_{\mathrm{U}}(\mathrm{V})^0 & & \\
 & & & \swarrow & \downarrow & & \\
1 \to \mathrm{H}_0(\mathrm{F}, \mathrm{A}) & \to & \tilde{\mathrm{G}}_{\mathrm{U},\mathrm{V}} & \to & \prod_{v \notin \mathrm{V}} \mathrm{G}(\mathrm{F}_v)^0 \times \prod_{v \in \mathrm{V}} \mathrm{G}_{\mathrm{U}}(\mathscr{O}_v) & \to & 1
\end{array}
\tag{2.13}
$$

The central extension (2.10), together with its splitting over $\mathrm{G}(\mathrm{F})^0$, is then obtained by taking colimit of (2.13) over $\mathrm{V} \subset \mathrm{U}$.

Example 2.2.7. Suppose that $\mathrm{N} \geq 1$ is an integer and S is a $\mathbb{Z}[\frac{1}{\mathrm{N}}]$-scheme. Consider the constant group scheme $\mathrm{G} = \mathrm{SL}_2$ over S. Viewed as an element of the reduced cohomology, the universal second Chern class $[c_2] \in \mathrm{H}^4_e(\mathrm{B}(\mathrm{SL}_2), \mu_{\mathrm{N}}^{\otimes 2})$ lifts to a pointed morphism $c_2 : \mathrm{B}(\mathrm{SL}_2) \to \mathrm{B}^4(\mu_{\mathrm{N}}^{\otimes 2})$; this follows from the vanishing of the reduced cohomology groups in degrees ≤ 3, see Proposition 5.1.11 below.

If $\mathrm{S} = \mathrm{Spec}(\mathrm{F})$ where F is a local or global field containing a primitive Nth root of unity, then c_2 defines an N-fold covering group of $\mathrm{SL}_2(\mathrm{F})$, respectively $\mathrm{SL}_2(\mathbb{A}_{\mathrm{F}})$. This is isomorphic to Kubota's central extension [Kub67].

Remark 2.2.8. More generally, classical examples of covering groups correspond to A being the constant sheaf of roots of unity $\mu(\mathrm{F})$ in a local or global field F (or a subgroup thereof). There is an alternative algebraic description of such covers, given by central extensions of G by the Zariski sheaf $\underline{\mathrm{K}}_2$, see §2.3 below.

In the study of genuine representations of covering groups, one typically fixes a coefficient ring, say $\mathbb{C}$, and an inclusion $\mu(\mathrm{F}) \subset \mathbb{C}^\times$. Our perspective is slightly different: without assuming that F contains enough roots of unity, we may still consider étale metaplectic covers with values in a constant abelian group $\mathrm{A} \subset \mathbb{C}^\times$ of order invertible in F. The covering groups arising this way appear in Kaletha's work on Langlands functoriality [Kal22].

2.3. Integral metaplectic covers.

2.3.1. Suppose that S is a smooth scheme over a field F. Let $\underline{\mathrm{K}}_2$ denote the Zariski sheafification of the functor $\mathrm{R} \mapsto \mathrm{K}_2(\mathrm{R})$ sending a commutative ring to its second algebraic K-group. The Picard groupoid of central extensions of sheaves of groups on the big Zariski site of S:

$$1 \to \underline{\mathrm{K}}_2 \to \mathrm{E} \to \mathrm{G} \to 1 \tag{2.14}$$

is equivalent to $\Gamma_e(\mathrm{BG}_{\mathrm{Zar}}, \mathrm{B}^2\underline{\mathrm{K}}_2)$, where $\mathrm{BG}_{\mathrm{Zar}}$ stands for the stack of Zariski locally trivial G-torsors and $\mathrm{B}^2\underline{\mathrm{K}}_2$ is the delooping of $\underline{\mathrm{K}}_2$ as a *Zariski sheaf*.

If $\mathrm{S} = \mathrm{Spec}(\mathrm{F})$ for a local or global field containing a primitive Nth root of unity, [BD01, §10] explains how to attach to (2.14) a topological central extension of $\mathrm{G}(\mathrm{F})$ (respectively $\mathrm{G}(\mathbb{A}_{\mathrm{F}})$) by $\mu_{\mathrm{N}}(\mathrm{F})$.

We shall show that when G is reductive, this process factors through the space of étale metaplectic covers of G with values in $\mathrm{A} := \mu_{\mathrm{N}}$.

2.3.2. Let F be a field and $\mathrm{Sm}_{/\mathrm{F}}$ be the category of smooth F-schemes. In this context, we shall construct a functor for any reductive group scheme $\mathrm{G} \to \mathrm{S}$ and $\mathrm{S} \in \mathrm{Sm}_{/\mathrm{F}}$:

$$\Gamma_e(\mathrm{BG}_{\mathrm{Zar}}, \mathrm{B}^2\underline{\mathrm{K}}_2) \to \Gamma_e(\mathrm{BG}, \mathrm{B}^4\mu_{\mathrm{N}}^{\otimes 2}). \tag{2.15}$$

The Galois symbol defines a morphism from $\underline{\mathrm{K}}_2$ to the Zariski sheafification of the presheaf $\mathrm{R} \mapsto \mathrm{H}^2_{\text{ét}}(\mathrm{R}, \mu_{\mathrm{N}}^{\otimes 2})$. However, this map does not lift to a morphism $\underline{\mathrm{K}}_2 \to \mathrm{B}^2\mu_{\mathrm{N}}^{\otimes 2}$.

The idea to circumvent this difficulty, due to Gaitsgory [Gai20, §6], is to interpret the left-hand-side of (2.15) as sections of the motivic complex $\mathbb{Z}(2)[4]$ over BG, which then admits an étale realization given by "mod N".

2.3.3. We temporarily assume that F is *perfect*. For $n \ge 0$, we denote by $\mathbb{Z}(n)$ the complex of presheaves of abelian groups on $\mathrm{Sm}_{/\mathrm{F}}$ defined in [MVW06, §3].

To briefly recall its construction, we write $\mathbb{Z}_{\mathrm{tr}}(\mathrm{X})$ (for $\mathrm{X} \in \mathrm{Sm}_{/\mathrm{F}}$) for the presheaf whose Y-points are finite correspondences from Y to X, i.e. the $\mathbb{Z}$-span of integral subschemes $\mathrm{Y}_1 \subset \mathrm{Y} \times \mathrm{X}$ such that $\mathrm{Y}_1 \to \mathrm{Y}$ is finite and surjective (called an "elementary correspondence"). Denote by $\mathbb{Z}_{\mathrm{tr}}(\wedge^n \mathbb{G}_m)$ the cokernel:

$$\bigoplus_{1 \le i \le n} \mathbb{Z}_{\mathrm{tr}}(\mathbb{G}_m \times \cdots \times \{1\} \times \cdots \times \mathbb{G}_m) \to \mathbb{Z}_{\mathrm{tr}}(\mathbb{G}_m^{\times n}) \to \mathbb{Z}_{\mathrm{tr}}(\wedge^n \mathbb{G}_m) \to 0.$$

The inclusion of $\mathbb{A}^1$-invariant complexes of presheaves into all complexes of presheaves on $\mathrm{Sm}_{/\mathrm{F}}$ admits a left adjoint $\mathrm{L}_{\mathbb{A}^1}$, and we set:

$$\mathbb{Z}(n) := \mathrm{L}_{\mathbb{A}^1} \mathbb{Z}_{\mathrm{tr}}(\wedge^n \mathbb{G}_m)[-n].$$

Then $\mathbb{Z}(n)$ is a complex of étale sheaves of abelian groups on $\mathrm{Sm}_{/\mathrm{F}}$ [MVW06, Corollary 6.4].

Remark 2.3.4. The complexes $\mathbb{Z}(n)$ are concentrated in cohomological degrees $\le n$, but are in general not left-bounded if $n \ge 2$.

Clearly $\mathbb{Z}(0) \cong \mathbb{Z}$. There is also an equivalence $\mathbb{Z}(1) \cong \mathbb{G}_m[-1]$ induced from the natural map $\mathbb{Z}_{\mathrm{tr}}(\mathbb{G}_m) \to \mathbb{G}_m$ sending an elementary correspondence $\mathrm{Y}_1 \subset \mathrm{Y} \times \mathbb{G}_m$ to the norm of the corresponding element of $\mathscr{O}_{\mathrm{Y}_1}^{\times}$ relative to $\mathrm{Y}_1 \to \mathrm{Y}$, see [MVW06, §4].

2.3.5. We shall construct a morphism of complexes of Zariski sheaves on $\mathrm{Sm}_{/\mathrm{F}}$:

$$\mathbb{Z}(n)[n] \to \underline{\mathrm{K}}_n. \tag{2.16}$$

Let us start with a map $\mathbb{Z}_{\mathrm{tr}}(\mathbb{G}_m^{\times n}) \to \underline{\mathrm{K}}_n$ of presheaves. For an elementary correspondence $\mathrm{Y}_1 \subset \mathrm{Y} \times \mathbb{G}_m^{\times n}$ corresponding to $f_1, \cdots, f_n \in \mathscr{O}_{\mathrm{Y}_1}^{\times}$, their norms define $\mathrm{Nm}(f_1), \cdots, \mathrm{Nm}(f_n) \in \mathscr{O}_{\mathrm{Y}}^{\times}$ and we form $\mathrm{Nm}(f_1) \otimes \cdots \otimes \mathrm{Nm}(f_n)$ using the product structure $\underline{\mathrm{K}}_1^{\otimes n} \to \underline{\mathrm{K}}_n$.

This map evidently factors through $\mathbb{Z}_{\mathrm{tr}}(\wedge^n \mathbb{G}_m)$. Since $\underline{\mathrm{K}}_n$ is $\mathbb{A}^1$-invariant on $\mathrm{Sm}_{/\mathrm{F}}$ [She79, Corollary 2.5], we obtain (2.16) by adjunction.

2.3.6. For a smooth affine group scheme $\mathrm{G} \to \mathrm{S}$, we write:

$$\Gamma(\mathrm{BG}, \mathbb{Z}(n)) := \lim_{[k] \in \Delta^{\mathrm{op}}} \Gamma(\mathrm{G}^{\times k}, \mathbb{Z}(n)), \tag{2.17}$$

as a complex of abelian groups. Here, the sections $\Gamma(\mathrm{G}^{\times k}, \mathbb{Z}(n))$ are computed by treating $\mathbb{Z}(n)$ as a complex of *étale* sheaves.

The corresponding Zariski version will be written as $\Gamma(\mathrm{BG}_{\mathrm{Zar}}, \mathbb{Z}(n))$. There is also the pointed version $\Gamma_e(\mathrm{BG}, \mathbb{Z}(n))$, i.e. the fiber along $e^* : \Gamma(\mathrm{BG}, \mathbb{Z}(n)) \to \Gamma(\mathrm{S}, \mathbb{Z}(n))$, and likewise for $\Gamma_e(\mathrm{BG}_{\mathrm{Zar}}, \mathbb{Z}(n))$.

2.3.7. Comparison of sites and the functor (2.16) induce two morphisms of complexes of abelian groups:

$$\Gamma_e(\mathrm{BG}_{\mathrm{Zar}}, \mathbb{Z}(2)[4]) \to \Gamma_e(\mathrm{BG}, \mathbb{Z}(2)[4]), \tag{2.18}$$

$$\Gamma_e(\mathrm{BG}_{\mathrm{Zar}}, \mathbb{Z}(2)[4]) \to \Gamma_e(\mathrm{BG}_{\mathrm{Zar}}, \underline{\mathrm{K}}_2[2]). \tag{2.19}$$

The following result shows that when $\mathrm{G} \to \mathrm{S}$ is reductive, central extensions of G by $\underline{\mathrm{K}}_2$ are equivalent to weight-2 motivic 4-cocycles on BG, computed in either the Zariski or the étale topology.

Our statement is more general than its counterparts in [EKLV98, §6] and [Gai20, §6.3-6.4], although the proof is essentially the same.

Theorem 2.3.8. *Let* F *be a perfect field. Suppose* $\mathrm{S} \in \mathrm{Sm}_{/\mathrm{F}}$ *and* $\mathrm{G} \to \mathrm{S}$ *is a reductive group scheme. Then both morphisms* (2.18), (2.19) *induce equivalences in degrees* ≤ 0.

2.3.9. Our strategy is to first prove that (2.18) induces an equivalence in degrees ≤ 0, which will imply that the association $\mathrm{S}_1 \mapsto \tau^{\leq 0}\Gamma_e(\mathrm{B}(\mathrm{G}_{\mathrm{S}_1})_{\mathrm{Zar}}, \mathbb{Z}(2)[4])$, where $\mathrm{G}_{\mathrm{S}_1} := \mathrm{G} \times_{\mathrm{S}} \mathrm{S}_1$, is an étale sheaf on smooth S-schemes.

Since $\mathrm{S}_1 \mapsto \tau^{\leq 0}\Gamma_e(\mathrm{B}(\mathrm{G}_{\mathrm{S}_1})_{\mathrm{Zar}}, \underline{\mathrm{K}}_2[2])$ also satisfies étale descent [BD01, §2], the assertion for (2.19) reduces to the case of a *split* reductive group G, for which we will prove an equivalence:

$$\Gamma_e(\mathrm{BG}_{\mathrm{Zar}}, \mathbb{Z}(2)[4]) \cong \Gamma_e(\mathrm{BG}_{\mathrm{Zar}}, \underline{\mathrm{K}}_2[2]),$$

using the rationality of G provided by the Bruhat decomposition.

Proof of Theorem 2.3.8. To prove that (2.18) induces an equivalence in degrees ≤ 0, it suffices to prove that for any smooth affine group scheme G, the analogous map:

$$\Gamma_e(\mathrm{G}_{\mathrm{Zar}}, \mathbb{Z}(2)) \to \Gamma_e(\mathrm{G}, \mathbb{Z}(2)) \tag{2.20}$$

induces isomorphisms on H^i for $i \leq 3$.

Indeed, the presentation (2.17) implies:

$$\Gamma_e(\mathrm{BG}, \mathbb{Z}(2)) \cong \lim_{[k]\in\Delta^{\mathrm{op}}} \Gamma_e(\mathrm{G}^{\times k}, \mathbb{Z}(2)),$$

so we obtain a spectral sequence $\mathrm{E}_1^{p,q} = \mathrm{H}_e^p(\mathrm{G}^{\times q}, \mathbb{Z}(2)) \Rightarrow \mathrm{H}_e^{p+q}(\mathrm{BG}, \mathbb{Z}(2))$ and analogously for the Zariski version $\mathrm{E}_{1,\mathrm{Zar}}^{p,q} = \mathrm{H}_e^p(\mathrm{G}_{\mathrm{Zar}}^{\times q}, \mathbb{Z}(2))$. Since $\mathrm{E}_{1,\mathrm{Zar}}^{4,0} \cong \mathrm{E}_1^{4,0} \cong 0$, the assertion on (2.20) implies that $\mathrm{E}_{1,\mathrm{Zar}}^{p,q} \cong \mathrm{E}_1^{p,q}$ for $p \leq 3$, which is sufficient to guarantee $\mathrm{H}_e^n(\mathrm{BG}_{\mathrm{Zar}}, \mathbb{Z}(2)) \cong \mathrm{H}_e^n(\mathrm{BG}, \mathbb{Z}(2))$ for $n \leq 4$.

The assertion on (2.20) in turn follows from Kahn's calculation of weight-2 motivic cohomology [Kah96, Theorem 1.1 & 1.6]. Namely, for a connected smooth scheme X over F, there hold:

$$\mathrm{H}^i(\mathrm{X}_{\mathrm{Zar}}, \mathbb{Z}(2)) \cong \mathrm{H}^i(\mathrm{X}, \mathbb{Z}(2)) \cong \begin{cases} 0 & i = 0 \\ \mathrm{K}_3(k(\mathrm{X}))_{\mathrm{ind}} & i = 1 \\ \mathrm{H}^0(\mathrm{X}_{\mathrm{Zar}}, \underline{\mathrm{K}}_2) & i = 2 \\ \mathrm{H}^1(\mathrm{X}_{\mathrm{Zar}}, \underline{\mathrm{K}}_2) & i = 3 \end{cases} \tag{2.21}$$

Here, $\mathrm{K}_3(k(\mathrm{X}))_{\mathrm{ind}}$ denotes the "indecomposable part" of $\mathrm{K}_3(k(\mathrm{X}))$, i.e. the quotient of the Quillen K_3 by the Milnor K_3 of the field $k(\mathrm{X})$. Note that Theorem 1.6 of *op.cit.* furthermore proves:

$$\mathrm{H}^i(\mathrm{X}_{\mathrm{Zar}}, \mathbb{Z}(2)) \cong 0, \quad i \geq 4. \tag{2.22}$$

We now turn to the proof that (2.19) induces an equivalence in degrees ≤ 0. By what we said in §2.3.9, it suffices to prove that for a *split* reductive group scheme $\mathrm{G} \to \mathrm{S}$, the map $\mathbb{Z}(2)[2] \to \underline{\mathrm{K}}_2$ of (2.16) induces an equivalence:

$$\Gamma_e(\mathrm{G}_{\mathrm{Zar}}, \mathbb{Z}(2)[2]) \cong \Gamma_e(\mathrm{G}_{\mathrm{Zar}}, \underline{\mathrm{K}}_2).$$

The calculations (2.21), (2.22) imply that the restriction of $\mathbb{Z}(2)[2] \to \underline{\mathrm{K}}_2$ to the small Zariski site of X has fiber being the (shifted) constant sheaf $\mathrm{K}_3(k(\mathrm{X}))_{\mathrm{ind}}[1]$. We thus obtain a morphism of triangles:

$$\begin{array}{ccccc} \mathrm{K}_3(k(\mathrm{G}))_{\mathrm{ind}}[1] & \longrightarrow & \Gamma(\mathrm{G}_{\mathrm{Zar}}, \mathbb{Z}(2)[2]) & \longrightarrow & \Gamma(\mathrm{G}_{\mathrm{Zar}}, \underline{\mathrm{K}}_2) \\ \downarrow{\scriptstyle e^*} & & \downarrow{\scriptstyle e^*} & & \downarrow{\scriptstyle e^*} \\ \mathrm{K}_3(k(\mathrm{S}))_{\mathrm{ind}}[1] & \longrightarrow & \Gamma(\mathrm{S}_{\mathrm{Zar}}, \mathbb{Z}(2)[2]) & \longrightarrow & \Gamma(\mathrm{S}_{\mathrm{Zar}}, \underline{\mathrm{K}}_2) \end{array} \tag{2.23}$$

It suffices to prove that the leftmost vertical arrow is an isomorphism. Since $\mathrm{G} \to \mathrm{S}$ is split reductive, it is birational to $\mathbb{A}^n_{\mathrm{S}}$ for some $n \geq 0$. Thus it suffices to prove the equivalence:

$$\mathrm{K}_3(k(\mathbb{A}^n_{\mathrm{S}})_{\mathrm{ind}}) \cong \mathrm{K}_3(k(\mathrm{S}))_{\mathrm{ind}}.$$

For this statement, we may return to (2.23) with $\mathrm{G} = \mathbb{A}^n_{\mathrm{S}}$, for which the middle and rightmost vertical arrows are equivalences, by the $\mathbb{A}^1$-invariance of motivic cohomology [MVW06, Theorem 13.8, Proposition 13.9] and $\underline{\mathrm{K}}_2$-cohomology [She79, Corollary 2.5]. □

Remark 2.3.10. Bryslinki–Deligne [BD01, Theorem 7.2] gives a complete description of the groupoid $\Gamma_e(\mathrm{BG}_{\mathrm{Zar}}, \mathrm{B}^2\underline{\mathrm{K}}_2)$. Theorem 2.3.8 implies that the same description is valid for the groupoids associated to $\tau^{\leq 0}\Gamma_e(\mathrm{BG}, \mathbb{Z}(2)[4])$ and $\tau^{\leq 0}\Gamma_e(\mathrm{BG}_{\mathrm{Zar}}, \mathbb{Z}(2)[4])$. In particular, they are 1-truncated.

If $\mathrm{S} = \mathrm{Spec}(\mathrm{F})$, we may choose a maximal torus $\mathrm{T} \subset \mathrm{G}$ with cocharacter lattice Λ_{T}. Then $\Gamma_e(\mathrm{BG}_{\mathrm{Zar}}, \mathrm{B}^2\underline{\mathrm{K}}_2)$ maps to the abelian group of Weyl and Galois-invariant quadratic forms on Λ_{T}, with fiber being $\Gamma(\mathrm{F}, \underline{\mathrm{Hom}}(\pi_1\mathrm{G}, \mathbb{G}_m[1]))$, where $\pi_1\mathrm{G}$ denotes the algebraic fundamental group of G. For semisimple G, the corresponding description of $\tau^{\leq 0}\Gamma_e(\mathrm{BG}, \mathbb{Z}(2)[4])$ has also been obtained by Merkurjev [Mer16, Theorem 5.3].

2.3.11. Let us use Theorem 2.3.8 to define the functor (2.15). Denote by F_1 the colimit perfection of F, i.e.

$$\mathrm{F}_1 \coloneqq \operatorname*{colim}_{x \mapsto x^p}(\mathrm{F}).$$

Then $\mathrm{Spec}(\mathrm{F}_1) \to \mathrm{Spec}(\mathrm{F})$ is a universal homeomorphism [Sta18, 0CNF]. Write S_1 (resp. G_1) for the base change of S (resp. G) along this map. Topological invariance of the étale site [Sta18, 04DZ] then implies that the pullback functor defines an equivalence:

$$\Gamma_e(\mathrm{BG}, \mathrm{B}^4\mu_{\mathrm{N}}^{\otimes 2}) \cong \Gamma_e(\mathrm{BG}_1, \mathrm{B}^4\mu_{\mathrm{N}}^{\otimes 2}).$$

Consequently, in order to define (2.15) we may assume that F is perfect. The desired functor is then given by composing the equivalences of Theorem 2.3.8 with the identification $\mathbb{Z}(2)/\mathrm{N} \cong \mu_{\mathrm{N}}^{\otimes 2}$ [MVW06, Theorem 10.3]:

$$\begin{aligned} \Gamma_e(\mathrm{BG}, \mathrm{B}^2\underline{\mathrm{K}}_2) &\cong \tau^{\leq 0}\Gamma_e(\mathrm{BG}_{\mathrm{Zar}}, \mathbb{Z}(2)[4]) \\ &\cong \tau^{\leq 0}\Gamma_e(\mathrm{BG}, \mathbb{Z}(2)[4]) \to \tau^{\leq 0}\Gamma_e(\mathrm{BG}, \mathbb{Z}(2)/\mathrm{N}[4]) \cong \Gamma_e(\mathrm{BG}, \mathrm{B}^4\mu_{\mathrm{N}}^{\otimes 2}). \end{aligned}$$

2.3.12. Suppose that F is a local field containing a primitive Nth root of unity. From any object of $\Gamma_e(\mathrm{BG}, \mathrm{B}^2\underline{\mathrm{K}}_2)$, one may construct a central extension of topological groups:

$$1 \to \mu_{\mathrm{N}}(\mathrm{F}) \to \tilde{\mathrm{G}} \to \mathrm{G}(\mathrm{F}) \to 1. \tag{2.24}$$

Namely, viewing this object as a central extension of G by $\underline{\mathrm{K}}_2$, we may evaluate at F and take the pushout along the Hilbert symbol $\mathrm{K}_2(\mathrm{F}) \to \mu_{\mathrm{N}}(\mathrm{F})$. The topology on $\tilde{\mathrm{G}}$ is constructed out of the corresponding central extension of $\mathrm{G}(\mathrm{F})$ by $\mathrm{H}^2(\mathrm{F}, \mu_{\mathrm{N}}^{\otimes 2})$ defined by the Galois symbol, see [BD01, §9].

Similarly, when F is a global field containing a primitive Nth root of unity, we obtain a central extension of $G(\mathbb{A}_F)$ by $\mu_N(F)$ as topological groups, equipped with a canonical splitting over $G(F)$. It is defined as the colimit, over $V \subset U \subset \mathrm{Spec}(\mathscr{O}_F)$, of extensions defined for an integral model G_U of G over U:

$$1 \to \mu_N(F) \to \tilde{G}_{U,V} \to \prod_{v \notin V} G(F_v) \times \prod_{v \in V} G_U(\mathscr{O}_v) \to 1,$$

which are in turn induced from the local extensions by the sum of the Hilbert symbol maps $\bigoplus_{v \notin V} K_2(F_v) \to \mu_N(F_v) \cong \mu_N(F)$, see *op.cit.* for details.

We shall now compare this construction with the ones in §2.1-2.2.

Proposition 2.3.13. *Suppose that* F *is a local field containing a primitive* N*th root of unity. Let* G *be a reductive group scheme over* Spec(F). *The following diagram is canonically commutative:*

$$\begin{array}{ccc} \Gamma_e(BG_{Zar}, B^2\underline{K}_2) & \searrow & \\ \Big\downarrow{\scriptstyle (2.15)} & & \left\{ \begin{array}{c} \text{topological covers of} \\ G(F) \text{ by } \mu_N(F) \end{array} \right\}. \\ \Gamma_e(BG, B^4\mu_N^{\otimes 2})^0 & \nearrow & \end{array} \tag{2.25}$$

where $\Gamma_e(BG, B^4\mu_N^{\otimes 2})^0 := \Gamma_e(BG, B^4\mu_N^{\otimes 2})$ *for* F *nonarchimedean or complex, and is the full subgroupoid for which the induced map* $G(F) \to H^3(F, \mu_N^{\otimes 2})$ (2.8) *vanishes for* F *real.*

The analogous statement holds for global fields F *containing a primitive* N*th root of unity, where one replaces the target of* (2.25) *by topological covers of* $G(\mathbb{A}_F)$ *by* $\mu_N(F)$*, equipped with a splitting over* $G(F)$.

Proof. The global statement follows from the local one. For the local statement, part of the assertion is that for F real, the image of (2.15) consists only of pointed morphisms $BG \to B^4\mu_N^{\otimes 2}$ whose induced map $G(F) \to H^3(F, \mu_N^{\otimes 2})$ vanishes. By construction, this map factors through $G(F) \to H^3(F_{Zar}, \mathbb{Z}(2))$, which vanishes because $\mathbb{Z}(2)$ is concentrated in cohomological degrees ≤ 2. (We write F_{Zar} as a shorthand for $\mathrm{Spec}(F)_{Zar}$.)

The commutativity of (2.25) unwinds into the commutativity of the square below:

$$\begin{array}{ccc} H^2(F_{Zar}, \mathbb{Z}(2)) & \xrightarrow{(2.16)} & K_2(F) \\ \Big\downarrow & \swarrow{\scriptstyle \text{Galois}} & \Big\downarrow{\scriptstyle \text{Hilbert}} \\ H^2(F, \mu_N^{\otimes 2}) & \xrightarrow{\text{Tate}} & \mu_N(F) \end{array}$$

This square is divided into two commutive triangles: the top arrow reduces to the map θ of [MVW06, §5] which defines the "inverse" $H^2(F, \mu_N^{\otimes 2}) \to K_2(F)/N$ of the Galois symbol. On the other hand, the Galois symbol, followed by the Tate duality map, coincides with the Nth Hilbert symbol. □

3. $\{a, a\} \cong \{a, -1\}$

From this section until §5, our goal is to describe the groupoid $\Gamma_e(\mathrm{BG}, \mathrm{B}^4\mathrm{A}(1))$ in explicit terms. This will rely on the description for a torus $\mathrm{T} \to \mathrm{S}$, which in turn requires us to study étale cochains on $\mathbb{G}_m$.

In this section, we construct a canonical isomorphism (Theorem 3.1.5) between two étale 2-cocycles on $\mathbb{G}_m$ rigidified along $e : \mathrm{S} \to \mathbb{G}_m$: the first one is the self-cup product $\Psi \cup \Psi$ of the Kummer 1-cocycle $\Psi : \mathbb{G}_m \to \mathrm{B}(\mu_{\mathrm{N}})$, and the second one is the Yoneda product of Ψ with $\Psi(-1)$, the value of Ψ at $-1 \in \mathbb{G}_m$.

The construction we give is surprisingly subtle, and the author would like to know if there is a more direct way of obtaining this isomorphism.

3.0.1. Throughout this section, we fix an integer $\mathrm{N} \geq 1$ and a $\mathbb{Z}[\frac{1}{\mathrm{N}}]$-scheme S.

3.1. 2-cocycles on $\mathbb{G}_m$.

3.1.1. We view $\mathbb{G}_m$ as a constant group scheme over S with structure map $p : \mathbb{G}_m \to \mathrm{S}$. Let $\underline{\Gamma}_e(\mathbb{G}_m, \mathrm{B}^2\mu_{\mathrm{N}}^{\otimes 2})$ denote the étale sheaf of groupoids over S whose sections over $\mathrm{S}_1 \to \mathrm{S}$ consist of sections of $\mathrm{B}^2(\mu_{\mathrm{N}}^{\otimes 2})$ over $\mathbb{G}_{m,\mathrm{S}_1} := \mathbb{G}_m \times_{\mathrm{S}} \mathrm{S}_1$ rigidified along $e : \mathrm{S}_1 \to \mathbb{G}_{m,\mathrm{S}_1}$.

Let $\Psi : \mathbb{G}_m \to \mathrm{B}(\mu_{\mathrm{N}})$ denote the Kummer torsor. The Yoneda product with Ψ defines a $\mathbb{Z}/\mathrm{N}$-linear morphism:

$$\begin{aligned} \Psi^* : \mathrm{B}(\mu_{\mathrm{N}}) &\to \underline{\mathrm{Maps}}_{\mathbb{Z}}(\mathbb{G}_m, \mathrm{B}^2\mu_{\mathrm{N}}^{\otimes 2}) \\ &\to \underline{\Gamma}_e(\mathbb{G}_m, \mathrm{B}^2\mu_{\mathrm{N}}^{\otimes 2}) \end{aligned} \tag{3.1}$$

where the second functor is the forgetful one.

Remark 3.1.2. The sheaf of $\mathbb{Z}$-linear maps $\mathbb{G}_m \to \mathrm{B}^2\mu_{\mathrm{N}}^{\otimes 2}$ is identified with the connective truncation of the internal Hom of complexes $\underline{\mathrm{Hom}}_{\mathbb{Z}}(\mathbb{G}_m, \mu_{\mathrm{N}}^{\otimes 2}[2])$, although we shall see in the proof of Lemma 3.1.3 that the latter is already connective.

Lemma 3.1.3. *Both functors in* (3.1) *are equivalences.*

Proof. To see that the first functor is an equivalence, it suffices to prove the analogous statement for $\Psi^* : \mathrm{B}(\mathbb{Z}/\mathrm{N}) \to \underline{\mathrm{Maps}}_{\mathbb{Z}}(\mathbb{G}_m, \mathrm{B}^2\mu_{\mathrm{N}})$. This follows at once after replacing μ_{N} in the target by the complex $\mathbb{G}_m \xrightarrow{\mathrm{N}} \mathbb{G}_m$.

The sheaf of $\mathbb{Z}/\mathrm{N}$-module spectra $\underline{\Gamma}_e(\mathbb{G}_m, \mathrm{B}^2\mu_{\mathrm{N}}^{\otimes 2})$ corresponds to the following complex of sheaves of $\mathbb{Z}/\mathrm{N}$-modules on S, in the sense of §1.3:

$$\mathrm{R}\tilde{p}_*(\mu_{\mathrm{N}}^{\otimes 2}[2]) := \mathrm{Fib}(\mathrm{R}p_*(\mu_{\mathrm{N}}^{\otimes 2}[2]) \to \mu_{\mathrm{N}}^{\otimes 2}[2]).$$

It suffices to prove that the map $\mu_{\mathrm{N}}[1] \to \mathrm{R}\tilde{p}_*(\mu_{\mathrm{N}}^{\otimes 2}[2])$ induced from Ψ is an isomorphism of complexes.

Since the étale cohomology of $\mathbb{G}_m \to \mathrm{S}$ commutes with arbitrary base change ([Del96, Rappel 1.5.1]), we may replace S by the spectrum of a separably closed field $\bar{s}$, where:

$$\mathrm{H}^i(\mathbb{G}_{m,\bar{s}}, \mu_{\mathrm{N}}^{\otimes 2}) \cong \begin{cases} \mu_{\mathrm{N}} & i = 1 \\ 0 & i \geq 2 \end{cases}$$

and the identification of H^1 is indeed induced from Ψ. □

3.1.4. Let us consider the self-cup product of Ψ:

$$\Psi \cup \Psi : \mathbb{G}_m \to \mathrm{B}^2(\mu_{\mathrm{N}}^{\otimes 2}), \tag{3.2}$$

which may be viewed as a section of $\underline{\Gamma}_e(\mathbb{G}_m, \mathrm{B}^2\mu_{\mathrm{N}}^{\otimes 2})$. Anti-symmetry of the cup product equips $\Psi \cup \Psi$ with a 2-torsion structure, i.e. a trivialization $2 \cdot (\Psi \cup \Psi) \cong *$.

Under the equivalences of Lemma 3.1.3, $\Psi \cup \Psi$ corresponds to a section of $\mathrm{B}(\mu_{\mathrm{N}})$ over S. Our goal is to determine this section.

To state the answer, we consider the section $\Psi(-1)$ of $\mathrm{B}(\mu_{\mathrm{N}})$, defined as the value of Ψ at $-1 \in \mathbb{G}_m$. It has a natural 2-torsion structure: linearity of Ψ yields $2 \cdot \Psi(-1) \cong \Psi(1)$ and 1 admits an Nth root $1^{\mathrm{N}} = 1$. Of course, if N is odd, then any 2-torsion section of $\mathrm{B}(\mu_{\mathrm{N}})$ is canonically trivial.

Theorem 3.1.5. *There is a canonical isomorphism:*

$$\Psi \cup \Psi \cong \Psi^*(\Psi(-1)) \tag{3.3}$$

in $\underline{\Gamma}_e(\mathbb{G}_m, \mathrm{B}^2\mu_{\mathrm{N}}^{\otimes 2})$ *compatible with the* 2*-torsion structures.*

Remark 3.1.6. The isomorphism (3.3) is an analogue of an equality in the second algebraic K-group. Indeed, suppose that $\mathrm{S} = \mathrm{Spec}(\mathrm{R})$. Given a section of $\mathbb{G}_m$ over S, represented by $a \in \mathrm{R}^\times$, there holds $\{a, a\} = \{a, -1\}$ in $\mathrm{K}_2(\mathrm{R})$.

If the base scheme S is smooth over a field, we can use this equality to construct an isomorphism $\Psi \cup \Psi \cong \Psi^*(\Psi(-1))$ as follows. In the proof of Theorem 2.3.8, we have constructed a morphism:

$$\underline{\Gamma}_e(\mathbb{G}_{m,\mathrm{Zar}}, \underline{\mathrm{K}}_2) \to \underline{\Gamma}_e(\mathbb{G}_m, \mathrm{B}^2\mu_{\mathrm{N}}^{\otimes 2}). \tag{3.4}$$

Let a denote the natural coordinate on $\mathbb{G}_m$. The image of $\{a, a\}$ under (3.4) is $\Psi \cup \Psi$, while that of $\{a, -1\}$ is $\Psi \cup p^*\Psi(-1)$ which is canonically isomorphic to $\Psi^*(\Psi(-1))$.

For our applications, we wish to construct (3.3) without any assumption on S. We have not checked that the isomorphism we will eventually construct coincides with the one from K-theory when S is smooth over a field.

3.2. Self-cup product.

3.2.1. Let us first explain the construction of (3.2) in more details. In particular, it will help to directly realize $\Psi \cup \Psi$ as a $\mathbb{Z}$-linear morphism $\mathbb{G}_m \to \mathrm{B}^2\mu_{\mathrm{N}}^{\otimes 2}$.

Let A be a sheaf of abelian groups on a site $\mathscr{C}$. Write $\mathrm{H}^{(1)}(\mathrm{A})$ for the extension of sheaves of abelian groups:

$$0 \to \mathrm{Sym}^2(\mathrm{A}) \to \mathrm{H}^{(1)}(\mathrm{A}) \to \mathrm{A} \to 0, \tag{3.5}$$

defined by the cocycle $a_1, a_2 \mapsto a_1 a_2$. Namely, there is a given set-theoretic splitting $\mathrm{A} \to \mathrm{H}^{(1)}(\mathrm{A})$, under which the product of the images of $a_1, a_2 \in \mathrm{A}$ in $\mathrm{H}^{(1)}(\mathrm{A})$ is the $(a_1 a_2)$-multiple of the image of $a_1 + a_2 \in \mathrm{A}$.

3.2.2. The relevant case for us is $\mathrm{A} = \mu_{\mathrm{N}}$ on the étale site of S. There holds $\mu_{\mathrm{N}}^{\otimes 2} \cong \mathrm{Sym}^2(\mu_{\mathrm{N}})$, so (3.5) defines a $\mathbb{Z}$-linear coboundary map:

$$\mathrm{H}^{(1)}(\mu_{\mathrm{N}}) : \mu_{\mathrm{N}} \to \mathrm{B}(\mu_{\mathrm{N}}^{\otimes 2}). \tag{3.6}$$

We write $\Psi \cup \Psi := \Psi^*\mathrm{H}^{(1)}(\mu_{\mathrm{N}})$ for the composition of Ψ with (3.6): it is naturally a section of $\underline{\mathrm{Maps}}_{\mathbb{Z}}(\mathbb{G}_m, \mathrm{B}^2\mu_{\mathrm{N}}^{\otimes 2})$.

Remark 3.2.3. The coboundary of (3.5) indeed encodes cup product on cohomology, i.e. for any object $c \in \mathscr{C}$, the coboundary $\mathrm{B}(\mathrm{A}) \to \mathrm{B}^2(\mathrm{Sym}^2(\mathrm{A}))$ induces the composition of:

$$\mathrm{H}^1(c, \mathrm{A}) \to \mathrm{H}^2(c, \mathrm{A}^{\otimes 2}), \quad x \mapsto x \cup x$$

with the natural map $\mathrm{H}^2(c, \mathrm{A}^{\otimes 2}) \to \mathrm{H}^2(c, \mathrm{Sym}^2(\mathrm{A}))$, see [PR11, Theorem 2.5].

3.2.4. Maps out of $\mathrm{H}^{(1)}(\mathrm{A})$ have a pleasant description as "quadratic functions without constant terms".

More precisely, let A_1 be a sheaf of abelian groups on $\mathscr{C}$. Write $\check{\mathrm{H}}^{(1)}(\mathrm{A},\mathrm{A}_1)$ for the sheaf of abelian groups whose sections are maps $\mathrm{Q}:\mathrm{A}\to\mathrm{A}_1$ satisfying the property that $a_1,a_2\mapsto \mathrm{Q}(a_1+a_2)-\mathrm{Q}(a_1)-\mathrm{Q}(a_2)$ defines a symmetric bilinear form on A. We refer to it as the symmetric form associated to Q.

Then there is a canonical isomorphism defined by restriction along the set-theoretic section $\mathrm{A}\subset\mathrm{H}^{(1)}(\mathrm{A})$:

$$\underline{\mathrm{Maps}}_{\mathbb{Z}}(\mathrm{H}^{(1)}(\mathrm{A}),\mathrm{A}_1)\cong\check{\mathrm{H}}^{(1)}(\mathrm{A},\mathrm{A}_1).$$

Under this bijection, restriction along $\mathrm{Sym}^2(\mathrm{A})\subset\mathrm{H}^{(1)}(\mathrm{A})$ corresponds to taking the *negative* of the symmetric form associated to a quadratic function $\mathrm{Q}:\mathrm{A}\to\mathrm{A}_1$.

3.2.5. The 2-torsion structure of self-cup product is encoded by a $\mathbb{Z}$-linear trivialization of the map:

$$2\cdot\mathrm{H}^{(1)}(\mathrm{A}):\mathrm{A}\to\mathrm{B}(\mathrm{Sym}^2(\mathrm{A})),$$

which is in turn defined by the quadratic function $\mathrm{A}\to\mathrm{Sym}^2(\mathrm{A})$, $a\mapsto -a^2$.

In particular, we obtain a trivialization of $2\cdot\Psi\cup\Psi$ as a section of $\underline{\mathrm{Maps}}_{\mathbb{Z}}(\mathbb{G}_m,\mathrm{B}^2\mu_{\mathrm{N}}^{\otimes 2})$.

3.3. The case $\mathrm{A}=\mu_{\mathrm{N}}$.

3.3.1. Let us now specialize (3.5) to the sheaf $\mathrm{A}:=\mu_{\mathrm{N}}$ on the étale site of S. For any homomorphism $\lambda:\mu_{\mathrm{N}}^{\otimes 2}\to\mathbb{G}_m$ (necessarily valued in $\mu_{\mathrm{N}}\subset\mathbb{G}_m$), we may induce (3.5) along λ to obtain a section $\lambda_*\mathrm{H}^{(1)}(\mu_{\mathrm{N}})\in\underline{\mathrm{Maps}}_{\mathbb{Z}}(\mu_{\mathrm{N}},\mathrm{B}\mu_{\mathrm{N}})$.

Resolving the target $\mathrm{B}(\mu_{\mathrm{N}})$ by the Kummer sequence yields a split triangle of sheaves of $\mathbb{Z}$-module spectra:

$$\underline{\mathrm{Maps}}_{\mathbb{Z}}(\mu_{\mathrm{N}},\mathbb{G}_m)\xrightarrow{\Psi_*}\underline{\mathrm{Maps}}_{\mathbb{Z}}(\mu_{\mathrm{N}},\mathrm{B}\mu_{\mathrm{N}})\to\underline{\mathrm{Maps}}_{\mathbb{Z}}(\mu_{\mathrm{N}},\mathrm{B}\mathbb{G}_m). \tag{3.7}$$

The first and last terms are isomorphic to $\mathbb{Z}/\mathrm{N}$, respectively $\mathrm{B}(\mathbb{Z}/\mathrm{N})$, giving:

$$\underline{\mathrm{Maps}}_{\mathbb{Z}}(\mu_{\mathrm{N}},\mathrm{B}\mu_{\mathrm{N}})\cong\mathbb{Z}/\mathrm{N}\oplus\mathrm{B}(\mathbb{Z}/\mathrm{N}). \tag{3.8}$$

For a nondegenerate pairing λ, we shall determine the image of $\lambda_*\mathrm{H}^{(1)}(\mu_{\mathrm{N}})$ under (3.8).

Remark 3.3.2. The inclusion $\mathrm{B}(\mathbb{Z}/\mathrm{N})\to\underline{\mathrm{Maps}}_{\mathbb{Z}}(\mu_{\mathrm{N}},\mathrm{B}\mu_{\mathrm{N}})$ in (3.8) may be described as follows: any section of $\mathrm{B}(\mathbb{Z}/\mathrm{N})$ defines a $\mathbb{Z}/\mathrm{N}$-linear map $\mathbb{Z}/\mathrm{N}\to\mathrm{B}(\mathbb{Z}/\mathrm{N})$ which we may tensor with μ_{N}.

The projection $\underline{\mathrm{Maps}}_{\mathbb{Z}}(\mu_{\mathrm{N}},\mathrm{B}\mu_{\mathrm{N}})\to\mathbb{Z}/\mathrm{N}$ can be identified with taking the isomorphism class and using $\underline{\mathrm{Ext}}^1_{\mathbb{Z}}(\mu_{\mathrm{N}},\mu_{\mathrm{N}})\cong\mathbb{Z}/\mathrm{N}$.

Remark 3.3.3. The sheaf of $\mathbb{Z}$-module spectra $\underline{\mathrm{Maps}}_{\mathbb{Z}}(\mu_{\mathrm{N}},\mathrm{B}\mu_{\mathrm{N}})$ has two $\mathbb{Z}/\mathrm{N}$-module structures, given by multiplication on the source or on the target. Their induced $\mathbb{Z}/\mathrm{N}^2$-module structures are naturally identified. In particular, we may view (3.8) unambiguously as an isomorphism of complexes of $\mathbb{Z}/\mathrm{N}^2$-modules.

3.3.4. Any primitive Nth root of unity $\zeta\in\mathscr{O}_{\mathrm{S}}^{\times}$ determines a nondegenerate pairing $\lambda:\mu_{\mathrm{N}}^{\otimes 2}\to\mathbb{G}_m$, satisfying the equality $\lambda(\zeta\otimes\zeta)=\zeta$.

Note that such ζ satisfies:

$$\zeta^{\binom{\mathrm{N}}{2}}=\begin{cases}1 & \mathrm{N}\text{ odd}\\ -1 & \mathrm{N}\text{ even}\end{cases}.$$

In particular, $\Psi(\zeta^{\binom{N}{2}}) \cong \Psi(-1)$ as sections of $B(\mu_N)$.

On the other hand, we equip $\Psi(\zeta^{\binom{N}{2}})$ with a "strange" 2-torsion structure: $2 \cdot \Psi(\zeta^{\binom{N}{2}}) \cong \Psi(1)$ and we use ζ^{-1} to trivialize $\Psi(1)$. This is generally distinct from the 2-torsion structure on $\Psi(-1)$ defined in §3.1.4.

The reason for introducing this 2-torsion structure is that it corresponds naturally to the 2-torsion structure on $H^{(1)}(\mu_N)$, which will play a role in the proof of Proposition 3.4.2.

Proposition 3.3.5. *Let $\zeta \in \mathscr{O}_S^\times$ be a primitive Nth root of unity with corresponding nondegenerate pairing λ. Under the splitting (3.8), $\lambda_* H^{(1)}(\mu_N)$ corresponds to:*

(1) $\binom{N}{2} \in \mathbb{Z}/N$*;*

(2) $\lambda_*\Psi(\zeta^{\binom{N}{2}}) \in B(\mathbb{Z}/N)$*, where λ is viewed as a map $\mu_N \cong \mathbb{Z}/N$.*

Proof. We shall use the exact sequences obtained by mapping (3.5) (for $A = \mu_N$) into μ_N and $\mathbb{G}_m$, as summarized in the diagram below:

$$\begin{array}{ccccccccc}
 & & & & \check{H}^{(1)}(\mu_N,\mu_N) & \longrightarrow & \underline{\mathrm{Maps}}_{\mathbb{Z}}(\mu_N^{\otimes 2},\mu_N) & \longrightarrow & \underline{\mathrm{Ext}}^1_{\mathbb{Z}}(\mu_N,\mu_N) \\
 & & & & \downarrow & & \downarrow{\scriptstyle\cong} & & \\
1 & \to & \underline{\mathrm{Maps}}_{\mathbb{Z}}(\mu_N,\mathbb{G}_m) & \to & \check{H}^{(1)}(\mu_N,\mathbb{G}_m) & \to & \underline{\mathrm{Maps}}_{\mathbb{Z}}(\mu_N^{\otimes 2},\mathbb{G}_m) & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle N} & & \downarrow{\scriptstyle N} & & \downarrow{\scriptstyle N} & & \\
1 & \to & \underline{\mathrm{Maps}}_{\mathbb{Z}}(\mu_N,\mathbb{G}_m) & \to & \check{H}^{(1)}(\mu_N,\mathbb{G}_m) & \to & \underline{\mathrm{Maps}}_{\mathbb{Z}}(\mu_N^{\otimes 2},\mathbb{G}_m) & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \mathrm{ev}_\zeta} & & \downarrow{\scriptstyle \mathrm{ev}_\zeta} & & \downarrow{\scriptstyle (\mathrm{ev}_{\zeta\otimes\zeta})^{\binom{N}{2}}} & & \\
 & & \mu_N & \longrightarrow & \mathbb{G}_m & \xrightarrow{N} & \mathbb{G}_m & &
\end{array} \tag{3.9}$$

Here, we have identified $\underline{\mathrm{Maps}}_{\mathbb{Z}}(H^{(1)}(\mu_N),\mathbb{G}_m)$ with $\check{H}^{(1)}(\mu_N,\mathbb{G}_m)$, the set of quadratic functions $\mu_N \to \mathbb{G}_m$, see §3.2.4. Hence the map $\check{H}^{(1)}(\mu_N,\mathbb{G}_m) \to \underline{\mathrm{Maps}}_{\mathbb{Z}}(\mu_N^{\otimes 2},\mathbb{G}_m)$ carries Q to the bilinear pairing $b(a_1 \otimes a_2) := Q(a_1)Q(a_2)Q(a_1a_2)^{-1}$.

The bottom vertical arrows in (3.9) are given by evaluating a linear (resp. quadratic) map $\mu_n \to \mathbb{G}_m$ at $\zeta \in \mu_N$, see §3.2.4. The lower right square commutes because of the binomial theorem:

$$Q(a)^k = b(a \otimes a)^{\binom{k}{2}} Q(a^k). \tag{3.10}$$

The extension class of $\lambda_* H^{(1)}(\mu_N)$ is the image of $\lambda \in \underline{\mathrm{Maps}}_{\mathbb{Z}}(\mu_N^{\otimes 2},\mu_N)$ in $\underline{\mathrm{Ext}}^1_{\mathbb{Z}}(\mu_N,\mu_N)$. The isomorphism $\underline{\mathrm{Ext}}^1_{\mathbb{Z}}(\mu_N,\mu_N) \cong \mathbb{Z}/N$ is induced from the connecting map of the Snake Lemma applied to the two middle rows of (3.9), so the extension class of λ is computed by choosing any quadratic function $Q : \mu_N \to \mathbb{G}_m$ lifting λ and taking Q^N: a linear form $\zeta \mapsto \lambda(\zeta\otimes\zeta)^{\binom{N}{2}} = \zeta^{\binom{N}{2}}$ according to (3.10). The expression (1) follows.

The section of $B(\mathbb{Z}/N)$ corresponding to $\lambda_* H^{(1)}(\mu_N)$ is equivalently described as the $\underline{\mathrm{Maps}}_{\mathbb{Z}}(\mu_N,\mathbb{G}_m)$-torsors of the quadratic lifts of λ, viewed as a $\mathbb{G}_m$-valued symmetric form. Under ev_ζ, it induces the μ_N-torsor $\Psi(\zeta^{\binom{N}{2}})$ by the commutativity of the two bottom rows in (3.9). Since the composition below is the identity:

$$\underline{\mathrm{Maps}}_{\mathbb{Z}}(\mu_N,\mathbb{G}_m) \xrightarrow{\mathrm{ev}_\zeta} \mu_N \xrightarrow{\lambda} \underline{\mathrm{Maps}}_{\mathbb{Z}}(\mu_N,\mathbb{G}_m),$$

expression (2) follows. □

3.3.6. We are now ready to construct an isomorphism in $\underline{\mathrm{Maps}}_{\mathbb{Z}}(\mathbb{G}_m, \mathrm{B}^2\mu_{\mathrm{N}})$:

$$\mathrm{T}_\zeta : \lambda_*(\Psi \cup \Psi) \cong \Psi^* \lambda_* \Psi(-1), \tag{3.11}$$

upon choosing a primitive Nth root of unity ζ (with corresponding pairing λ).

Construction. Recall the isomorphism $\Psi \cup \Psi \cong \Psi^* \mathrm{H}^{(1)}(\mu_{\mathrm{N}})$, see §3.2. Hence T_ζ may be produced from the two isomorphisms below, taking place in $\underline{\mathrm{Maps}}_{\mathbb{Z}}(\mu_{\mathrm{N}}, \mathrm{B}\mu_{\mathrm{N}})$ respectively $\underline{\mathrm{Maps}}_{\mathbb{Z}}(\mathbb{G}_m, \mathrm{B}^2\mu_{\mathrm{N}})$:

$$\lambda_* \mathrm{H}^{(1)}(\mu_{\mathrm{N}}) - \lambda_* \Psi(-1) \cong \Psi_* \binom{\mathrm{N}}{2}, \tag{3.12}$$

$$\Psi^* \Psi_* \binom{\mathrm{N}}{2} \cong *. \tag{3.13}$$

(In (3.12), $\lambda_* \Psi(-1) \in \mathrm{B}(\mathbb{Z}/\mathrm{N})$ is understood as its image under the inclusion in (3.8).)

The first isomorphism (3.12) follows from Proposition 3.3.5 and the isomorphism $\Psi(\zeta^{\binom{\mathrm{N}}{2}}) \cong \Psi(-1)$, see §3.3.4.

To define (3.13), we trivialize the following composition of $\mathbb{Z}/\mathrm{N}^2$-linear maps:

$$\mathbb{Z}/\mathrm{N}^2 \twoheadrightarrow \mathbb{Z}/\mathrm{N} \xrightarrow{\Psi_*} \underline{\mathrm{Maps}}_{\mathbb{Z}}(\mu_{\mathrm{N}}, \mathrm{B}\mu_{\mathrm{N}}) \to \underline{\mathrm{Maps}}_{\mathbb{Z}}(\mu_{\mathrm{N}^2}, \mathrm{B}\mu_{\mathrm{N}}). \tag{3.14}$$

Indeed, by adjunction it suffices to trivialize the image of $1 \in \mathbb{Z}/\mathrm{N}^2$. Under the composition (3.14), $1 \in \mathbb{Z}/\mathrm{N}^2$ maps to the extension $\mu_{\mathrm{N}} \to \mu_{\mathrm{N}^2} \to \mu_{\mathrm{N}}$ induced along $\mu_{\mathrm{N}^2} \twoheadrightarrow \mu_{\mathrm{N}}$; it naturally splits. Since the element $\binom{\mathrm{N}}{2} \in \mathbb{Z}/\mathrm{N}$ lifts along the surjection $\mathbb{Z}/\mathrm{N}^2 \twoheadrightarrow \mathbb{Z}/\mathrm{N}$, the image of $\Psi_*\binom{\mathrm{N}}{2}$ in $\underline{\mathrm{Maps}}_{\mathbb{Z}}(\mu_{\mathrm{N}^2}, \mathrm{B}\mu_{\mathrm{N}})$ is thus trivialized.

Finally, we observe that the map:

$$\Psi^* : \underline{\mathrm{Maps}}_{\mathbb{Z}}(\mu_{\mathrm{N}}, \mathrm{B}\mu_{\mathrm{N}}) \to \underline{\mathrm{Maps}}_{\mathbb{Z}}(\mathbb{G}_m, \mathrm{B}\mu_{\mathrm{N}})$$

factors through the last map in (3.14). This defines the isomorphism (3.13). □

Remark 3.3.7. We emphasize that $\lambda_* \mathrm{H}^{(1)}(\mu_{\mathrm{N}})$ is *not* isomorphic to $\lambda_* \Psi(-1)$, i.e. the isomorphism T_ζ (3.11) only exists after applying Ψ^* to both sides.

3.4. Properties of T_ζ.

3.4.1. After choosing a primitive Nth root of unity ζ with corresponding nondegenerate pairing $\lambda : \mu_{\mathrm{N}}^{\otimes 2} \to \mathbb{G}_m$, we have constructed an isomorphism T_ζ (3.11) of two sections of $\underline{\mathrm{Maps}}_{\mathbb{Z}}(\mathbb{G}_m, \mathrm{B}^2\mu_{\mathrm{N}})$.

Inducing along the inverse of λ, i.e. the mapping $\mu_{\mathrm{N}} \cong \mu_{\mathrm{N}}^{\otimes 2}$, $\zeta \mapsto \zeta \otimes \zeta$, we obtain an isomorphism:

$$\lambda^{-1}\mathrm{T}_\zeta : \Psi \cup \Psi \cong \Psi^*(\Psi(-1)). \tag{3.15}$$

This is our candidate for (3.3), meaningful over the base scheme $\mathrm{S}_1 := \mu_{\mathrm{N},\mathrm{S}}$. In order to show that it descends to S and satisfies the requirement of Theorem 3.1.5, we need to prove that it is compatible with the 2-torsion structures and is independent of the choice of ζ.

These properties are established below.

Proposition 3.4.2. *The isomorphism* (3.15) *relates the* 2*-torsion structure on* $\Psi \cup \Psi$ *to the* 2*-torsion structure on* $\Psi(-1)$.

Proof. It suffices to prove that (3.11) has this property. An isomorphism $f : a_1 \cong a_2$ of objects equipped with 2-torson structures in a Picard groupoid A defines an element in $\pi_1(\mathrm{A})$: the composition $* \cong 2 \cdot a_1 \xrightarrow{2f} 2 \cdot a_2 \cong *$. Let us call it the "2-torsion error" of f. It vanishes if and only if f is compatible with the 2-torsion structures.

Recall that (3.11) is defined by the composition of isomorphisms in the Picard groupoid $\mathrm{A} := \underline{\mathrm{Maps}}_{\mathbb{Z}}(\mathbb{G}_m, \mathrm{B}^2\mu_{\mathrm{N}})$:

$$\Psi^*(\lambda_*\mathrm{H}^{(1)}(\mu_{\mathrm{N}}) - \lambda_*\Psi(-1)) \cong \Psi^*\Psi_*\binom{\mathrm{N}}{2} \cong *. \tag{3.16}$$

Under the isomorphism $\pi_1(\mathrm{A}) \cong \mathbb{Z}/\mathrm{N}$, we shall prove that the first isomorphism has 2-torsion error -1 and the second isomorphism has 2-torsion error 1.

The first isomorphism comes from the isomorphism (3.12) of sections of $\underline{\mathrm{Maps}}_{\mathbb{Z}}(\mu_{\mathrm{N}}, \mathrm{B}\mu_{\mathrm{N}})$, and it suffices to calculate the 2-torsion error there. To do so, we first argue that the isomorphism:

$$\lambda_*\mathrm{H}^{(1)}(\mu_{\mathrm{N}}) - \lambda_*\Psi(\zeta^{\binom{\mathrm{N}}{2}}) \cong \Psi_*\binom{\mathrm{N}}{2}, \tag{3.17}$$

where $\Psi(\zeta^{\binom{\mathrm{N}}{2}})$ is equipped with the 2-torsion structure defined using ζ^{-1} (see §3.3.4), is *compatible* with the 2-torsion structures. The desired claim will follow since the isomorphism $\Psi(-1) \cong \Psi(\zeta^{\binom{\mathrm{N}}{2}})$ of sections of $\mathrm{B}(\mu_{\mathrm{N}})$ has 2-torsion error ζ, so $-\lambda_*\Psi(-1) \cong -\lambda_*\Psi(\zeta^{\binom{\mathrm{N}}{2}})$ has 2-torsion error $-\lambda_*(\zeta) = -1$.

The isomorphism (3.17) comes from the proof of Proposition 3.3.5. Its compatibility with 2-torsion structures will follow once we prove that along the projection of (3.8):

$$\underline{\mathrm{Maps}}_{\mathbb{Z}}(\mu_{\mathrm{N}}, \mathrm{B}\mu_{\mathrm{N}}) \to \mathrm{B}(\mathbb{Z}/\mathrm{N}),$$

the identification of the image of $\lambda_*\mathrm{H}^{(1)}(\mu_{\mathrm{N}})$ with $\lambda_*\Psi(\zeta^{\binom{\mathrm{N}}{2}})$ is compatible with the 2-torsion structures. Recall that the image of $\lambda_*\mathrm{H}^{(1)}(\mu_{\mathrm{N}})$ is the $\mathbb{Z}/\mathrm{N}$-torsor of quadratic lifts of λ, viewed as a $\mathbb{G}_m$-valued symmetric form. Its 2-torsion structure is given by the canonical quadratic lift of 2λ:

$$\mu_{\mathrm{N}} \to \mathbb{G}_m, \quad a \mapsto \lambda(a \otimes a)^{-1}, \tag{3.18}$$

see §3.2.5. Under the bijection $\mathrm{Q} \mapsto \mathrm{Q}(\zeta)$ between quadratic lifts of 2λ and Nth roots of $(\zeta^{\binom{\mathrm{N}}{2}})^2 = 1$, the form (3.18) passes to $\lambda(\zeta \otimes \zeta)^{-1} = \zeta^{-1}$. The desired conclusion follows.

The second isomorphism in (3.16) comes from the isomorphism (3.13), which already occurs in $\underline{\mathrm{Maps}}_{\mathbb{Z}}(\mu_{\mathrm{N}^2}, \mathrm{B}\mu_{\mathrm{N}})$, so we will calculate the 2-torsion error there instead. More precisely, the image of $\Psi_*\binom{\mathrm{N}}{2}$ in $\underline{\mathrm{Maps}}_{\mathbb{Z}}(\mu_{\mathrm{N}^2}, \mathrm{B}\mu_{\mathrm{N}})$ is the extension induced along:

$$\begin{array}{ccccccccc}
 & & & & & & \mu_{\mathrm{N}^2} & & \\
 & & & & & & \downarrow\!\!\downarrow & & \\
 & & & & & & \mu_{\mathrm{N}} & & \\
 & & & & & & \downarrow{\scriptstyle a \mapsto a^{\binom{\mathrm{N}}{2}}} & & \\
1 & \longrightarrow & \mu_{\mathrm{N}} & \longrightarrow & \mu_{\mathrm{N}^2} & \longrightarrow & \mu_{\mathrm{N}} & \longrightarrow & 1
\end{array}$$

It is trivialized by the section $\mu_{\mathrm{N}^2} \to \mu_{\mathrm{N}^2}$, $a \mapsto a^{\binom{\mathrm{N}}{2}}$, using the lift of $\binom{\mathrm{N}}{2}$ along $\mathbb{Z}/\mathrm{N}^2 \twoheadrightarrow \mathbb{Z}/\mathrm{N}$. In particular, its square is the trivialization of $2 \cdot \Psi_*\binom{\mathrm{N}}{2} \cong \Psi_*(2 \cdot \binom{\mathrm{N}}{2})$ given by the section $\mu_{\mathrm{N}^2} \to \mu_{\mathrm{N}^2}$, $a \mapsto a^{2 \cdot \binom{\mathrm{N}}{2}}$.

On the other hand, the 2-torsion structure on $\Psi_*\binom{N}{2}$ is defined by the vanishing of $2\cdot\binom{N}{2}$ in $\mathbb{Z}/N$ and the linearity of Ψ_*, so it corresponds to the section $\mu_{N^2}\to\mu_{N^2}$, $a\mapsto 1$. The 2-torsion error of the trivialization of $\Psi_*\binom{N}{2}$ in $\underline{\mathrm{Maps}}_{\mathbb{Z}}(\mu_{N^2}, B\mu_N)$ is thus induced from the difference of these two sections:

$$0 - 2\cdot\binom{N}{2} = N \in \mathbb{Z}/N^2,$$

along the map:

$$\mathrm{Ker}(\mathbb{Z}/N^2 \twoheadrightarrow \mathbb{Z}/N) \to \pi_1\underline{\mathrm{Maps}}_{\mathbb{Z}}(\mu_{N^2}, B\mu_N) \cong \mathbb{Z}/N,$$

which carries N to 1. □

Proposition 3.4.3. *The isomorphism* (3.15) *is independent of* ζ.

Proof. For another primitive Nth root of unity ζ_1 with corresponding nongenerate pairing λ_1, we need to show an equality of the isomorphisms (3.15) defined by ζ and ζ_1.

Let $k\in(\mathbb{Z}/N)^\times$ be the unique element so that $\zeta=\zeta_1^k$, or equvalently $\lambda_1=\lambda^k$. We need to show that the following diagram in $\underline{\mathrm{Maps}}_{\mathbb{Z}}(\mathbb{G}_m, B^2\mu_N)$ is commutative:

$$\begin{array}{ccc} k_*\lambda_*(\Psi\cup\Psi) & \xrightarrow{k_*T_\zeta} & \Psi^*k_*\lambda_*\Psi(\zeta^{\binom{N}{2}}) \\ \downarrow & & \downarrow \\ (\lambda_1)_*(\Psi\cup\Psi) & \xrightarrow{T_{\zeta_1}} & \Psi^*(\lambda_1)_*\Psi(\zeta^{\binom{N}{2}}) \end{array} \tag{3.19}$$

where k_* means inducing along the map $\mu_N\to\mu_N$, $a\mapsto a^k$.

We shall deduce (3.19) from the $\mathbb{Z}/N^2$-linear structure of the splitting (3.8), see Remark 3.3.3. Indeed, let $k_1\in\mathbb{Z}/N^2$ be a lift of k. The operation k_* amounts to multiplication by k_1. Thus (3.12) yields a commutative diagram:

$$\begin{array}{ccc} k_1\cdot\lambda_*H^{(1)}(\mu_N) - k_1\cdot\lambda_*\Psi(\zeta^{\binom{N}{2}}) & \xrightarrow{\cong} & k_1\cdot\Psi_*\binom{N}{2} \\ \downarrow{\cong} & & \downarrow{\cong} \\ (\lambda_1)_*H^{(1)}(\mu_N) - (\lambda_1)_*\Psi(\zeta_1^{\binom{N}{2}}) & \xrightarrow{\cong} & \Psi_*\binom{N}{2} \end{array}$$

To obtain the commutativity of (3.19), it remains to prove that the trivialization of $\Psi^*\Psi_*\binom{N}{2}$ is compatible with multiplication by k_1. This in turn follows from the $\mathbb{Z}/N^2$-linearity of the null-homotopy of (3.14). □

4. Tori

The goal of this section is to classify étale metaplectic covers of a torus $T\to S$ in terms of its sheaf of cocharacters Λ. The main result is Theorem 4.3.2, which states the answer in terms of (even) ϑ-data with coefficients in $A(-1)$.

The notion of ϑ-data is introduced by Beilinson–Drinfeld [BD04, §3.10]. It also appeared in Brylinski–Deligne [BD01, §3], although this name was not used. The difference between their ϑ-data and ours is that we allow torsion coefficients instead of $\mathbb{Z}$. In this situation, ϑ-data naturally form a 2-groupoid, and the usual definition needs to be reformulated in a way that is homotopy coherent. This will be carried out in §4.1-4.2.

The main classification result is contained in §4.3, and the two sections which follow §4.4-4.5 are elaborations on it.

Then in §4.6-4.7, we study étale metaplectic covers of T which are "commutative". These induce, for example, commutative topological covers of T(F) for a local field F. Such covers are used in the definition of the L-group of an étale metaplectic cover.

Although the classification Theorem 4.3.2 makes essential use of Theorem 3.1.5, the study of the commutative covers in §4.6 (hence the definition of the L-group as well) does not rely on Theorem 3.1.5.

4.1. Linear algebra.

4.1.1. Let Λ be a locally constant sheaf of finite free $\mathbb{Z}$-modules on a site $\mathscr{C}$. The permutation group Σ_2 acts on $\Lambda \otimes \Lambda$ by exchanging its factors.

Derived Σ_2-coinvariants $(\Lambda \otimes \Lambda)_{\Sigma_2}$ are computed by the complex of sheaves in cohomological degrees ≤ 0:

$$[\cdots \xrightarrow{\text{Ant}} \Lambda \otimes \Lambda \xrightarrow{\text{Sym}} \Lambda \otimes \Lambda \xrightarrow{\text{Ant}} \Lambda \otimes \Lambda]. \tag{4.1}$$

Here, Sym (resp. Ant) denotes the (anti-)symmetrizer sending $x_1 \otimes x_2$ to $x_1 \otimes x_2 + x_2 \otimes x_1$ (resp. $x_1 \otimes x_2 - x_2 \otimes x_1$).

4.1.2. Let us calculate the cohomology groups H^i of (4.1).

By definition, we have $\mathrm{H}^0 \cong \mathrm{Sym}^2(\Lambda)$.

Note that the anti-symmetrizer $\Lambda \otimes \Lambda \to \Lambda \otimes \Lambda$ has image $\wedge^2(\Lambda)$. In particular, we have a short exact sequence:

$$0 \to \wedge^2(\Lambda) \to \Lambda \otimes \Lambda \to \mathrm{Sym}^2(\Lambda) \to 0. \tag{4.2}$$

We find $\mathrm{H}^{-1} \cong \Lambda/2$, according to a commutative diagram of four additional short exact sequences:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathrm{Sym}^2(\Lambda) & \xrightarrow{\cong} & \mathrm{Sym}^2(\Lambda) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \Gamma^2(\Lambda) & \longrightarrow & \Lambda \otimes \Lambda & \longrightarrow & \wedge^2(\Lambda) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle\cong} & & \\
0 & \longrightarrow & \Lambda/2 & \longrightarrow & \mathrm{Ant}^2(\Lambda) & \longrightarrow & \wedge^2(\Lambda) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \tag{4.3}$$

Next, $\mathrm{H}^{-2} = 0$ again by the exact sequence (4.2). The rest of the negative cohomology groups alternate between $\Lambda/2$ and 0.

Remark 4.1.3. Let $\check{\Lambda}$ denote the sheaf of $\mathbb{Z}$-modules dual to Λ. Items in (4.3) admit descriptions as integral "forms" on $\check{\Lambda}$. Bilinear forms are classified by $\Lambda \otimes \Lambda$ in the obvious way, $\wedge^2(\Lambda)$ consists of alternating forms via the inclusion in (4.2) and $\Gamma^2(\Lambda)$ consists of symmetric bilinear forms via the inclusion in (4.3).

Next, $\mathrm{Sym}^2(\Lambda)$ consists of quadratic forms Q whose value at $x \in \check{\Lambda}$ is given by $c(x,x)$ for any lift $c \in \Lambda \otimes \Lambda$ along (4.2). Finally, $\mathrm{Ant}^2(\Lambda)$ consists of pairs (a, f) where a is an alternating form on $\check{\Lambda}$ and $f : \check{\Lambda} \to \mathbb{Z}/2$ is a function satisfying:

$$f(x_1 + x_2) - f(x_1) - f(x_2) = a(x_1, x_2) \mod 2.$$

The maps $\mathrm{Sym}^2(\Lambda) \to \Gamma^2(\Lambda)$, respectively $\mathrm{Ant}^2(\Lambda) \to \wedge^2(\Lambda)$ in (4.3) encode the following operations: going from a quadratic form Q to its symmetric bilinear form b, and going from a pair (a, f) to its alternating form a.

Remark 4.1.4. We may also consider the Σ_2 action on $\Lambda \otimes \Lambda$ twisted by the sign character. The derived coinvariants are computed by a complex similar to (4.1), but which starts with Sym as the differential d^{-1}. In particular, we find $\mathrm{Ant}^2(\Lambda)$ in H^0. Then we have $\mathrm{H}^{-1} = 0$, $\mathrm{H}^{-2} \cong \Lambda/2$, and the rest of the H^{-n} alternate between them.

The vanishing of H^{-1} has a practical consequence: given a complex A of sheaves of abelian groups in cohomological degrees $[-1, 0]$, the sheaf of Σ_2-equivariant maps $\Lambda \otimes \Lambda \to \mathrm{A}$, *a priori* given by $\underline{\mathrm{Maps}_{\mathbb{Z}}}((\Lambda \otimes \Lambda)_{\Sigma_2}, \mathrm{A})$ for the derived Σ_2-coinvariants, is actually identified with $\underline{\mathrm{Maps}_{\mathbb{Z}}}(\mathrm{Ant}^2(\Lambda), \mathrm{A})$.

4.1.5. We shall also meet two kinds of "quadratic functions" on Λ, one of which has already made an appearance in (3.2).

Namely, for a sheaf A of abelian groups, we write $\mathrm{H}^{(1)}(\mathrm{A})$ for the extension of A by $\mathrm{Sym}^2(\mathrm{A})$ defined by the cocycle $a_1, a_2 \mapsto a_1 a_2$.

Define $\mathrm{H}^{(2)}(\mathrm{A})$ to be the chain complex in cohomological degrees $[-1, 0]$:

$$\mathrm{H}^{(2)}(\mathrm{A}) := [\mathrm{A} \otimes \mathrm{A} \to \mathrm{H}^{(1)}(\mathrm{A})],$$

where the differential is given by the projection $\mathrm{A} \otimes \mathrm{A} \twoheadrightarrow \mathrm{Sym}^2(\mathrm{A})$ followed by the natural inclusion of $\mathrm{Sym}^2(\mathrm{A})$ in $\mathrm{H}^{(1)}(\mathrm{A})$.

4.1.6. We shall only need the case $\mathrm{A} := \Lambda$, in which case $\mathrm{H}^{(1)}(\Lambda)$ is torsion-free. Consider the duals as chain complexes of sheaves of $\mathbb{Z}$-modules:

$$\check{\mathrm{H}}^{(1)}(\Lambda) := \underline{\mathrm{Hom}}(\mathrm{H}^{(1)}(\Lambda), \mathbb{Z}),$$
$$\check{\mathrm{H}}^{(2)}(\Lambda) := \underline{\mathrm{Hom}}(\mathrm{H}^{(2)}(\Lambda), \mathbb{Z}).$$

The complex $\check{\mathrm{H}}^{(2)}(\Lambda)$ is concentrated in degrees $[0, 1]$.

Recall that $\check{\mathrm{H}}^{(1)}(\Lambda)$ is identified with the abelian group of $\mathbb{Z}$-valued quadratic functions on Λ, and restriction along $\mathrm{Sym}^2(\Lambda) \to \mathrm{H}^{(1)}(\Lambda)$ yields the *negative* of the associated symmetric form, see §3.2.4 for details.

4.1.7. Note that the short exact sequence (4.2) gives rise to a fiber sequence:

$$\wedge^2(\Lambda)[1] \to \mathrm{H}^{(2)}(\Lambda) \to \Lambda. \tag{4.4}$$

Let us identify the strictly commutative Picard groupoid associated to $\mathrm{H}^{(2)}(\Lambda)$.

Lemma 4.1.8. *There is a* monoidal *equivalence* $\mathrm{H}^{(2)}(\Lambda) \cong \Lambda \times \mathrm{B}(\wedge^2(\Lambda))$*, under which the commutativity constraint on* $\mathrm{H}^{(2)}(\Lambda)$ *corresponds to the isomorphism* $\lambda_1 + \lambda_2 \cong \lambda_2 + \lambda_1$ *defined by* $\lambda_1 \wedge \lambda_2 \in \wedge^2(\Lambda)$*, for all* $\lambda_1, \lambda_2 \in \Lambda$ *viewed as objects of* $\mathrm{H}^{(2)}(\Lambda)$.

Proof. As a pointed morphism, the section $\Lambda \to \mathrm{H}^{(2)}(\Lambda)$ of (4.4) is induced from the natural map $v : \mathrm{H}^{(1)}(\Lambda) \to \mathrm{H}^{(2)}(\Lambda)$, restricted along the section $\Lambda \subset \mathrm{H}^{(1)}(\Lambda)$. It is equipped with an $\mathbb{E}_1$-monoidal structure coming from the trivialization of v over $\Lambda \otimes \Lambda$.

The commutativity constraint in $\mathrm{H}^{(2)}(\Lambda)$ for two objects λ_1, λ_2 coming from Λ is then the automorphism of $\lambda_1 + \lambda_2$ determined by $\lambda_1 \otimes \lambda_2 - \lambda_2 \otimes \lambda_1$ in the subgroup $\wedge^2(\Lambda) \subset \Lambda \otimes \Lambda$. □

4.1.9. By Lemma 4.1.8, for any sheaf of abelian groups A_1, $\mathbb{Z}$-linear maps $\mathrm{H}^{(2)}(\Lambda) \to \mathrm{B}(\mathrm{A}_1)$ correspond to pairs (a, Λ_1) where $a : \wedge^2(\Lambda) \to \mathrm{A}_1$ is an alternating form and Λ_1 is a central extension of Λ by A_1 with commutator $\lambda_1, \lambda_2 \mapsto a(\lambda_1, \lambda_2)$.

Restriction along the first map in (4.4) corresponds to the functor $(a, \Lambda_1) \mapsto a$. The central extension Λ_1 is recovered by the restriction along the *monoidal* section $\Lambda \to \mathrm{H}^{(2)}(\Lambda)$.

4.1.10. The sheaf of $\mathbb{Z}$-modules $\check{\Lambda}$ has the natural coalgebra structure with respect to the Cartesian symmetric monoidal structure. It is defined by the cosimplicial object $[n] \mapsto \check{\Lambda}^{\oplus n}$, whose coface maps are insertions and whose degeneracy maps are projections. We display the maps between $\check{\Lambda}$ and $\check{\Lambda}^{\oplus 2}$:

$$d_i^2(x) = \begin{cases} (0, x) & i = 0 \\ (x, x) & i = 1 \\ (x, 0) & i = 2 \end{cases}; \quad s_i^1(x_0, x_1) = \begin{cases} x_1 & i = 0 \\ x_0 & i = 1 \end{cases}.$$

Functoriality of the constructions $\check{\Lambda} \mapsto \mathrm{F}(\check{\Lambda}) := \check{\Lambda} \otimes \check{\Lambda}$, $\Gamma^2(\check{\Lambda})$, $\wedge^2(\check{\Lambda})$, $\mathrm{Sym}^2(\check{\Lambda})$, $\mathrm{Ant}^2(\check{\Lambda})$, $\check{\mathrm{H}}^{(1)}(\Lambda)$, $\check{\mathrm{H}}^{(2)}(\Lambda)$ determine cosimplicial objects $[n] \mapsto \mathrm{F}(\check{\Lambda}^{\oplus n})$ in the category of complexes of sheaves of $\mathbb{Z}$-modules.

Lemma 4.1.11. *There are canonical isomorphisms of complexes:*

(1) $\lim_{[n]}(\check{\Lambda}^{\oplus n}) \cong \check{\Lambda}[-1]$;
(2) $\lim_{[n]}(\check{\Lambda}^{\oplus n} \otimes \check{\Lambda}^{\oplus n}) \cong \check{\Lambda} \otimes \check{\Lambda}[-2]$;
(3) $\lim_{[n]} \mathrm{F}_1(\check{\Lambda}^{\oplus n}) \cong \mathrm{F}_2(\check{\Lambda})[-2]$ *for all functors notated* $\mathrm{F}_1 \Rightarrow \mathrm{F}_2$ *below:*
$$\Gamma^2 \Rightarrow \wedge^2 \Rightarrow \mathrm{Sym}^2 \Rightarrow \mathrm{Ant}^2;$$
(4) $\lim_{[n]}(\check{\mathrm{H}}^{(1)}(\Lambda^{\oplus n})) \cong \check{\mathrm{H}}^{(2)}(\Lambda)[-1]$.

Proof. The cosimplicial limit of $\mathrm{F}(\check{\Lambda}^{\oplus n})$ is equivalent to the complex in degrees ≥ 0:

$$[0 \xrightarrow{d} \mathrm{F}(\check{\Lambda}) \xrightarrow{d} \mathrm{F}(\check{\Lambda}^{\oplus 2}) \xrightarrow{d} \cdots], \tag{4.5}$$

where d denotes the alternating sum of the face maps. Inclusion of the subcomplex $\bigcap \ker(s_i)$ of nondegenerate cochains is a homotopy equivalence. This already implies (1), since the subcomplex of nondegenerate cochains is equivalent to $\check{\Lambda}[-1]$. For (2), the subcomplex of nondegenerate cochains vanish in degrees ≥ 3, while in degrees $[1, 2]$ we find:

$$\check{\Lambda} \otimes \check{\Lambda} \xrightarrow{d} (\check{\Lambda} \otimes \check{\Lambda}) \oplus (\check{\Lambda} \otimes \check{\Lambda}), \tag{4.6}$$

where the differential is the diagonal inclusion. Its quotient is then $\check{\Lambda} \otimes \check{\Lambda}$.

For (3) with $\mathrm{F} = \Gamma^2$, $\wedge^2$, or Sym^2, the essential observation is that nondegenerate n-cochains vanish for $n \geq 3$ and nondegenerate 2-cochains are equivalent to $\check{\Lambda} \otimes \check{\Lambda}$. The differential it receives from nondegenerate 1-cochains $d : \mathrm{F}(\check{\Lambda}) \to \check{\Lambda} \otimes \check{\Lambda}$ is the natural inclusion, so we conclude by the short exact sequences in (4.2), (4.3) (see also [BD01, §3.9]).

For (4), we identify elements of $\check{\mathrm{H}}^{(1)}(\Lambda^{\oplus n})$ with quadratic functions $\mathrm{Q}^{(n)} : \Lambda^{\oplus n} \to \mathbb{Z}$. Thus nondegenerate cochains for $n \geq 3$ vanish and are isomorphic to $\check{\Lambda} \otimes \check{\Lambda}$ for $n = 2$: those

$\mathrm{Q}^{(2)}$ which vanish on $\Lambda \oplus 0$ and $0 \oplus \Lambda$ are uniquely determined by the pairing between $\Lambda \oplus 0$ and $0 \oplus \Lambda$, which can be arbitrary. The nondegenerate cochains form the complex in cohomological degrees $[1, 2]$:

$$[\check{\mathrm{H}}^{(1)}(\Lambda) \xrightarrow{d} \check{\Lambda} \otimes \check{\Lambda}],$$

where d takes $\mathrm{Q}^{(1)}$ to the bilinear form $\lambda_1, \lambda_2 \mapsto \mathrm{Q}^{(1)}(\lambda_1) + \mathrm{Q}^{(1)}(\lambda_2) - \mathrm{Q}^{(1)}(\lambda_1 + \lambda_2)$ on Λ. Taking the negative sign into account, this is precisely $\check{\mathrm{H}}^{(2)}(\Lambda)[-1]$. $\square$

4.2. Even ϑ-data.

4.2.1. Let S be a scheme, and let Λ be a locally constant sheaf of finite free $\mathbb{Z}$-modules on its étale site.

Denote by $\vartheta^{(1)}(\Lambda)$ the sheaf of abelian groups whose sections are pairs (a, f) where $a \in \wedge^2(\check{\Lambda})$ is an integral alternating form on Λ, and $f : \Lambda \to \mathbb{G}_m$ is a function with $f(\lambda_1 + \lambda_2)f(\lambda_1)^{-1}f(\lambda_2)^{-1} = (-1)^{a(\lambda_1,\lambda_2)}$.

Then $\vartheta^{(1)}(\Lambda)$ is simultaneously a pullback and a pushout:

$$\begin{array}{ccccc}
\Gamma^2(\check{\Lambda}) & \longrightarrow & \check{\Lambda} \otimes \check{\Lambda} & & \\
{\scriptstyle b \mapsto (\lambda \mapsto (-1)^{b(\lambda,\lambda)})}\downarrow & & \downarrow & & \\
\check{\Lambda} \otimes \mathbb{G}_m & \longrightarrow & \vartheta^{(1)}(\Lambda) & \longrightarrow & \wedge^2(\check{\Lambda}) \\
& & \downarrow & & \downarrow{\scriptstyle a \mapsto (\lambda_1, \lambda_2 \mapsto (-1)^{a(\lambda_1,\lambda_2)})} \\
& & \check{\mathrm{H}}^{(1)}(\Lambda) \otimes \mathbb{G}_m & \longrightarrow & \Gamma^2(\check{\Lambda}) \otimes \mathbb{G}_m
\end{array} \tag{4.7}$$

Here, the morphism $\check{\Lambda} \otimes \check{\Lambda} \to \vartheta^{(1)}(\Lambda)$ sends a bilinear form c to the pair (a, f), where a is the anti-symmetrization of c and f is the map $\lambda \mapsto (-1)^{c(\lambda,\lambda)}$.

The fact that $\wedge^2(\check{\Lambda})$ is identified with the cokernel of $\Gamma^2(\check{\Lambda}) \to \check{\Lambda} \otimes \check{\Lambda}$ implies that the middle sequence in (4.7) is exact.

Remark 4.2.2. Sections of $\vartheta^{(1)}(\Lambda)$ are in canonical bijection with pointed morphisms from $\mathrm{T} := \Lambda \otimes \mathbb{G}_m$ to $\underline{\mathrm{K}}_2$, when the base scheme S is regular of finite type over a field, see [BD01, Construction 3.5].

4.2.3. We define $\vartheta^{(2)}(\Lambda)$ to be the limit of the cosimplicial diagram $[n] \mapsto \mathrm{B}^2(\vartheta^{(1)}(\Lambda^{\oplus n}))$ of sheaves of connective $\mathbb{Z}$-module spectra, see §4.1.10.

Applying the same limit to the diagram (4.7) and using Lemma 4.1.11, we see that $\vartheta^{(2)}(\Lambda)$ is simultaneously a pullback and a pushout:

$$\begin{array}{ccccc}
\wedge^2(\check{\Lambda}) & \longrightarrow & \check{\Lambda} \otimes \check{\Lambda} & & \\
\downarrow{\scriptstyle (1)} & & \downarrow & & \\
\check{\Lambda} \otimes \mathrm{B}\mathbb{G}_m & \longrightarrow & \vartheta^{(2)}(\Lambda) & \longrightarrow & \mathrm{Sym}^2(\check{\Lambda}) \\
& & \downarrow & & \downarrow{\scriptstyle (2)} \\
& & \check{\mathrm{H}}^{(2)}(\Lambda) \otimes \mathrm{B}\mathbb{G}_m & \longrightarrow & \wedge^2(\check{\Lambda}) \otimes \mathbb{G}_m
\end{array} \tag{4.8}$$

The arrow labeled (1) is given by the compositions:

$$\wedge^2(\check{\Lambda}) \xrightarrow{\mathrm{Ant}(\check{\Lambda})} \mathrm{B}(\check{\Lambda}/2) \xrightarrow{(-1)} \check{\Lambda} \otimes \mathrm{B}\mathbb{G}_m,$$

where the first map is the coboundary of the bottom exact sequence of (4.3) and the second map is the tensor product of $\check{\Lambda}$ with $\mathbb{Z}/2 \to \mathbb{G}_m$, $a \mapsto (-1)^a$.

The arrow labeled (2) is the composition:

$$\mathrm{Sym}^2(\check{\Lambda}) \xrightarrow{\mathrm{Q}\mapsto b} \wedge^2(\check{\Lambda})/2 \xrightarrow{(-1)} \wedge^2(\check{\Lambda}) \otimes \mathbb{G}_m, \tag{4.9}$$

where the first map sends Q to its symmetric form b, viewed as an *alternating* form valued in $\mathbb{Z}/2$, and the second map is the tensor product of $\wedge^2(\check{\Lambda})$ with $\mathbb{Z}/2 \to \mathbb{G}_m$, $a \mapsto (-1)^a$.

Furthermore, the middle sequence in (4.8) is canonically a triangle.

Remark 4.2.4. Interpreting sections of $\check{\mathrm{H}}^{(2)}(\Lambda) \otimes \mathrm{B}\mathbb{G}_m$ as central extensions of Λ by $\mathbb{G}_m$ with prescribed commutators (§4.1.9), the pullback square in (4.8) says that sections of $\vartheta^{(2)}(\Lambda)$ are pairs (Q, Λ_1), where Q is a quadratic form on Λ, and Λ_1 is a central extension of Λ by $\mathbb{G}_m$ whose commutator is $\lambda_1, \lambda_2 \mapsto (-1)^{b(\lambda_1,\lambda_2)}$, for b being the symmetric form associated to Q.

This description shows that sections of $\vartheta^{(2)}(\Lambda)$ recover the "even ϑ-data" of Beilinson–Drinfeld [BD04, §3.10.3] except the terms involving ω_{X}. They are also equivalent to central extensions of $\mathrm{T} := \Lambda \otimes \mathbb{G}_m$ by $\underline{\mathrm{K}}_2$, when the base scheme S is regular of finite type over a field, see [BD01, Theorem 3.16].

Remark 4.2.5. The composition $\check{\Lambda} \otimes \check{\Lambda} \to \check{\mathrm{H}}^{(2)}(\Lambda) \otimes \mathrm{B}\mathbb{G}_m$ in (4.8) can be described as follows: it sends a bilinear form c to the central extension of Λ by $\mathbb{G}_m$ defined by the cocycle $\lambda_1, \lambda_2 \mapsto (-1)^{c(\lambda_1,\lambda_2)}$.

4.2.6. Let A be a locally constant étale sheaf of finite abelian groups over S of invertible order. Then we may form $\vartheta^{(1)}(\Lambda) \otimes \mathrm{A}(-1)$ and $\vartheta^{(2)}(\Lambda) \otimes \mathrm{A}(-1)$. (The Tate twists are introduced to conform with the conventions in §4.3 below.)

There are obvious analogues of the pullback/pushout diagrams (4.7), (4.8). In particular, sections of $\vartheta^{(2)}(\Lambda) \otimes \mathrm{A}(-1)$ consist of triples $(\mathrm{Q}, \mathrm{F}, h)$ where:

1. Q is an $\mathrm{A}(-1)$-valued quadratic form on Λ;
2. $\mathrm{F} : \mathrm{H}^{(2)}(\Lambda) \to \mathrm{B}^2\mathrm{A}$ is a $\mathbb{Z}$-linear morphism;
3. h is a $\mathbb{Z}$-linear isomorphism between the restriction of F to $\mathrm{B}(\wedge^2(\Lambda))$ and the image of Q along the map coming from tensoring (4.9) with $\mathrm{A}(-1)$:

$$\mathrm{Sym}^2(\check{\Lambda}) \otimes \mathrm{A}(-1) \to \wedge^2(\check{\Lambda}) \otimes \mathrm{BA}. \tag{4.10}$$

Furthermore, we have a canonical triangle coming from the middle triangle of (4.8):

$$\check{\Lambda} \otimes \mathrm{B}^2\mathrm{A} \to \vartheta^{(2)}(\Lambda) \otimes \mathrm{A}(-1) \to \mathrm{Sym}^2(\check{\Lambda}) \otimes \mathrm{A}(-1). \tag{4.11}$$

Remark 4.2.7. If $\mathrm{A}(-1)$ has no 2-torsion, then Q has zero image in $\wedge^2(\check{\Lambda}) \otimes \mathrm{BA}$; indeed, (4.10) factors through $\wedge^2(\check{\Lambda})/2 \otimes \mathrm{A}(-1) = 0$. In this case, (4.11) canonically splits.

Remark 4.2.8. If Λ has rank 1, then $\wedge^2(\check{\Lambda}) = 0$ and (4.11) also splits: the map $\mathrm{Sym}^2(\check{\Lambda}) \otimes \mathrm{A}(-1) \to \vartheta^{(2)}(\Lambda) \otimes \mathrm{A}(-1)$ is defined by lifting a quadratic form uniquely to $\check{\Lambda}^{\otimes 2} \otimes \mathrm{A}(-1)$ and applying the map induced from (4.8).

In fact, any basis $\Lambda \cong \bigoplus_{i\in\mathrm{I}} \mathbb{Z}e_i$ induces a splitting of (4.11) by the restriction maps composed with the splitting for $\Lambda = \mathbb{Z}$:

$$\begin{aligned}\bigoplus_{i\in\mathrm{I}} e_i^* : \vartheta^{(2)}(\Lambda) \otimes \mathrm{A}(-1) &\to \bigoplus_{i\in\mathrm{I}} \vartheta^{(2)}(\mathbb{Z}) \otimes \mathrm{A}(-1) \\ &\to \bigoplus_{i\in\mathrm{I}} \mathrm{B}^2\mathrm{A} \cong \check{\Lambda} \otimes \mathrm{B}^2\mathrm{A}.\end{aligned}$$

4.3. Classification: tori.

4.3.1. Let S be a scheme. Suppose that A is a locally constant étale sheaf of abelian groups over S of invertible order. Let $\mathrm{T} \to \mathrm{S}$ be a torus with sheaf of cocharacters Λ, so $\mathrm{T} \cong \Lambda \otimes \mathbb{G}_m$.

Let $\underline{\Gamma}_e(\mathrm{BT}, \mathrm{B}^4\mathrm{A}(1))$ denote the étale sheaf of connective $\mathbb{Z}$-module spectra over the base scheme S, whose sections are rigidified maps $\mathrm{BT} \to \mathrm{B}^4\mathrm{A}(1)$. The classification of étale metaplectic covers of T amounts to describing the étale sheaf $\underline{\Gamma}_e(\mathrm{BT}, \mathrm{B}^4\mathrm{A}(1))$.

Theorem 4.3.2. *There is a canonical equivalence of étale sheaves of connective $\mathbb{Z}$-module spectra:*

$$\vartheta^{(2)}(\Lambda) \otimes \mathrm{A}(-1) \cong \underline{\Gamma}_e(\mathrm{BT}, \mathrm{B}^4\mathrm{A}(1)). \tag{4.12}$$

In particular, $\underline{\Gamma}_e(\mathrm{BT}, \mathrm{B}^4\mathrm{A}(1))$ is situated in a pushout square, a pullback square, and a cofiber sequence:

$$\begin{array}{ccccc} \wedge^2(\check{\Lambda}) \otimes \mathrm{A}(-1) & \longrightarrow & \check{\Lambda}^{\otimes 2} \otimes \mathrm{A}(-1) & & \\ \downarrow & & \downarrow{\scriptstyle (1)} & & \\ \check{\Lambda} \otimes \mathrm{B}^2\mathrm{A} & \xrightarrow{(2)} & \underline{\Gamma}_e(\mathrm{BT}, \mathrm{B}^4\mathrm{A}(1)) & \to & \mathrm{Sym}^2(\check{\Lambda}) \otimes \mathrm{A}(-1) \\ & & \downarrow & & \downarrow \\ & & \check{\mathrm{H}}^{(2)}(\Lambda) \otimes \mathrm{B}^2\mathrm{A} & \longrightarrow & \wedge^2(\check{\Lambda}) \otimes \mathrm{BA} \end{array} \tag{4.13}$$

4.3.3. Let us explicitly describe the two labeled maps in (4.13). Recall that the Kummer torsor $\Psi : \mathbb{G}_m \to \lim_{\mathrm{N}} \mathrm{B}(\mu_{\mathrm{N}})$ (where N ranges over integers ≥ 1 invertible on S) induces a $\mathbb{Z}$-linear map $\mathrm{B}\mathbb{G}_m \to \lim_{\mathrm{N}} \mathrm{B}^2(\mu_{\mathrm{N}})$, so in particular a rigidified section over $\mathrm{B}\mathbb{G}_m$.

Our descriptions of the maps (1) & (2) are as follows:

(1) $x_1 \otimes x_2 \otimes a \mapsto (x_1^*\Psi \cup x_2^*\Psi) \otimes a$, for $x_1, x_2 \in \check{\Lambda}$ and $a \in \mathrm{A}(-1)$;
(2) $x \otimes t \mapsto x^*\Psi^*(t)$, where $t \in \mathrm{B}^2\mathrm{A}$ is viewed as a map $\lim_{\mathrm{N}} \mathrm{B}^2(\mu_{\mathrm{N}}) \to \mathrm{B}^4\mathrm{A}(1)$.

It is not *a priori* clear how to identify them over $\wedge^2(\check{\Lambda}) \otimes \mathrm{A}(-1)$. This identification will be constructed in the proof of Theorem 4.3.2.

In this proof, we shall first construct the analogous diagram for $\underline{\Gamma}_e(\mathrm{T}, \mathrm{B}^2\mathrm{A}(1))$, where the necessary identification will come from the self-cup product formula $\Psi \cup \Psi \cong \Psi^*(\Psi(-1))$ of §3. The corresponding diagram for $\underline{\Gamma}_e(\mathrm{BT}, \mathrm{B}^4\mathrm{A}(1))$ follows from a cosimplicial limit.

Remark 4.3.4. Informally, the functor (1) produces extensions of T by $\mathrm{B}^2\mathrm{A}(1)$ which are "defined by cocycles", while the functor (2) produces $\mathbb{Z}$-linear extensions. We justify the second statement now but defer the first to §4.4.

Indeed, the functor (2) in (4.13) defines morphisms of complexes $\mathrm{T}[1] \to \mathrm{A}(1)[4]$, i.e. $\mathbb{Z}$-linear maps $\mathrm{BT} \to \mathrm{B}^4\mathrm{A}(1)$. In fact, it is an equivalence onto the groupoid of $\mathbb{Z}$-linear maps by the following calculation using Lemma 3.1.3:

$$\begin{aligned} \underline{\mathrm{Maps}}_{\mathbb{Z}}(\mathrm{BT}, \mathrm{B}^4\mathrm{A}(1)) &\cong \check{\Lambda} \otimes \underline{\mathrm{Maps}}_{\mathbb{Z}}(\mathrm{B}\mathbb{G}_m, \mathrm{B}^4\mathrm{A}(1)) \\ &\cong \check{\Lambda} \otimes \mathrm{B}^2\mathrm{A}. \end{aligned} \tag{4.14}$$

The cofiber sequence in (4.13) implies that $\mathbb{Z}$-linear maps $\mathrm{BT} \to \mathrm{B}^4\mathrm{A}(1)$ form a *full* subgroupoid of the goupoid of all pointed maps.

Proposition 4.3.5. *There is a canonical equivalence of étale sheaves of connective $\mathbb{Z}$-module spectra:*

$$\vartheta^{(1)}(\Lambda) \otimes \mathrm{A}(-1) \cong \underline{\Gamma}_e(\mathrm{T}, \mathrm{B}^2\mathrm{A}(1)). \tag{4.15}$$

Proof. We shall first construct a functor from $\vartheta^{(1)}(\Lambda)\otimes \mathrm{A}(-1)$ to $\underline{\Gamma}_e(\mathrm{T},\mathrm{B}^2\mathrm{A}(1))$ using the description of the former as a pushout, see (4.7). Indeed, let us first construct maps:

$$\begin{array}{ccc} & & \check{\Lambda}^{\otimes 2}\otimes \mathrm{A}(-1) \\ & & \big\downarrow{\scriptstyle (1)} \\ \check{\Lambda}\otimes \mathrm{A}[1] & \xrightarrow{(2)} & \underline{\Gamma}_e(\mathrm{T},\mathrm{B}^2\mathrm{A}(1)) \end{array} \tag{4.16}$$

Writing $\Psi:\mathbb{G}_m\to \lim_{\mathrm{N}}\mathrm{B}(\mu_{\mathrm{N}})$ for the Kummer torsor, we describe these maps as follows:

(1) $x_1\otimes x_2\otimes a\mapsto (x_1^*\Psi\cup x_2^*\Psi)\otimes a$, for $x_1,x_2\in\check{\Lambda}$ and $a\in \mathrm{A}(-1)$;
(2) $x\otimes t\mapsto x^*\Psi^*(t)$, where $t\in \mathrm{BA}$ is viewed as a map $\lim_{\mathrm{N}}\mathrm{B}(\mu_{\mathrm{N}})\to \mathrm{B}^2\mathrm{A}(1)$.

Let us construct an identification of their restrictions to $\Gamma^2(\check{\Lambda})\otimes \mathrm{A}(-1)$.

It suffices to treat the case $\mathrm{A}=\mu_{\mathrm{N}}$ for some $\mathrm{N}\geq 1$ invertible on S. In this case, (4.16) consists of $\mathbb{Z}/\mathrm{N}$-linear morphisms, so by adjunction, it suffices to identify the two circuits of the following diagram:

$$\begin{array}{ccc} \Gamma^2(\check{\Lambda}) & \longrightarrow & \check{\Lambda}^{\otimes 2} \\ \big\downarrow & & \big\downarrow \\ \check{\Lambda}/2 & \xrightarrow{x\mapsto x\otimes x} & \mathrm{Ant}^2(\check{\Lambda}) \\ {\scriptstyle \Psi(-1)}\big\downarrow & & \big\downarrow{\scriptstyle (1)} \\ \check{\Lambda}\otimes \mathrm{B}\mu_{\mathrm{N}} & \xrightarrow{(2)} & \underline{\Gamma}_e(\mathrm{T},\mathrm{B}^2\mu_{\mathrm{N}}^{\otimes 2}) \end{array} \tag{4.17}$$

Here, we have factored the morphism (1) through $\mathrm{Ant}^2(\check{\Lambda})$ using the canonical anti-symmetric structure on $x_1\otimes x_2\mapsto x_1^*\Psi\cup x_2^*\Psi$ and the vanishing of H^{-1} in Remark 4.1.4.

The commutativity of the lower square in (4.17) can in turn be constructed by identifying the two circuits after pre-composing with $\check{\Lambda}\twoheadrightarrow\check{\Lambda}/2$, and showing that this identification is compatible with 2-torsion. Over $\check{\Lambda}$, the two circuits are defined by the Yoneda pairing:

$$\check{\Lambda}\otimes\underline{\Gamma}_e(\mathbb{G}_m,\mathrm{B}^2\mu_{\mathrm{N}}^{\otimes 2})\to\underline{\Gamma}_e(\mathrm{T},\mathrm{B}^2\mu_{\mathrm{N}}^{\otimes 2}),\quad x\otimes t\mapsto x^*(t)$$

applied to sections $t:=\Psi\cup\Psi$ respectively $\Psi^*(\Psi(-1))$. They are identified by Theorem 3.1.5 compatibly with the 2-torsion structures.

Having constructed the functor from $\vartheta^{(1)}(\Lambda)\otimes\mathrm{A}(-1)$ to $\underline{\Gamma}_e(\mathrm{T},\mathrm{B}^2\mathrm{A}(1))$, we prove that it is an equivalence by calculating the induced maps on π_n. Since the cohomology of $\mathrm{T}\to\mathrm{S}$ commutes with arbitrary base change [Del96, Rappel 1.5.1], we may replace S by the spectrum of a separably closed field. In this case, there holds:

$$\mathrm{H}^i(\mathrm{T},\mathrm{A}(1))\cong\begin{cases}\check{\Lambda}\otimes\mathrm{A} & i=1\\ \wedge^2(\check{\Lambda})\otimes\mathrm{A}(-1) & i=2\end{cases}$$

and the isomorphisms are defined by Ψ and the cup product.

On the other hand, $\vartheta^{(1)}(\Lambda)\otimes\mathrm{A}(-1)$ is an extension of $\wedge^2(\check{\Lambda})\otimes\mathrm{A}(-1)$ by $\check{\Lambda}\otimes\mathrm{BA}$ by the maps defined in (4.7). This finishes the proof. □

Proof of Theorem 4.3.2. Consider the cosimplicial system $[n]\mapsto\check{\Lambda}^{\oplus n}$ defined by the coalgebra structure on $\check{\Lambda}$, see §4.1.10. Since $\vartheta^{(2)}(\Lambda)\otimes\mathrm{A}(-1)$ is identified with $\lim_{[n]}(\vartheta^{(1)}(\Lambda^{\oplus n})\otimes$

$A(-1)[2]$) and Proposition 4.3.5 identifies $\vartheta^{(1)}(\Lambda)\otimes A(-1)$ with $\underline{\Gamma}_e(T, B^2A(1))$ naturally in T, it suffices to establish an equivalence:

$$\underline{\Gamma}_e(BT, B^4A(1)) \cong \lim_{[n]} B^2\underline{\Gamma}_e(T^{\times n}, B^2A(1)). \tag{4.18}$$

We argue that expressing BT as the colimit of the simplicial system $[n]\mapsto T^{\times n}$ defines equivalences whose composition gives (4.18):

$$\underline{\Gamma}_e(BT, B^4A(1)) \cong \lim_{[n]} \underline{\Gamma}_e(T^{\times n}, B^4A(1)) \tag{4.19}$$

$$\cong \lim_{[n]} B\underline{\Gamma}_e(T^{\times n}, B^3A(1)) \tag{4.20}$$

$$\cong \lim_{[n]} B^2\underline{\Gamma}_e(T^{\times n}, B^2A(1)). \tag{4.21}$$

Indeed, (4.19) follows from writing $\underline{\Gamma}(S, B^4A(1))$ as the limit of $[n]\mapsto\underline{\Gamma}(S, B^4A(1))$ and commuting the limit with fiber product. The isomorphism (4.20) follows because the zeroth simplex $\underline{\Gamma}_e(S, B^3A(1))$ is contractible. Next, we observe that:

$$\pi_0\underline{\Gamma}_e(T^{\times n}, B^3A(1)) \cong \wedge^3(\Lambda^{\oplus n})\otimes A(-2).$$

However, we have $\lim_{[n]}(\wedge^3(\check{\Lambda}^{\oplus n}))\cong \mathrm{Sym}^3(\check{\Lambda})[-3]$ as chain complexes, so the limit of spaces $\lim_{[n]} B(\pi_0\underline{\Gamma}_e(T^{\times n}, B^3A(1)))$ is contractible, giving (4.21). □

4.4. Covers defined by cocycles.

4.4.1. Let us study the morphism (1) in (4.13) more closely. We may regard any pointed morphism $BT\to B^4A(1)$ as an $\mathbb{E}_1$-monoidal morphism $T\to B^3A(1)$. Taking the fiber at the neutral point of $B^3A(1)$, we obtian an $\mathbb{E}_1$-monoidal extension $T^\dagger$ of T by $B^2A(1)$.

If the underlying pointed morphism of $T\to B^3A(1)$ is trivialized, then $T^\dagger\to T$ admits a section s as a pointed stack. Consequently, we obtain a morphism:

$$T\times T\to B^2A(1),\quad (t_1,t_2)\mapsto s(t_1t_2)s(t_1)^{-1}s(t_2)^{-1}. \tag{4.22}$$

4.4.2. Suppose that X_1, X_2 are stacks over S pointed by $e_1 : S\to X_1$, $e_2 : S\to X_2$ and $n\geq 0$ is an integer.

We call a section of $B^nA(1)$ over $X_1\times X_2$ *bi-rigidified* if it is equipped with trivializations along $e_1\times X_2$, $X_1\times e_2$ and an isomorphism of the two induced trivializations along $e_1\times e_2$. We denote the étale sheaf of bi-rigidified morphism by:

$$\underline{\Gamma}_{e_1,e_2}(X_1\times X_2, B^nA(1)).$$

Note that any section $x_1\otimes x_2\otimes a$ of $\check{\Lambda}^{\otimes 2}\otimes A(-1)$ naturally defines a bi-rigidified morphism $T\times T\to B^2A(1)$ given by $(p_1^*x_1^*\Psi\cup p_2^*x_2^*\Psi)\otimes a$, where p_1, p_2 denote the two projections from $T\times T$ to T. Here, Ψ denotes the Kummer torsor on $\mathbb{G}_m$.

Lemma 4.4.3. *The above functor and its analogue for* $BT\times BT$ *define equivalences:*

$$\check{\Lambda}^{\otimes 2}\otimes A(-1)\cong \underline{\Gamma}_{e,e}(T\times T, B^2A(1)),$$

$$\check{\Lambda}^{\otimes 2}\otimes A(-1)\cong \underline{\Gamma}_{e,e}(BT\times BT, B^4A(1)).$$

Proof. Indeed, morphisms $T\times T\to B^2A(1)$ rigidified along $T\times e$ are equivalent to morphisms

$$T\to \underline{\Gamma}_e(T, B^2A(1)). \tag{4.23}$$

Hence rigidifying it along $e \times \mathrm{T}$, *as* a morphism rigidified along $\mathrm{T} \times e$, is equivalent to rigidifying the corresponding morphism (4.23).

By Proposition 4.3.5, the neutral component of $\underline{\Gamma}_e(\mathrm{T}, \mathrm{B}^2\mathrm{A}(1))$ is identified with $\mathrm{B}^2(\check{\Lambda} \otimes \mathrm{A})$. Hence, a rigidified morphism (4.23) is equivalent to a rigidified morphism $\mathrm{BT} \to \mathrm{B}^2(\check{\Lambda} \otimes \mathrm{A})$, which by Proposition 4.3.5 again (applied to coefficient group $\check{\Lambda} \otimes \mathrm{A}$) is equivalent to a section of $\check{\Lambda}^{\otimes 2} \otimes \mathrm{A}(-1)$.

The statment for $\underline{\Gamma}_{e,e}(\mathrm{BT} \times \mathrm{BT}, \mathrm{B}^4\mathrm{A}(1))$ is proved in the same manner, substituting Proposition 4.3.5 by Theorem 4.3.2. □

4.4.4. Consider the functor of forgetting the $\mathbb{E}_1$-monoidal structure:

$$\begin{aligned}\underline{\Gamma}_e(\mathrm{BT}, \mathrm{B}^4\mathrm{A}(1)) &\cong \underline{\mathrm{Maps}}_{\mathbb{E}_1}(\mathrm{T}, \mathrm{B}^3\mathrm{A}(1)) \\ &\to \underline{\Gamma}_e(\mathrm{T}, \mathrm{B}^3\mathrm{A}(1)).\end{aligned} \tag{4.24}$$

An object in the fiber of (4.24) naturally defines a bi-rigidified morphism (4.22) as the cocycle of the corresponding extension of T by $\mathrm{B}^2\mathrm{A}(1)$, which gives a section of $\check{\Lambda}^{\otimes 2} \otimes \mathrm{A}(-1)$ by Lemma 4.4.3: we call it the *cocycle* of this object.

The following result is a formulation of the idea that sections of $\check{\Lambda}^{\otimes 2} \otimes \mathrm{A}(-1)$ yield étale metaplectic covers "defined by cocycles".

Proposition 4.4.5. *The functor* (1) *of* (4.13) *defines an equivalence between* $\check{\Lambda}^{\otimes 2} \otimes \mathrm{A}(-1)$ *and the fiber of* (4.24)*. Its inverse is given by the association of cocycles.*

Proof. Recall from the proof of Theorem 4.3.2 that the functor (1) of (4.13) is defined by the cosimplicial limit of a system of maps of complexes parametrized by $[n] \in \Delta$:

$$(\check{\Lambda}^{\oplus n})^{\otimes 2} \otimes \mathrm{A}(-1)[2] \to \underline{\Gamma}_e(\mathrm{T}^{\times n}, \mathrm{B}^2\mathrm{A}(1))[2],$$

using Lemma 4.1.11(2). The forgetful functor (4.24) is defined by the evaluation at $[1] \in \Delta$:

$$\begin{aligned}\lim_{[n]} \underline{\Gamma}_e(\mathrm{T}^{\times n}, \mathrm{B}^2\mathrm{A}(1))[2] &\to \underline{\Gamma}_e(\mathrm{T}, \mathrm{B}^2\mathrm{A}(1))[1] \\ &\cong \mathrm{B}\underline{\Gamma}_e(\mathrm{T}, \mathrm{B}^2\mathrm{A}(1)) \subset \underline{\Gamma}_e(\mathrm{T}, \mathrm{B}^3\mathrm{A}(1)).\end{aligned}$$

In particular, we obtain a commutative diagram:

$$\begin{array}{ccc} \check{\Lambda}^{\otimes 2} \otimes \mathrm{A}(-1) & \xrightarrow{\mathrm{ev}_{[1]}} & \mathrm{B}(\check{\Lambda}^{\otimes 2} \otimes \mathrm{A}(-1)) \\ \big\downarrow{\scriptstyle (1)} & & \big\downarrow \\ \underline{\Gamma}_e(\mathrm{BT}, \mathrm{B}^4\mathrm{A}(1)) & \xrightarrow{\mathrm{ev}_{[1]}} & \mathrm{B}\underline{\Gamma}_e(\mathrm{T}, \mathrm{B}^2\mathrm{A}(1)) \end{array} \tag{4.25}$$

The top horizontal arrow is canonically trivialized by splitting the quotient of (4.6). Therefore, the lower circuit of (4.25) is canonically trivialized, so we obtain a functor from $\check{\Lambda}^{\otimes 2} \otimes \mathrm{A}(-1)$ to the fiber of the forgetful functor.

To conclude that it is an equivalence, it suffices to observe that this lower circuit induces a long exact sequence of homotopy groups:

$$\begin{aligned}0 \to \mathrm{A} \cong \mathrm{A} \to 0 \to 0 \to &\wedge^2\check{\Lambda} \otimes \mathrm{A}(-1) \\ &\to \check{\Lambda}^{\otimes 2} \otimes \mathrm{A}(-1) \to \mathrm{Sym}^2(\check{\Lambda}) \otimes \mathrm{A}(-1) \to 0.\end{aligned}$$

The assertion on the inverse is left to the interested reader. □

4.5. The $\mathbb{E}_1$-monoidal morphism $\mathrm{F}:\Lambda\to\mathrm{B}^2\mathrm{A}$.

4.5.1. Viewing a pointed morphism $\mu:\mathrm{BT}\to\mathrm{B}^4\mathrm{A}(1)$ as an $\mathbb{E}_1$-monoidal morphism $\mathrm{T}\to\mathrm{B}^3\mathrm{A}(1)$, we may apply the functor $\underline{\Gamma}_e(\mathbb{G}_m,-)$ to obtain an $\mathbb{E}_1$-monoidal morphism:

$$\begin{aligned}\Lambda &\to \underline{\Gamma}_e(\mathbb{G}_m,\mathrm{T})\\ &\to \underline{\Gamma}_e(\mathbb{G}_m,\mathrm{B}^3\mathrm{A}(1))\cong\mathrm{B}^2(\mathrm{A}),\end{aligned}\tag{4.26}$$

where the last isomorphism follows from the proof of Lemma 3.1.3, using the vanishing of $\mathrm{H}^3(\mathbb{G}_{m,\bar{s}},\mathrm{A}(1))$ at a geometric point $\bar{s}$.

The following Proposition tells us how to read off the $\mathbb{E}_1$-monoidal morphism (4.26) from the ϑ-datum corresponding to μ.

Proposition 4.5.2. *The following diagram is canonically commutative:*

$$\begin{array}{ccc}\underline{\Gamma}_e(\mathrm{BT},\mathrm{B}^4\mathrm{A}(1)) & \cong & \underline{\mathrm{Maps}}_{\mathbb{E}_1}(\mathrm{T},\mathrm{B}^3\mathrm{A}(1))\\ \downarrow & & \downarrow{\scriptstyle\underline{\Gamma}_e(\mathbb{G}_m,-)}\\ \check{\mathrm{H}}^{(2)}(\Lambda)\otimes\mathrm{B}^2\mathrm{A} & \longrightarrow & \underline{\mathrm{Maps}}_{\mathbb{E}_1}(\Lambda,\mathrm{B}^2\mathrm{A})\end{array}\tag{4.27}$$

where the left vertical arrow is as in (4.13) *and the bottom horizontal arrow is the restriction along the monoidal section of the map* $\mathrm{H}^{(2)}(\Lambda)\to\Lambda$ *in* (4.4).

Proof. As in the proof of Theorem 4.3.2, we shall produce this commutative diagram from an analogous commutative diagram for $\underline{\Gamma}_e(\mathrm{T},\mathrm{B}^2\mathrm{A}(1))$ via a cosimplicial limit.

The relevant diagram for $\underline{\Gamma}_e(\mathrm{T},\mathrm{B}^2\mathrm{A}(1))$ is the following one:

$$\begin{array}{ccc}\underline{\Gamma}_e(\mathrm{T},\mathrm{B}^2\mathrm{A}(1)) & \cong & \underline{\Gamma}_e(\mathrm{T},\mathrm{B}^2\mathrm{A}(1))\\ \downarrow & & \downarrow{\scriptstyle\underline{\Gamma}_e(\mathbb{G}_m,-)}\\ \check{\mathrm{H}}^{(1)}(\Lambda)\otimes\mathrm{BA} & \longrightarrow & \underline{\Gamma}_e(\Lambda,\mathrm{BA})\end{array}\tag{4.28}$$

To prove that it is commutative, we use the description of $\underline{\Gamma}_e(\mathrm{T},\mathrm{B}^2\mathrm{A}(1))$ as a pushout, provided by Proposition 4.3.5 and (4.7). Namely, we identify the two circuits of (4.28) after pre-composing with the two maps of (4.16), and check that these identifications agree over $\Gamma^2(\check{\Lambda})\otimes\mathrm{A}(-1)$.

The identification over $\check{\Lambda}\otimes\mathrm{BA}$ is clear. To make the identification over $\check{\Lambda}^{\otimes 2}\otimes\mathrm{A}(-1)$, we may assume $\mathrm{A}=\mu_{\mathrm{N}}$ for some N invertible on S. Along the upper circuit of (4.28), we thus find the following map by adjunction:

$$\check{\Lambda}^{\otimes 2}\to\underline{\Gamma}_e(\Lambda,\mathrm{BA}),\quad x_1\otimes x_2\mapsto(\lambda\mapsto\Psi^{x_1(\lambda)}\cup\Psi^{x_2(\lambda)}),\tag{4.29}$$

where $\Psi^{x_1(\lambda)}\cup\Psi^{x_2(\lambda)}$ is viewed as a section of $\mathrm{B}(\mathrm{A})$ via the equivalence $\underline{\Gamma}_e(\mathbb{G}_m,\mathrm{B}^2\mathrm{A}(1))\cong\mathrm{B}(\mathrm{A})$ of Lemma 3.1.3, defined by Ψ^*.

On the other hand, Theorem 3.1.5 gives us an identification:

$$\Psi^{x_1(\lambda)}\cup\Psi^{x_2(\lambda)}\cong\Psi^*(\Psi(-1)^{x_1(\lambda)x_2(\lambda)}),$$

so (4.29) sends a bilinear form c to the pointed map $\lambda\mapsto\Psi(-1)^{c(\lambda,\lambda)}$. This is precisely the restriction of the lower circuit of (4.28) to $\check{\Lambda}^{\otimes 2}$, see the description after (4.7). We omit the verification that these identifications agree over $\Gamma^2(\check{\Lambda})\otimes\mathrm{A}(-1)$. □

4.6. $\mathbb{E}_\infty$-monoidal covers.

4.6.1. Let S, A, and $\mathrm{T} = \Lambda \otimes \mathbb{G}_m$ be as in §4.3.1. Viewing sections of $\underline{\Gamma}_e(\mathrm{BT}, \mathrm{B}^4\mathrm{A}(1))$ as $\mathbb{E}_0$-monoidal morphisms $\mathrm{BT} \to \mathrm{B}^4\mathrm{A}(1)$, we see that it receives forgetful functors:

$$\underline{\mathrm{Maps}}_{\mathbb{E}_\infty}(\mathrm{BT}, \mathrm{B}^4\mathrm{A}(1)) \to \cdots \to \underline{\mathrm{Maps}}_{\mathbb{E}_1}(\mathrm{BT}, \mathrm{B}^4\mathrm{A}(1)) \tag{4.30}$$

$$\to \underline{\Gamma}_e(\mathrm{BT}, \mathrm{B}^4\mathrm{A}(1)). \tag{4.31}$$

According to Theorem 4.3.2 and the triangle (4.11), $\underline{\Gamma}_e(\mathrm{BT}, \mathrm{B}^4\mathrm{A}(1))$ admits a canonical functor to $\mathrm{Sym}^2(\check{\Lambda}) \otimes \mathrm{A}(-1)$; we call the image of μ the quadratic form Q *associated to* μ. Then Q defines an $\mathrm{A}(-1)$-valued symmetric form b.

Proposition 4.6.2. *The following statements hold:*

(1) the functors in (4.30) *are equivalences;*

(2) the functor (4.31) *is fully faithful and its essential image consists of sections* $\mu \in \underline{\Gamma}_e(\mathrm{BT}, \mathrm{B}^4\mathrm{A}(1))$ *whose associated symmetric form* b *vanishes.*

Proof. The Bar construction defines an equivalence for all $k \geq 1$:

$$\underline{\mathrm{Maps}}_{\mathbb{E}_{k-1}}(\mathrm{BT}, \mathrm{B}^4\mathrm{A}(1)) \cong \underline{\Gamma}_e(\mathrm{B}^k\mathrm{T}, \mathrm{B}^{k+4}\mathrm{A}(1)).$$

For $k = 1$, the simplicial system $[n] \mapsto \mathrm{BT}^{\times n}$ yields an equivalence by descent:

$$\underline{\Gamma}_e(\mathrm{B}^2\mathrm{T}, \mathrm{B}^5\mathrm{A}(1)) \cong \lim_{[n]} \underline{\Gamma}_e(\mathrm{BT}^{\times n}, \mathrm{B}^5\mathrm{A}(1))$$
$$\cong \lim_{[n]} \mathrm{B}\underline{\Gamma}_e(\mathrm{BT}^{\times n}, \mathrm{B}^4\mathrm{A}(1)).$$

The triangle in (4.13) shows that we have a triangle of *complexes*:

$$\lim_{[n]}(\check{\Lambda}^{\oplus n} \otimes \mathrm{A}[3]) \to \lim_{[n]}(\Gamma_e(\mathrm{BT}^{\times n}, \mathrm{B}^4\mathrm{A}(1))[1]) \to \lim_{[n]}(\mathrm{Sym}^2(\check{\Lambda}^{\oplus n}) \otimes \mathrm{A}(-1)[1]), \tag{4.32}$$

Applying the isomorphisms $\lim_{[n]}(\check{\Lambda}^{\oplus n}) \cong \check{\Lambda}[-1]$, $\lim_{[n]}(\mathrm{Sym}^2(\check{\Lambda})^{\oplus n}) \cong \mathrm{Ant}^2(\check{\Lambda})[-2]$ of Lemma 4.1.11, we obtain a cofiber sequence of connective complexes by truncating (4.32) in degrees ≤ 0:

$$\check{\Lambda} \otimes \mathrm{B}^2\mathrm{A} \to \underline{\Gamma}_e(\mathrm{B}^2\mathrm{T}, \mathrm{B}^5\mathrm{A}(1)) \to \mathrm{Tor}^1(\check{\Lambda}/2, \mathrm{A}(-1)). \tag{4.33}$$

Here, the last term appears through the isomorphism below, as $\wedge^2(\check{\Lambda})$ is torsion-free:

$$\mathrm{Tor}^1(\check{\Lambda}/2, \mathrm{A}(-1)) \cong \mathrm{Tor}^1(\mathrm{Ant}^2(\check{\Lambda}), \mathrm{A}(-1)).$$

Now, the cofiber sequence (4.33) is fixed under the operation $\lim_{[n]} \mathrm{B}(-)$ applied to the cosimplicial system $[n] \mapsto \check{\Lambda}^{\oplus n}$. The same descent argument then gives statement (1).

For statement (2), we observe that the canonical functor:

$$\underline{\Gamma}_e(\mathrm{B}^2\mathrm{T}, \mathrm{B}^5\mathrm{A}(1)) \to \underline{\Gamma}_e(\mathrm{BT}, \mathrm{B}^4\mathrm{A}(1)) \tag{4.34}$$

comes from the morphism of triangles from (4.32) to the triangle in (4.13) defined by evaluation at the first simplex [1]. The isomorphisms exhibited in the proof of Lemma 4.1.11 then shows that (4.34) fits into a map of cofiber sequences:

$$\begin{array}{ccccc} \check{\Lambda} \otimes \mathrm{B}^2\mathrm{A} & \to & \underline{\Gamma}_e(\mathrm{B}^2\mathrm{T}, \mathrm{B}^5\mathrm{A}(1)) & \to & \mathrm{Tor}^1(\check{\Lambda}/2, \mathrm{A}(-1)) \\ \downarrow \cong & & \downarrow & & \downarrow \\ \check{\Lambda} \otimes \mathrm{B}^2\mathrm{A} & \to & \underline{\Gamma}_e(\mathrm{BT}, \mathrm{B}^4\mathrm{A}(1)) & \to & \mathrm{Sym}^2(\check{\Lambda}) \otimes \mathrm{A}(-1) \end{array} \tag{4.35}$$

where the rightmost arrow comes from the short exact sequence in (4.3):

$$0 \to \mathrm{Sym}^2(\check{\Lambda}) \to \Gamma^2(\check{\Lambda}) \to \check{\Lambda}/2 \to 0.$$

(It is deduced from the extension of $\mathrm{Ant}^2(\check{\Lambda})$ by $\mathrm{Sym}^2(\check{\Lambda})$ defined by $\check{\Lambda}^{\otimes 2}$, the nondegenerate cochains in $\mathrm{Sym}^2(\check{\Lambda}^{\oplus 2})$.) In other words, $\mathrm{Tor}^1(\check{\Lambda}/2, \mathrm{A}(-1))$ appears in (4.35) as the subgroup of $\mathrm{A}(-1)$-valued quadratic form Q on Λ whose symmetric form vanishes. □

Remark 4.6.3. Using the conclusion of Proposition 4.6.2, we may reproduce the cofiber sequence (4.33) as follows:

$$\underline{\mathrm{Maps}}_{\mathbb{Z}}(\mathrm{BT}, \mathrm{B}^4\mathrm{A}(1)) \to \underline{\mathrm{Maps}}_{\mathbb{E}_\infty}(\mathrm{BT}, \mathrm{B}^4\mathrm{A}(1)) \to \underline{\mathrm{Maps}}_{\mathbb{Z}}(\Lambda/2, \mathrm{A}(-1)), \tag{4.36}$$

where the second functor takes μ to its associated quadratic form Q, viewed as a 2-torsion homomorphism $\Lambda \to \mathrm{A}(-1)$.

4.6.4. If $\mu : \mathrm{BT} \to \mathrm{B}^4\mathrm{A}(1)$ is an $\mathbb{E}_\infty$-monoidal morphism, then the corresponding composition (4.26) inherits an $\mathbb{E}_\infty$-monoidal structure.

In other words, taking $\underline{\Gamma}_e(\mathbb{G}_m, -)$ defines a functor:

$$\begin{aligned} \underline{\mathrm{Maps}}_{\mathbb{E}_\infty}(\mathrm{BT}, \mathrm{B}^4\mathrm{A}(1)) &\cong \underline{\mathrm{Maps}}_{\mathbb{E}_\infty}(\mathrm{T}, \mathrm{B}^3\mathrm{A}(1)) \\ &\to \underline{\mathrm{Maps}}_{\mathbb{E}_\infty}(\Lambda, \mathrm{B}^2\mathrm{A}). \end{aligned} \tag{4.37}$$

4.6.5. On the other hand, there is a canonical functor:

$$\mathrm{inv} : \underline{\mathrm{Maps}}_{\mathbb{E}_\infty}(\Lambda, \mathrm{B}^2\mathrm{A}) \to \underline{\mathrm{Maps}}_{\mathbb{Z}}(\Lambda/2, \mathrm{A}), \tag{4.38}$$

with fiber $\underline{\mathrm{Maps}}_{\mathbb{Z}}(\Lambda, \mathrm{B}^2\mathrm{A})$.

To define it, we observe that an $\mathbb{E}_\infty$-monoidal morphism $\mathrm{F} : \Lambda \to \mathrm{B}^2(\mathrm{A})$ defines a symmetric monoidal extension of Λ by $\mathrm{B}(\mathrm{A})$, whose commutativity constraint is an anti-symmetric map $c : \Lambda \otimes \Lambda \to \mathrm{A}$. Then (4.38) sends F to the homomorphism $\lambda \mapsto c(\lambda, \lambda)$.

Proposition 4.6.6. *The functor* (4.37) *is an equivalence and identifies the cofiber sequence* (4.36) *with that defined by* (4.38)*:*

$$\begin{array}{ccccc} \underline{\mathrm{Maps}}_{\mathbb{Z}}(\mathrm{BT}, \mathrm{B}^4\mathrm{A}(1)) & \to & \underline{\mathrm{Maps}}_{\mathbb{E}_\infty}(\mathrm{BT}, \mathrm{B}^4\mathrm{A}(1)) & \xrightarrow{\mu \mapsto \mathrm{Q}} & \underline{\mathrm{Maps}}_{\mathbb{Z}}(\Lambda/2, \mathrm{A}(-1)) \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ \check{\Lambda} \otimes \mathrm{B}^2\mathrm{A} & \longrightarrow & \underline{\mathrm{Maps}}_{\mathbb{E}_\infty}(\Lambda, \mathrm{B}^2\mathrm{A}) & \xrightarrow{\mathrm{inv}} & \underline{\mathrm{Maps}}_{\mathbb{Z}}(\Lambda/2, \mathrm{A}) \end{array} \tag{4.39}$$

Here, the right vertical isomorphism uses the canonical identification between the 2*-torsion subgroup of* $\mathrm{A}(-1)$ *and that of* A.

Proof. The statement is tautological when A has no 2-torsion. In what follows, we assume that A has 2-torsion. In particular, 2 is invertible over the base scheme S.

We begin by giving a different interpretation to both rows in (4.39), where their identification is tautological. Let $\mathbb{S}$ denote the sphere spectrum. Using the identification between sheaves of connective spectra and sheaves of grouplike $\mathbb{E}_\infty$-monoids, we have:

$$\begin{aligned} \underline{\mathrm{Maps}}_{\mathbb{E}_\infty}(\mathrm{BT}, \mathrm{B}^4\mathrm{A}(1)) &\cong \underline{\mathrm{Maps}}_{\mathbb{S}}(\mathrm{T}[1], \mathrm{A}(1)[4]) \\ &\cong \underline{\mathrm{Maps}}_{\mathbb{Z}}(\mathrm{T} \otimes_{\mathbb{Z}} (\mathbb{Z} \otimes_{\mathbb{S}} \mathbb{Z})[1], \mathrm{A}(1)[4]) \end{aligned}$$

Similarly, we have:

$$\underline{\mathrm{Maps}}_{\mathbb{Z}}(\Lambda, \mathrm{B}^2\mathrm{A}) \cong \underline{\mathrm{Maps}}_{\mathbb{Z}}(\mathrm{T} \otimes_{\mathbb{Z}} (\mathbb{Z} \otimes_{\mathbb{S}} \mathbb{Z}), \mathrm{A}[2]).$$

Consider the canonical triangle given by cohomological truncation of $\mathbb{Z} \otimes_{\mathbb{S}} \mathbb{Z}$:

$$\mathrm{L} \to \mathbb{Z} \otimes_{\mathbb{S}} \mathbb{Z} \to \mathbb{Z} \tag{4.40}$$

where L has cohomology in degrees ≤ -2, with $\mathrm{H}^{-2}(\mathrm{L}) \cong \mathbb{Z}/2$. Thus $\mathrm{T} \otimes_{\mathbb{Z}} \mathrm{L}$ has cohomology in degrees ≤ -3, with $\mathrm{H}^{-3}(\mathrm{T} \otimes_{\mathbb{Z}} \mathrm{L}) \cong \Lambda \otimes \{\pm 1\}$.

Tensoring (4.40) with T[1], respectively Λ, and mapping the results into A(1)[4], respectively A[2], we obtain a morphism of triangles:

$$\begin{array}{ccccc} \underline{\mathrm{Maps}}_{\mathbb{Z}}(\mathrm{T}[1], \mathrm{A}(1)[4]) & \to & \underline{\mathrm{Maps}}_{\mathbb{S}}(\mathrm{T}[1], \mathrm{A}(1)[4]) & \to & \underline{\mathrm{Maps}}_{\mathbb{Z}}(\Lambda \otimes \{\pm 1\}, \mathrm{A}(1)) \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ \underline{\mathrm{Maps}}_{\mathbb{Z}}(\Lambda, \mathrm{A}[2]) & \longrightarrow & \underline{\mathrm{Maps}}_{\mathbb{S}}(\Lambda, \mathrm{A}[2]) & \longrightarrow & \underline{\mathrm{Maps}}_{\mathbb{Z}}(\Lambda \otimes \mathbb{Z}/2, \mathrm{A}) \end{array} \tag{4.41}$$

It remains to identify (4.41) with (4.39). This is clear for the left squares. For the right squares, we reduce to the case $\Lambda = \mathbb{Z}$ and $\mathrm{A} = \mathbb{Z}/2$. Then the rightmost columns consist of copies of $\mathbb{Z}/2$, so the required identification follows from that of the left squares. ☐

Remark 4.6.7. Let us give an alternative definition of the morphism (4.37) which is often useful in practice. Namely, taking $\underline{\Gamma}_e(\mathrm{B}\mathbb{G}_m, -)$ yields a functor:

$$\underline{\mathrm{Maps}}_{\mathbb{E}_\infty}(\mathrm{BT}, \mathrm{B}^4\mathrm{A}(1)) \to \underline{\mathrm{Maps}}_{\mathbb{E}_\infty}(\Lambda, \mathrm{B}^2\mathrm{A} \oplus \mathrm{A}(-1)), \tag{4.42}$$

using the facts that Λ is isomorphic to the abelian group of rigidified sections of BT over $\mathrm{B}\mathbb{G}_m$ and $\underline{\Gamma}_e(\mathrm{B}\mathbb{G}_m, \mathrm{B}^4\mathrm{A}(1))$ is canonically isomorphic to $\mathrm{B}^2\mathrm{A} \oplus \mathrm{A}(-1)$ (Remark 4.2.8).

Composing (4.42) with the projection onto $\underline{\mathrm{Maps}}_{\mathbb{E}_\infty}(\Lambda, \mathrm{B}^2\mathrm{A})$, we obtain (4.37). On the other hand, composing (4.42) with the projection onto $\underline{\mathrm{Maps}}_{\mathbb{E}_\infty}(\Lambda, \mathrm{A}(-1))$, i.e. the sheaf of homomorphisms $\Lambda \to \mathrm{A}(-1)$, we find the second arrow in (4.36) mapping μ to Q.

4.7. Quadratic structure.

4.7.1. We have seen in Proposition 4.6.2 that a pointed morphism $\mu : \mathrm{BT} \to \mathrm{B}^4\mathrm{A}(1)$ whose symmetric form b vanishes acquires an $\mathbb{E}_1$ (or equivalently $\mathbb{E}_\infty$)-monoidal structure.

It turns out that in general, μ has a "quadratic structure" whose associated bilinear form is a pairing $\mathrm{BT} \times \mathrm{BT} \to \mathrm{B}^4\mathrm{A}(1)$ determined by b. This quadratic structure allows us to calculate the commutator of the $\mathbb{E}_1$-monoidal morphism $\mathrm{T} \to \mathrm{B}^3\mathrm{A}(1)$ corresponding to μ (Corollary 4.7.6). Similar calculations have also been performed in [Del96, §4].

4.7.2. Indeed, a pointed morphism $\mu : \mathrm{BT} \to \mathrm{B}^4\mathrm{A}(1)$ defines a bi-rigidified morphism (see §4.4.2):

$$(m^*\mu) \otimes (p_1^*\mu^{\otimes -1}) \otimes (p_2^*\mu^{\otimes -1}) : \mathrm{BT} \times \mathrm{BT} \to \mathrm{B}^4\mathrm{A}(1), \tag{4.43}$$

where m denotes the product morphism $\mathrm{BT} \times \mathrm{BT} \to \mathrm{BT}$.

The following result shows that μ is automatically a "quadratic function" whose associated symmetric form is defined by b.

Proposition 4.7.3. *Let $\mu : \mathrm{BT} \to \mathrm{B}^4\mathrm{A}(1)$ be a pointed morphism with associated symmetric form b. Then the bi-rigidified morphism* (4.43) *corresponds to b under Lemma 4.4.3.*

Proof. If μ comes from a $\mathbb{Z}$-linear morphism $\mathrm{BT} \to \mathrm{B}^4\mathrm{A}(1)$, then (4.43) is trivial and $b = 0$. Hence it suffices to determine the morphism (4.43) when μ is defined by a bilinear form $c \in \check{\Lambda}^{\otimes 2} \otimes \mathrm{A}(-1)$, see §4.3.3.

Since the statement is compatible with tensor product of the coefficient group A, we may assume that $\mathrm{A} = \mu_{\mathrm{N}}$ for an integer $\mathrm{N} \ge 1$ invertible on S. Because we wish to prove the equality of two elements of a discrete groupoid, namely $\check{\Lambda}\otimes\check{\Lambda}/\mathrm{N}$, we may consider an integral lift $c = x_1\otimes x_2$ for $x_1, x_2\in\check{\Lambda}$.

The morphism (4.43) is then given by:

$$\begin{aligned} e_1, e_2 \mapsto\ & (\Psi_{x_1(e_1e_2)}\cup\Psi_{x_2(e_1e_2)})\cdot(\Psi_{x_1(e_1)}\cup\Psi_{x_2(e_1)})^{-1}\cdot(\Psi_{x_1(e_2)}\cup\Psi_{x_2(e_2)})^{-1}\\ &\cong (\Psi_{x_1(e_1)}\cup\Psi_{x_2(e_2)})\cdot(\Psi_{x_1(e_2)}\cup\Psi_{x_2(e_1)}),\end{aligned}$$

using the linearity of x_1, x_2, Ψ, and the bilinearity of the cup product. Since the cup product of sections of $\mathrm{B}^2\mu_{\mathrm{N}}$ is *symmetric*, the last expression is precisely the value of the map $\mathrm{BT}\times\mathrm{BT}\to\mathrm{B}^4\mathrm{A}(1)$ at (e_1, e_2) corresponding to $x_1\otimes x_2 + x_2\otimes x_1$ under Lemma 4.4.3, i.e. the symmetrization of c. □

4.7.4. A pointed morphism $\mu : \mathrm{BT}\to\mathrm{B}^4\mathrm{A}(1)$ is equivalently described by an $\mathbb{E}_1$-monoidal morphism $\mathrm{T}\to\mathrm{B}^3\mathrm{A}(1)$ which we denote by the same letter μ.

Such a morphism has a "commutator", which is a bi-rigidified morphism:

$$\mathrm{com}(\mu) : \mathrm{T}\times\mathrm{T}\to\mathrm{B}^2\mathrm{A}(1), \tag{4.44}$$

It is defined as the following loop in the groupoid of bi-rigidified morphisms $\mathrm{T}\times\mathrm{T}\to\mathrm{B}^3\mathrm{A}(1)$ (using the equivalence between the loop space of such bi-rigidified morphisms with bi-rigidified morphisms $\mathrm{T}\times\mathrm{T}\to\mathrm{B}^2\mathrm{A}(1)$):

$$\begin{array}{ccc} \mu\circ m & \xrightarrow{\cong} & m\circ(\mu\times\mu)\\ {\scriptstyle\cong}\uparrow & & \downarrow{\scriptstyle\cong}\\ \mu\circ m^{\mathrm{op}} & \xleftarrow{\cong} & m^{\mathrm{op}}\circ(\mu\times\mu)\end{array}$$

where m^{op} denotes the reversed multiplication $a_1, a_2\mapsto a_2a_1$ on T and $\mathrm{B}^3\mathrm{A}(1)$, the horizontal arrows are defined by the $\mathbb{E}_1$-monoidal structure of μ, and the vertical maps are defined by the commutative structures $m\cong m^{\mathrm{op}}$.

Remark 4.7.5. Regarding an $\mathbb{E}_1$-monoidal morphism $\mathrm{T}\to\mathrm{B}^3\mathrm{A}(1)$ as an extension of $\mathbb{E}_1$-monoidal groupoids $\mathrm{B}^2\mathrm{A}(1)\to\mathrm{T}^\dagger\to\mathrm{T}$, the value of $\mathrm{com}(\mu)$ at sections $t_1, t_2\in\mathrm{T}$ is identified with the A(1)-gerbe of isomorphisms $\tilde{t}_1\tilde{t}_2\cong\tilde{t}_2\tilde{t}_1$, for any of their lifts $\tilde{t}_1, \tilde{t}_2\in\mathrm{T}^\dagger$.

Corollary 4.7.6. *Let $\mu : \mathrm{BT}\to\mathrm{B}^4\mathrm{A}(1)$ be a pointed morphism with associated symmetric form b. Then the commutator (4.44) of its induced $\mathbb{E}_1$-monoidal morphism $\mathrm{T}\to\mathrm{B}^3\mathrm{A}(1)$ correponds to b under Lemma 4.4.3.*

Proof. Given a bi-rigidified morphism $\mathrm{BT}\times\mathrm{BT}\to\mathrm{B}^4\mathrm{A}(1)$, we may iteratively consider loop spaces along the two factors to obtain a bi-rigidified morphism $\mathrm{T}\times\mathrm{T}\to\mathrm{B}^2\mathrm{A}(1)$.

This operation renders the following diagram commutative, where the horizontal arrows associate to μ the bi-rigidified morphisms (4.43) and (4.44):

$$\begin{array}{ccccc} \underline{\Gamma}_e(\mathrm{BT},\mathrm{B}^4\mathrm{A}(1)) & \longrightarrow & \underline{\Gamma}_{e,e}(\mathrm{BT}\times\mathrm{BT},\mathrm{B}^4\mathrm{A}(1)) & \xrightarrow{\cong} & \check{\Lambda}^{\otimes 2}\otimes\mathrm{A}(-1)\\ \downarrow{\scriptstyle\Omega} & & \downarrow{\scriptstyle\Omega\times\Omega} & & \downarrow{\scriptstyle\cong}\\ \underline{\mathrm{Maps}}_{\mathbb{E}_1}(\mathrm{T},\mathrm{B}^3\mathrm{A}(1)) & \longrightarrow & \underline{\Gamma}_{e,e}(\mathrm{T}\times\mathrm{T},\mathrm{B}^2\mathrm{A}(1)) & \xrightarrow{\cong} & \check{\Lambda}^{\otimes 2}\otimes\mathrm{A}(-1)\end{array}$$

Thus the assertion follows from Proposition 4.7.3. □

5. Reductive group schemes

The first goal of this section is to complete the classification of étale metaplectic covers for a reductive group scheme. We have already treated the case of a torus in the previous section. The new input is a canonical fiber sequence (Proposition 5.1.11):

$$\underline{\mathrm{Maps}}_{\mathbb{Z}}(\pi_1(\mathrm{G}), \mathrm{B}^2\mathrm{A}) \to \underline{\Gamma}_e(\mathrm{BG}, \mathrm{B}^4\mathrm{A}(1)) \to \mathrm{Quad}(\Lambda, \mathrm{A}(-1))_{\mathrm{st}},$$

linking étale metaplectic covers to "abelianized cohomology" and "strict" (or "strictly Weyl-invariant") quadratic forms on its sheaf of cocharacters.

The statement of our classification Theorem 5.1.13 requires either choosing a Borel subgroup or a maximal torus: it cannot be formulated intrinsically to G. However we prove in §5.2 that after restriction to a sublattice inside Λ, the dependence goes away. This fact is important for the definition of the metaplectic L-group.

In §5.3, we study étale metaplectic covers of G which descend to its abelianization. One natural source for such covers is the restriction of certain étale metaplectic covers of G to suitable parabolic subgroups.

Sections §5.4-5.5 address some technical points about the center Z_{G} and the adjoint group G_{ad}. They are included for future applications.

In §5.6, we make some remarks about the case $\mathrm{S} = \mathrm{Spec}(\mathbb{R})$.

5.0.1. In this section, S denotes a scheme and A denotes a locally constant étale sheaf of abelian groups of order invertible on S.

5.1. Classification.

5.1.1. Suppose that $\mathrm{G} \to \mathrm{S}$ is a reductive group scheme. Denote by $\mathrm{G}_{\mathrm{der}} \subset \mathrm{G}$ its derived subgroup and $\mathrm{G}_{\mathrm{sc}} \twoheadrightarrow \mathrm{G}_{\mathrm{der}}$ its simply connected cover.

Denote by T the *universal Cartan* of G: it is a torus canonically attached to G. To define T, we first assume that G is split and we set $\mathrm{T} := \mathrm{T}_1$ to be the quotient of a Borel subgroup $\mathrm{B}_1 \subset \mathrm{G}$ by its unipotent radical. Another Borel subgroup $\mathrm{B}_2 \subset \mathrm{G}$ is étale locally conjugate to the previous one by a section of G/B_2 [ABD+66, Exposé XXII, Corollaire 5.8.3], which uniquely determines an isomorphism $\mathrm{T}_1 \cong \mathrm{T}_2$. This isomorphism satisfies the cocycle condition for a third Borel subgroup, defining T canonically for split G. The general case then follows from étale descent.

Write Λ for the cocharacter lattice of T and $\check{\Lambda}$ for its dual. Then Λ forms part of the locally constant étale sheaf of based (reduced) root data defined by G:

$$(\Delta \subset \Phi \subset \Lambda, \check{\Delta} \subset \check{\Phi} \subset \check{\Lambda}), \quad \Phi \cong \check{\Phi}. \tag{5.1}$$

Indeed, (5.1) is constructed from étale local splittings of G, which includes the data of a Killing couple identifying the split maximal torus of G with T.

We also write Λ_{sc} for the sheaf of cocharacters of the universal Cartan T_{sc} of G_{sc}. Then the canonical map $\mathrm{T}_{\mathrm{sc}} \to \mathrm{T}$ identifies Λ_{sc} with the span of Φ.

Denote by W the Weyl group of the root datum (5.1), viewed as a locally constant étale sheaf of finite groups.

5.1.2. Suppose that G has a Borel subgroup $\mathrm{B} \subset \mathrm{G}$, with unipotent radical $\mathrm{N} \subset \mathrm{B}$. Then we may restrict along $\mathrm{B}(\mathrm{B}) \to \mathrm{B}(\mathrm{G})$ to obtain a morphism:

$$\begin{aligned} \mathrm{res}_{\mathrm{B}} : \underline{\Gamma}_e(\mathrm{BG}, \mathrm{B}^4\mathrm{A}(1)) &\to \underline{\Gamma}_e(\mathrm{B}(\mathrm{B}), \mathrm{B}^4\mathrm{A}(1)) \\ &\cong \underline{\Gamma}_e(\mathrm{BT}, \mathrm{B}^4\mathrm{A}(1)) \end{aligned} \tag{5.2}$$

where the isomorphism is due to the fact $\mathrm{B(N)} \to \mathrm{S}$ has vanishing higher direct image in étale cohomology.

Remark 5.1.3. The functor $\mathrm{res_B}$ *depends* on the choice of the Borel subgroup. This dependency will be studied in more details in §5.2 below.

5.1.4. Recall the morphism $\underline{\Gamma}_e(\mathrm{BT}, \mathrm{B^4A(1)}) \to \mathrm{Sym}^2(\Lambda) \otimes \mathrm{A}(-1)$ from Theorem 4.3.2 associating a quadratic form Q to every étale metaplecic cover of T. Its composition with $\mathrm{res_B}$ (5.2) defines a map:

$$\underline{\Gamma}_e(\mathrm{BG}, \mathrm{B^4A(1)}) \to \mathrm{Sym}^2(\Lambda) \otimes \mathrm{A}(-1), \quad \mu \mapsto \mathrm{Q}. \tag{5.3}$$

It is independent of the choice of B, since the target $\mathrm{Sym}^2(\Lambda) \otimes \mathrm{A}(-1)$ is a sheaf of discrete groupoids and every two choices of Borel subgroups are étale locally conjugate and we may appeal to the isomorphism (5.12) below.

In particular, (5.3) is a well-defined morphism for any reductive group scheme $\mathrm{G} \to \mathrm{S}$, without assuming the existence of a Borel subgroup. We refer to the image of μ under (5.3) as its *associated* quadratic form.

5.1.5. Define the subsheaf of *strict* quadratic forms:

$$\mathrm{Quad}(\Lambda, \mathrm{A}(-1))_{\mathrm{st}} \subset \mathrm{Sym}^2(\Lambda) \otimes \mathrm{A}(-1)$$

to consist of sections Q whose associated symmetric forms b satisfy:

$$b(\alpha, \lambda) = \langle \check{\alpha}, \lambda \rangle \mathrm{Q}(\alpha), \quad (\alpha \in \Delta, \lambda \in \Lambda). \tag{5.4}$$

Denote by $\underline{\Gamma}_e(\mathrm{BT}, \mathrm{B^4A(1)})_{\mathrm{st}}$ the full subgroupoid of $\underline{\Gamma}_e(\mathrm{BT}, \mathrm{B^4A(1)})$ whose associated quadratic form Q belongs to $\mathrm{Quad}(\Lambda; \mathrm{A}(-1))_{\mathrm{st}}$.

Remark 5.1.6. The equality (5.4) can be seen as a strengthened "W-invariance" property. Indeed, we make the following observations:

(1) (5.4) implies W-invariance, as one sees from computing $\mathrm{Q}(s_\alpha(\lambda))$ for the simple reflection s_α associated to $\alpha \in \Delta$. The converse is not true in general, as one sees in the example $\mathrm{G} = \mathrm{SL}_2 \times \mathbb{G}_m$ and $\mathrm{A} = \{\pm 1\}$.

However, if A has no 2-torsion, then the converse holds. Indeed, after multiplying (5.4) by 2, the resulting equality is equivalent to $b(s_\alpha(\alpha), s_\alpha(\lambda)) = b(\alpha, \lambda)$.

(2) if Q belongs to $\mathrm{Sym}^2(\Lambda)^{\mathrm{W}} \otimes \mathrm{A}(-1)$, then the equality (5.4) holds, according to the observation (1) for $\mathrm{A} = \mathbb{Z}$. The converse is also not true in general, as one sees in the example $\mathrm{G} = \mathrm{SO}_5$ and $\mathrm{A} = \{\pm 1\}$.

However, if G is simply connected, i.e. Φ spans Λ, then the converse holds. Indeed, the hypothesis implies that $(\Delta \subset \Phi \subset \Lambda, \check{\Delta} \subset \check{\Phi} \subset \check{\Lambda})$ admits a W-invariant decomposition into a direct sum of based irreducible reduced root data, and (5.4) implies that distinct summands are orthogonal under b. Hence the problem reduces to a single summand, where we have an isomorphism $\mathrm{Quad}(\Lambda, \mathrm{A}(-1))_{\mathrm{st}} \cong \mathrm{A}(-1)$ given by evaluating Q at a short coroot.

Furthermore, once the equality (5.4) holds for $\alpha \in \Delta$, it also holds for all $\alpha \in \Phi$, so the sheaf of strict quadratic forms is defined independently of the base of the root data (5.1).

5.1.7. Denote by $\pi_1(\mathrm{G}) \coloneqq \Lambda/\Lambda_{\mathrm{sc}}$ the algebraic fundamental group of G, viewed as a locally constant étale sheaf of abelian groups on S. We shall construct a functor:

$$\underline{\mathrm{Maps}_{\mathbb{Z}}}(\pi_1(\mathrm{G}), \mathrm{B^2A}) \to \underline{\Gamma}_e(\mathrm{BG}, \mathrm{B^4A(1)}), \tag{5.5}$$

whose targets consists of étale metaplectic covers coming from "abelianized cohomology" in the sense of Borovoi [Bor98].

Recall that the conjugation action of G on G_{der} induces one on G_{sc} by functoriality of the simply connected cover, which we continue to call the *conjugation action* of G on G_{sc}. In particular, the quotient stack G/G_{sc} inherits a monoidal structure.

Remark 5.1.8. As a sheaf of monoidal groupoids, objects of G/G_{sc} are those of G and for $g_1, g_2 \in G$, a morphism $g_1 \to g_2$ is a lift of $g_2 g_1^{-1}$ to G_{sc}.

The monoidal operation on G/G_{sc} carries objects g_1, g_2 to $g_1 g_2$. On morphisms, it carries a lift $\tilde{g}$ of $g_2 g_1^{-1}$ and a lift $\tilde{g}'$ of $g_2' g_1'^{-1}$ to the lift $(g_2 \tilde{g}' g_2^{-1})\tilde{g}$ of $(g_2 g_2')(g_1 g_1')^{-1}$, where we have used the conjugation action of G on G_{sc}.

5.1.9. Let us construct a canonical isomorphism of monoidal stacks:

$$T/T_{sc} \cong G/G_{sc}. \tag{5.6}$$

(Note: the left-hand-side is isomorphic to $\pi_1(G) \otimes \mathbb{G}_m$.)

Construction. This is a variant of [Bor98, Lemma 3.8.1]. We first treat the case where G splits. For any Killing couple $T_1 \subset B_1 \subset G$, we let (5.6) be the map induced from $T_1 \to G$, in view of the isomorphism $T \cong T_1$. The map $f_{B_1} : T_1/T_{1,sc} \to G/G_{sc}$ is an isomorphism of monoidal stacks, as $T_{1,sc}$ is the preimage of T along $G_{sc} \to G$. Hence, we only need to treat its independence on the Killing couple (as a monoidal morphism).

Let $T_2 \subset B_2 \subset G$ be another Killing couple. Since Killing couples are parametrized by $G/T_1 \cong G_{sc}/T_{1,sc}$, étale locally on S we may choose $\tilde{g} \in G_{sc}$ which conjugates $T_1 \subset B_1$ into $T_2 \subset B_2$. There is a commutative square:

$$\begin{array}{ccc} T_1/T_{1,sc} & \xrightarrow{f_{B_1}} & G/G_{sc} \\ \downarrow{\scriptstyle \alpha_{1,2}} & & \downarrow{\scriptstyle \mathrm{int}_{\tilde{g}}} \\ T_2/T_{2,sc} & \xrightarrow{f_{B_2}} & G/G_{sc} \end{array}$$

The left vertical arrow is induced from the canonical identifications $T_1 \cong T_2$, $T_{1,sc} \cong T_{2,sc}$, which depend only on the class of $\tilde{g}$ in $G_{sc}/T_{1,sc}$. We shall construct a 2-isomorphism between $\mathrm{int}_{\tilde{g}}$ and the identity endomorphism on G/G_{sc}:

$$\mathrm{int}_{\tilde{g}} \cong \mathrm{id}_{G/G_{sc}} \tag{5.7}$$

The value of (5.7) at $g \in G$ is defined by the lift $\tilde{g}(g\tilde{g}^{-1}g^{-1})$ of $\mathrm{int}_{\tilde{g}}(g)g^{-1}$ to G_{sc} using the G-action on G_{sc} by conjugation. This lift satisfies the following properties:

(1) for g coming from G_{sc}, it agrees with the lift of $\mathrm{int}_{\tilde{g}}(g)g^{-1}$ given by lifting g;
(2) for $g \in T_1$ and $\tilde{g} \in T_{1,sc}$, it agrees with the lift of $\mathrm{int}_{\tilde{g}}(g)g^{-1}$ given by the identity, using the equality $\mathrm{int}_{\tilde{g}}(g) = g$ in G.
(3) for $g_1, g_2 \in G$, the lifts $\tilde{g}_1$ of $\mathrm{int}_{\tilde{g}}(g_1)g_1^{-1}$ and $\tilde{g}_2$ of $\mathrm{int}_{\tilde{g}}(g_2)g_2^{-1}$ satisfy the property that $(\mathrm{int}_{\tilde{g}}(g_1)\tilde{g}_2\mathrm{int}_{\tilde{g}}(g_1)^{-1})\tilde{g}_1$ agrees with the chosen lift of $\mathrm{int}_{\tilde{g}}(g_1 g_2)(g_1 g_2)^{-1}$.

Property (1) shows that the isomorphism $g \cong \mathrm{int}_{\tilde{g}}(g)$ specified above for $g \in G$ defines a 2-isomorphism (5.7). Property (2) shows that the induced 2-isomorphism:

$$f_{B_1} \cong f_{B_2} \circ \alpha_{1,2} \tag{5.8}$$

depends only on the class of $\tilde{g}$ in $G_{sc}/T_{1,sc}$. Property (3) shows that (5.8) is a monoidal isomorphism, in view of the description in Remark 5.1.8.

It remains to show that for a third Killing couple $T_3 \subset B_3 \subset G$, the 2-isomorphisms $f_{B_1} \cong f_{B_2} \circ \alpha_{1,2}$ and $f_{B_2} \cong f_{B_3} \circ \alpha_{2,3}$ constructed above concatenate into the 2-isomorphism $f_{B_1} \cong f_{B_3} \circ \alpha_{1,3}$. The desired statement being an equality, we may choose $\tilde{g}_1, \tilde{g}_2 \in G_{sc}$ which conjugate $T_1 \subset B_1$ into $T_2 \subset B_2$, respectively $T_2 \subset B_2$ into $T_3 \subset B_3$. This follows from the equality $\mathrm{int}_{\tilde{g}_2} \circ \mathrm{int}_{\tilde{g}_1} = \mathrm{int}_{\tilde{g}_2\tilde{g}_1}$ and the fact that (5.7) is compatible with compositions. The latter compatibility amounts to the equality:

$$(\tilde{g}_2[\tilde{g}_1, g]\tilde{g}_2^{-1})[\tilde{g}_2, g] = [\tilde{g}_2\tilde{g}_1, g]$$

for each $g \in G$, where $[-,-]$ is a shorthand for the commutator.

Having constructed the canonical isomorphism $T/T_{sc} \cong G/G_{sc}$ for a split reductive group G, the general case follows from étale descent. ☐

5.1.10. The isomorphism $\check{\Lambda} \otimes B^2A \cong \underline{\mathrm{Maps}}_{\mathbb{Z}}(BT, B^4A(1))$ of Remark 4.3.4 and its analogue for Λ_{sc} then define (5.5) as the following composition:

$$\begin{aligned}\underline{\mathrm{Maps}}_{\mathbb{Z}}(\pi_1(G), B^2A) &\cong \underline{\mathrm{Maps}}_{\mathbb{Z}}(B(T/T_{sc}), B^4A(1)) \\ &\to \underline{\Gamma}_e(B(T/T_{sc}), B^4A(1)) \\ &\cong \underline{\Gamma}_e(B(G/G_{sc}), B^4A(1)) \to \Gamma_e(BG, B^4A(1)),\end{aligned}$$

where the second isomorphism appeals to (5.6).

The following result is a consequence of the calculation of $H^4(BG)$ with torsion coefficients, which is supposedly well known.

Proposition 5.1.11. *The morphisms* (5.5) *and* (5.3) *determine a cofiber sequence:*

$$\underline{\mathrm{Maps}}_{\mathbb{Z}}(\pi_1(G), B^2A) \to \underline{\Gamma}_e(BG, B^4A(1)) \to \mathrm{Quad}(\Lambda, A(-1))_{st}. \tag{5.9}$$

Proof. We first observe that the composition is indeed the zero map. Thus it remains to show that (5.9) induces a long exact sequence on cohomology groups. Since the étale cohomology of $BG \to S$ commutes with arbitrary base change ([Del96, Rappel 1.5.1]), we may replace S with the spectrum of $\mathbb{C}$.

Furthermore, we may replace A by the constant sheaf of abelian groups $\frac{1}{N}\mathbb{Z}/\mathbb{Z}$, and using the short exact sequence:

$$0 \to \frac{1}{N}\mathbb{Z}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z} \xrightarrow{\cdot N} \mathbb{Q}/\mathbb{Z} \to 0,$$

the assertion follows from the analogous one for the coefficient group $\mathbb{Q}/\mathbb{Z}$. For a *divisible* abelian group A (e.g. $\mathbb{Q}/\mathbb{Z}$), we have the calculation:

$$H^i(BG, A(1)) \cong \begin{cases} 0 & i = 1; \\ \mathrm{Hom}(\pi_1(G), A) & i = 2; \\ \mathrm{Ext}^1(\pi_1(G), A) & i = 3; \\ \mathrm{Quad}(\Lambda, A(-1))_{st} & i = 4, \end{cases}$$

performed in the *corrected version* of [GL18, Appendix B]. Note that these isomorphisms of *op.cit.* are defined by pulling back to the classifying stack of a chosen Borel subgroup, which agree with the maps induced from (5.5) and (5.3). ☐

Remark 5.1.12. If G is simply connected, Proposition 5.1.11 asserts that $\underline{\Gamma}_e(BG, B^4A(1))$ is canonically equivalent to $\mathrm{Quad}(\Lambda, A(-1))_{st}$, or $\mathrm{Sym}^2(\check{\Lambda})^W \otimes A(-1)$ according to Remark 5.1.6(2). In particular, $\underline{\Gamma}_e(BG, B^4A(1))$ is discrete for simply connected G.

On the other hand, if G is a torus, then the cofiber sequence (5.9) is identified with the cofiber sequence in (4.13).

Theorem 5.1.13. *Suppose that* $\mathrm{G} \to \mathrm{S}$ *is a reductive group scheme equipped with a Borel subgroup* $\mathrm{B} \subset \mathrm{G}$*. Then* (5.2) *gives rise to a Cartesian diagram:*

$$\begin{array}{ccc} \underline{\Gamma}_e(\mathrm{BG}, \mathrm{B}^4\mathrm{A}(1)) & \xrightarrow{\mathrm{res}_{\mathrm{B}}} & \underline{\Gamma}_e(\mathrm{BT}, \mathrm{B}^4\mathrm{A}(1))_{\mathrm{st}} \\ \downarrow & & \downarrow \\ \underline{\Gamma}_e(\mathrm{BG}_{\mathrm{sc}}, \mathrm{B}^4\mathrm{A}(1)) & \xrightarrow{\mathrm{res}_{\mathrm{B}_{\mathrm{sc}}}} & \underline{\Gamma}_e(\mathrm{BT}_{\mathrm{sc}}, \mathrm{B}^4\mathrm{A}(1))_{\mathrm{st}} \end{array} \tag{5.10}$$

where B_{sc} *denotes the induced Borel subgroup of* G_{sc}*.*

Proof. The square (5.10) is commutative by functoriality of the construction of $\mathrm{res}_{\mathrm{B}}$. To show that it is Cartesian, it suffices to prove that the horizontal morphisms induce an isomorphism on fibers.

We apply Proposition 5.1.11 to all four terms in (5.10), so these fibers fit into a map of fiber sequences:

$$\begin{array}{ccccc} \mathrm{Fib}(\mathrm{res}_{\mathrm{B}}) & \longrightarrow & \underline{\mathrm{Maps}}_{\mathbb{Z}}(\pi_1(\mathrm{G}), \mathrm{B}^2\mathrm{A}) & \longrightarrow & \check{\Lambda} \otimes \mathrm{B}^2\mathrm{A} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Fib}(\mathrm{res}_{\mathrm{B}_{\mathrm{sc}}}) & \longrightarrow & 0 & \longrightarrow & \check{\Lambda}_{\mathrm{sc}} \otimes \mathrm{B}^2\mathrm{A} \end{array}$$

The right square is Cartesian by the definition of $\pi_1(\mathrm{G})$, implying that the leftmost vertical arrow is an isomorphism. □

Remark 5.1.14. Theorem 5.1.13, combined with Theorem 4.3.2, gives a complete classification of étale metaplectic covers of reductive group schemes G equipped with a Borel subgroup B.

Namely, it shows that any pointed morphism $\mu : \mathrm{BG} \to \mathrm{B}^4\mathrm{A}(1)$ is uniquely determined by quadruple $(\mathrm{Q}, \mathrm{F}, h, \varphi)$ where:

(1) Q is a strict $\mathrm{A}(-1)$-valued quadratic form on Λ;
(2) $\mathrm{F} : \mathrm{H}^{(2)}(\Lambda) \to \mathrm{B}^2\mathrm{A}$ is a $\mathbb{Z}$-linear morphism;
(3) h is an isomorphism between the restriction of F to $\wedge^2(\Lambda)$ and the map determined by Q, see §4.2.6;
(4) φ is an isomorphism between the restriction of (F, h) to Λ_{sc} and the pair defined by the restriction of Q to Λ_{sc} under $\mathrm{res}_{\mathrm{B}_{\mathrm{sc}}}$.

Remark 5.1.15. If, instead of choosing a Borel subgroup B, we fix a maximal torus $\mathrm{T}_1 \subset \mathrm{G}$. Then there is a Cartesian diagram analogous to (5.10), where the horizontal morphisms are replaced by restrictions along $\mathrm{B}(\mathrm{T}_1) \to \mathrm{BG}$, respectively $\mathrm{B}(\mathrm{T}_{1,\mathrm{sc}}) \to \mathrm{BG}$, for $\mathrm{T}_{1,\mathrm{sc}} \subset \mathrm{G}_{\mathrm{sc}}$ being the induced maximal torus. In view of the explicit description in Remark 5.1.14, this statement gives an analogue of [BD01, Theorem 7.2] in étale cohomology.

5.1.16. Recall from Remark 4.2.8 that the inclusion $\mathrm{B}^2\mathrm{A} \to \underline{\Gamma}_e(\mathrm{B}\mathbb{G}_m, \mathrm{B}^4\mathrm{A}(1))$ has a canonical splitting.

If G is split, then its sheaf of based root data is constant. Hence we may view each $\alpha \in \Delta$ as a homomorphism $\mathbb{G}_m \to \mathrm{T}$ defined globally over S. Pulling back along it and composing with the above splitting defines a functor:

$$\bigoplus_{\alpha\in\Delta} \alpha^* : \underline{\Gamma}_e(\mathrm{BT}, \mathrm{B}^4\mathrm{A}(1))_{\mathrm{st}} \to \bigoplus_{\alpha\in\Delta} \mathrm{A}[2]. \tag{5.11}$$

When G is equipped with a pinning, the classification Theorem 5.1.13 can be reformulated in much more explicit terms.

Corollary 5.1.17. *Suppose that* $\mathrm{G} \to \mathrm{S}$ *is a pinned split reductive group scheme. Then* $\underline{\Gamma}_e(\mathrm{BG}, \mathrm{B}^4\mathrm{A}(1))$ *is canonically identified with the fiber of* (5.11).

5.1.18. In order to prove Corollary 5.1.17, we need an observation about SL_2 and its diagonal maximal torus $\mathbb{G}_m \subset \mathrm{SL}_2$, $a \mapsto \mathrm{diag}(a, a^{-1})$.

Being simply connected, $\underline{\Gamma}_e(\mathrm{B}(\mathrm{SL}_2), \mathrm{B}^4\mathrm{A}(1))$ is equivalent to $\mathrm{A}(-1)$, by evaluating the corresponding quadratic form on a coroot. Consider the étale metaplectic cover μ_a corresponding to $a \in \mathrm{A}(-1)$.

Lemma 5.1.19. *The restriction of* μ_a *along* $\mathbb{G}_m \subset \mathrm{SL}_2$ *is naturally identified with the image of* a *under the map* (1) *of* (4.13).

Proof. Indeed, there are equivalences:

$$\begin{aligned}\underline{\Gamma}_e(\mathrm{B}(\mathrm{SL}_2), \mathrm{B}^4\mathrm{A}(1)) &\cong \underline{\Gamma}_e(\mathrm{SL}_2, \mathrm{B}^3\mathrm{A}(1)) \\ &\cong \underline{\Gamma}_e(\mathbb{A}^2\backslash\{0\}, \mathrm{B}^3\mathrm{A}(1)),\end{aligned}$$

where the second map is induced from the map $\mathrm{SL}_2 \to \mathbb{A}^2\backslash\{0\}$, defined by the natural action on $(1,0)$, where $\mathbb{A}^2\backslash\{0\}$ is pointed by $e = (1,0)$. The étale metaplectic cover μ_a is thus defined by gluing the trivial sections of $\mathrm{B}^3\mathrm{A}(1)$ on the charts $x \neq 0$ and $y \neq 0$ along the section of $\mathrm{B}^2\mathrm{A}(1)$ over $\mathbb{G}_m \times \mathbb{G}_m$ defined by a.

The restriction of this section of $\mathrm{B}^3\mathrm{A}(1)$ along $\mathbb{G}_m \subset \mathrm{SL}_2 \to \mathbb{A}^2\backslash\{0\}$ admits a canonical trivialization, since the image belongs to the chart $x \neq 0$.

Note that $\mathbb{E}_1$-monoidal morphisms $\mathbb{G}_m \to \mathrm{B}^3\mathrm{A}(1)$ equipped with a trivialization of the underlying pointed morphism are in bijection with $\mathrm{A}(-1)$: the maps in two directions are given by (1) of (4.13), respectively taking cocycle which defines a bi-rigidified morphism $\mathbb{G}_m \times \mathbb{G}_m \to \mathrm{B}^2\mathrm{A}(1)$, or equivalently an element of $\mathrm{A}(-1)$ (Proposition 4.4.5). The fact that the cocycle defined by $a \in \mathrm{A}(-1)$ has quadratic form a shows that the cocycle corresponding to the restriction of μ_a must be a. ☐

Remark 5.1.20. The proof of Lemma 5.1.19 also shows that the section of $\mathrm{B}^3\mathrm{A}(1)$ over SL_2 underlying μ_a is nontrivial, i.e. the induced $\mathbb{E}_1$-monoidal extension of SL_2 by $\mathrm{B}^2\mathrm{A}(1)$ is not "defined by a cocycle." This stands in contrast with the corresponding central extension of $\mathrm{SL}_2(\mathrm{F})$ over a local field F, which can be defined by a cocycle, see [Kub67].

Proof of Corollary 5.1.17. In view of the Cartesian diagram (5.10), it suffices to argue that the row below forms a fiber sequence:

$$\begin{array}{ccccc} & & \check{\Lambda}_{\mathrm{sc}} \otimes \mathrm{B}^2\mathrm{A} & & \\ & & \downarrow & & \\ \underline{\Gamma}_e(\mathrm{BG}_{\mathrm{sc}}, \mathrm{B}^4\mathrm{A}(1)) & \xrightarrow{\mathrm{res}_{\mathrm{B}_{\mathrm{sc}}}} & \underline{\Gamma}_e(\mathrm{BT}_{\mathrm{sc}}, \mathrm{B}^4\mathrm{A}(1))_{\mathrm{st}} & \xrightarrow{\oplus_{\alpha\in\Delta}\alpha^*} & \oplus_{\alpha\in\Delta} \mathrm{B}^2\mathrm{A}. \\ & & \downarrow & & \\ & & \mathrm{Quad}(\Lambda_{\mathrm{sc}}, \mathrm{A}(-1))_{\mathrm{st}} & & \end{array}$$

Since we already know that the column forms a fiber sequence (Theorem 4.3.2) and the compositions given equivalences:

$$\check{\Lambda}_{\mathrm{sc}} \otimes \mathrm{B}^2\mathrm{A} \cong \bigoplus_{\alpha \in \Delta} \mathrm{B}^2\mathrm{A}$$

$$\underline{\Gamma}_e(\mathrm{BG}_{\mathrm{sc}}, \mathrm{B}^4\mathrm{A}(1)) \cong \mathrm{Quad}(\Lambda_{\mathrm{sc}}, \mathrm{A}(-1))_{\mathrm{st}}$$

it only remains to construct a null-homotopy of the composition $(\bigoplus_{\alpha\in\Delta} \alpha^*) \circ \mathrm{res}_{\mathrm{B}_{\mathrm{sc}}}$.

For this statement, we may work with an individual α. Let $\mathrm{N}_\alpha \subset \mathrm{G}_{\mathrm{sc}}$ denote the root subgroup associated to α. There exists a map $f_\alpha : \mathrm{SL}_2 \to \mathrm{G}_{\mathrm{sc}}$ inducing α on the maximal tori and sending the unipotent radical N^+ of the upper triangular Borel subgroup $\mathrm{B}^+ \subset \mathrm{SL}_2$ to N_α. Furthermore, requiring f_α to match the standard pinning $\mathrm{N}^+ \cong \mathbb{G}_a$ with the given one on N_α determines it uniquely.

Therefore, we may identify $\alpha^* \circ \mathrm{res}_{\mathrm{B}_{\mathrm{sc}}}$ with the restriction along:

$$\mathrm{B}(\mathbb{G}_m) \to \mathrm{B}(\mathrm{SL}_2) \xrightarrow{f_\alpha} \mathrm{BG}_{\mathrm{sc}},$$

so the desired null-homotopy follows from Lemma 5.1.19. □

5.2. Dependence on B.

5.2.1. Suppose that $\mathrm{G} \to \mathrm{S}$ is a reductive group scheme. We keep the notations of §5.1.1 associated to G.

If G is equipped with a Borel subgroup $\mathrm{B}_1 \subset \mathrm{G}$, then we have a functor $\mathrm{res}_{\mathrm{B}_1}$ as in (5.2). If $\mathrm{B}_2 \subset \mathrm{G}$ is another Borel subgroup and $g \in \mathrm{G}$ is a section such that $g\mathrm{B}_1 g^{-1} = \mathrm{B}_2$, then we shall construct an isomorphism of functors from $\underline{\Gamma}_e(\mathrm{BG}, \mathrm{B}^4\mathrm{A}(1))$ to $\underline{\Gamma}_e(\mathrm{BT}, \mathrm{B}^4\mathrm{A}(1))$:

$$\mathrm{F}_g : \mathrm{res}_{\mathrm{B}_1} \cong \mathrm{res}_{\mathrm{B}_2}. \tag{5.12}$$

Construction. Inner automorphism by g induces a commutative diagram:

$$\begin{array}{ccccccc} \mathrm{G} & \longleftarrow & \mathrm{B}_1 & \longrightarrow & \mathrm{T}_1 & \xrightarrow{\cong} & \mathrm{T} \\ \downarrow{\scriptstyle \mathrm{int}_g} & & \downarrow{\scriptstyle \mathrm{int}_g} & & \downarrow{\scriptstyle \mathrm{int}_g} & & \downarrow{\scriptstyle \cong} \\ \mathrm{G} & \longleftarrow & \mathrm{B}_2 & \longrightarrow & \mathrm{T}_2 & \xrightarrow{\cong} & \mathrm{T} \end{array}$$

where T_1, T_2 denotes the maximal quotient tori of B_1, B_2. Hence it suffices to construct an isomorphism between $(\mathrm{int}_g)^*$ and the identity endofunctor on $\underline{\Gamma}_e(\mathrm{BG}, \mathrm{B}^4\mathrm{A}(1))$.

Recall that a pointed morphism $\mu : \mathrm{BG} \to \mathrm{B}^4\mathrm{A}(1)$ is equivalent to an $\mathbb{E}_1$-monoidal morphism $\mathrm{G} \to \mathrm{B}^3\mathrm{A}(1)$. Denoting the latter by the same letter μ, we have a commutative diagram of $\mathbb{E}_1$-monoidal stacks:

$$\begin{array}{ccc} \mathrm{G} & \xrightarrow{\mu} & \mathrm{B}^3\mathrm{A}(1) \\ \downarrow{\scriptstyle \mathrm{int}_g} & & \downarrow{\scriptstyle \mathrm{int}_{\mu(g)}} \\ \mathrm{G} & \xrightarrow{\mu} & \mathrm{B}^3\mathrm{A}(1) \end{array}$$

Since the $\mathbb{E}_1$-monoidal structure on $\mathrm{B}^3\mathrm{A}(1)$ lifts to an $\mathbb{E}_\infty$-monoidal one, the automorphism $\mathrm{int}_{\mu(g)}$ is trivialized, giving the desired isomorphism $(\mathrm{int}_g)^*(\mu) \cong \mu$. □

Remark 5.2.2. The isomorphism (5.12) is compatible with compositions in the following sense: given a third Borel subgroup $\mathrm{B}_3 \subset \mathrm{G}$ with $g' \in \mathrm{G}$ such that $g'\mathrm{B}_2(g')^{-1} = \mathrm{B}_3$, there is a canonical isomorphism between $\mathrm{F}_{g'} \circ \mathrm{F}_g$ and $\mathrm{F}_{g'g}$. For a fourth Borel subgroup $\mathrm{B}_4 \subset \mathrm{G}$ with $g'' \in \mathrm{G}$ such that $g''\mathrm{B}_3(g'')^{-1} = \mathrm{B}_4$, a cocycle condition is satisfied.

5.2.3. Suppose that $\mathrm{B} = \mathrm{B}_1 = \mathrm{B}_2$ and $g \in \mathrm{B}$. The isomorphism F_g (5.12) is in general not the identity automorphism of $\mathrm{res}_{\mathrm{B}}$.

For $\mu \in \underline{\Gamma}_e(\mathrm{BG}, \mathrm{B}^4\mathrm{A}(1))$, the association $g \mapsto \mathrm{F}_g(\mu)$ defines a pointed morphism from B to the sheaf of automorphisms of $\mathrm{res}_{\mathrm{B}}(\mu)$. By Theorem 4.3.2, the latter is canonically identified with $\check{\Lambda} \otimes \mathrm{A}[1]$, so we obtain a pointed morphism:

$$\mathrm{B} \to \check{\Lambda} \otimes \mathrm{A}[1]. \tag{5.13}$$

By the vanishing of étale cohomology of $\mathrm{N} \to \mathrm{S}$ and the calculation of étale cohomology of $\mathrm{T} \to \mathrm{S}$, pointed morphisms $\mathrm{B} \to \check{\Lambda} \otimes \mathrm{A}[1]$ are parametrized by $\check{\Lambda}^{\otimes 2} \otimes \mathrm{A}(-1)$ (see the proof of Lemma 4.4.3). In particular, associating (5.13) to μ defines a map:

$$\underline{\Gamma}_e(\mathrm{BG}, \mathrm{B}^4\mathrm{A}(1)) \to \check{\Lambda}^{\otimes 2} \otimes \mathrm{A}(-1). \tag{5.14}$$

It may be viewed as the obstruction of the isomorphism F_g to depend *only* on the Borel subgroups $\mathrm{B}_1, \mathrm{B}_2$, as opposed to the section $g \in \mathrm{G}$.

Lemma 5.2.4. *The image of μ under* (5.14) *is given by the symmetric form b associated to its quadratic form* Q.

Proof. Since the target of (5.14) is discrete, we may work étale locally on S and choose a splitting of $\mathrm{B}_1 \to \mathrm{T}_1$, so T is identified with a subgroup of G. The functor $\mathrm{res}_{\mathrm{B}}$ is canonically identified with $\mathrm{res}_{\mathrm{T}}$: restriction along $\mathrm{BT} \to \mathrm{BG}$. The pointed morphism (5.13) may also be calculated after pre-composing with $\mathrm{T} \to \mathrm{B}$.

For a section $t \in \mathrm{T}$, the automorphism $\mathrm{F}_t(\mu)$ of $\mathrm{res}_{\mathrm{T}}(\mu)$ is given by the following loop in $\underline{\Gamma}_e(\mathrm{BT}, \mathrm{B}^4\mathrm{A}(1))$, or equivalently in $\underline{\mathrm{Maps}}_{\mathbb{E}_1}(\mathrm{T}, \mathrm{B}^3\mathrm{A}(1))$:

$$\begin{array}{ccc} \mathrm{res}_{\mathrm{T}}(\mathrm{int}_t^*\mu) & \xrightarrow{\cong} & \mathrm{res}_{\mathrm{T}}(\mu) \\ \cong\uparrow & & \downarrow\cong \\ \mathrm{int}_t^*\mathrm{res}_{\mathrm{T}}(\mu) & \xleftarrow[\cong]{} & \mathrm{res}_{\mathrm{T}}(\mu) \end{array} \tag{5.15}$$

where the top isomorphism comes from identification $\mathrm{int}_t^*(\mu) \cong \mu$ in $\underline{\Gamma}_e(\mathrm{BG}, \mathrm{B}^4\mathrm{A}(1))$ constructed in §5.2.1 and the bottom isomorphism comes from the commutativity of T.

The loop (5.15), evaluated at a section $t_1 \in \mathrm{T}$, is precisely the commutator of $\mathrm{res}_{\mathrm{T}}(\mu) : \mathrm{T} \to \mathrm{B}^3\mathrm{A}(1)$ evaluated at the sections $t, t_1 \in \mathrm{T}$, so the claim follows from Corollary 4.7.6. □

5.2.5. Suppose that $\mathrm{G} \to \mathrm{S}$ is a reductive group scheme (without assuming the existence of a Borel subgroup). For a strict quadratic form $\mathrm{Q} \in \mathrm{Quad}(\Lambda, \mathrm{A}(-1))_{\mathrm{st}}$, we denote by:

$$\underline{\Gamma}_e(\mathrm{BG}, \mathrm{B}^4\mathrm{A}(1))_{\mathrm{Q}} \subset \underline{\Gamma}_e(\mathrm{BG}, \mathrm{B}^4\mathrm{A}(1)).$$

the full subgroupoid of étale metaplectic covers of G with associated quadratic form Q (in the sense of §5.1.4).

On the other hand, we may let $\Lambda^\sharp \subset \Lambda$ denote the kernel of the symmetric form b associated to Q. It defines a torus $\mathrm{T}^\sharp := \Lambda^\sharp \otimes \mathbb{G}_m$ equipped with an isogeny to T.

5.2.6. We shall construct a canonical morphism:

$$\mathrm{res}_{\mathrm{T}^\sharp} : \underline{\Gamma}_e(\mathrm{BG}, \mathrm{B}^4\mathrm{A}(1))_{\mathrm{Q}} \to \underline{\mathrm{Maps}}_{\mathbb{E}_\infty}(\mathrm{BT}^\sharp, \mathrm{B}^4\mathrm{A}(1)). \tag{5.16}$$

Construction. We first work étale locally on S and assume the existence of a Borel subgroup $B_1 \subset G$. In this case, we define $\mathrm{res}_{T^\sharp}$ as the composition of res_{B_1} and the restriction along $B(T^\sharp) \to BT$:

$$\begin{aligned}\underline{\Gamma}_e(BG, B^4A(1))_Q \subset \underline{\Gamma}_e(BG, B^4A(1)) \\ \xrightarrow{\mathrm{res}_{B_1}} \underline{\Gamma}_e(BT, B^4A(1)) \to \underline{\Gamma}_e(BT^\sharp, B^4A(1)).\end{aligned} \tag{5.17}$$

Since the image of this composition has vanishing symmetric form, it belongs to the full subgroupoid of $\mathbb{E}_\infty$-monoidal morphisms $BT^\sharp \to B^4A(1)$ by Proposition 4.6.2.

The key point to address is the independence of the Borel subgroup. For two Borel subgroups $B_1, B_2 \subset G$ and a section $g \in G$ such that $gB_1g^{-1} = B_2$, we have the isomorphism:

$$F_g : \mathrm{res}_{B_1} \cong \mathrm{res}_{B_2} \tag{5.18}$$

from §5.2.1. We claim that the composition of F_g with the restriction along $BT^\sharp \to BT$ depends only on B_1, B_2 (and not on the section $g \in G$).

Indeed, any other section $g' \in G$ such that $g'B_1(g')^{-1} = B_2$ differs from g by an element of B_1. By the isomorphism $F_{gg'} \cong F_g \circ F_{g'}$ (see Remark 5.2.2), it suffices to prove that for $B = B_1 = B_2$ and $g \in B$, the composition of F_g with the restriction along $BT^\sharp \to BT$ is the identity. For this statement, we recall that as an automorphism of $\mathrm{res}_B(\mu)$, the section:

$$F_g(\mu) \in \check{\Lambda} \otimes A[1]$$

is the image of $g \in B$ under (5.13), which is in turn given by the symmetric form associated to Q (Lemma 5.2.4). This section restricts to zero in $\check{\Lambda}^\sharp \otimes A[1]$, as desired.

Therefore, for two Borel subgroups $B_1, B_2 \subset G$, we have constructed a canonical identifications between the compositions (5.17) defined by B_1, B_2. The coherence data for three Borel subgroups and the cocycle condition for a fourth one follow from the corresponding facts about the isomorphism (5.18), see Remark 5.2.2. □

5.3. Covers of $\pi_1(G) \otimes \mathbb{G}_m$.

5.3.1. Let $G \to S$ be a reductive group scheme. We keep the notations of §5.1.1 and recall the canonical morphism of monoidal stacks constructed in §5.1.9:

$$G \to \pi_1(G) \otimes \mathbb{G}_m \cong T/T_{sc}. \tag{5.19}$$

It is surjective in the étale topology, with fiber canonically identified with G_{sc}. If G_{der} is simply connected, then $\pi_1(G) \otimes \mathbb{G}_m$ is identified with the maximal quotient torus (i.e. coradical) of G.

The following result, combined with Proposition 5.1.11 for G_{sc}, identifies "étale metaplectic covers" of T/T_{sc}, i.e. rigidified sections of $B^4A(1)$ over $B(T/T_{sc})$, as those of G characterized by the vanishing of the associated quadratic form on Λ_{sc}.

Proposition 5.3.2. *Pulling back along* (5.19) *defines a fiber sequence:*

$$\underline{\Gamma}_e(B(T/T_{sc}), B^4A(1)) \to \underline{\Gamma}_e(BG, B^4A(1)) \to \underline{\Gamma}_e(BG_{sc}, B^4A(1)). \tag{5.20}$$

Proof. The last term $\underline{\Gamma}_e(BG_{sc}, B^4A(1))$ is discrete and the composition is evidently zero. Thus the construction of (5.20) requires no additional data and can be treated étale local on S.

By Theorem 5.1.13, it suffices to identify $\underline{\Gamma}_e(B(T/T_{sc}), B^4A(1))$ as the fiber of:

$$\underline{\Gamma}_e(BT, B^4A(1))_{st} \to \underline{\Gamma}_e(BT_{sc}, B^4A(1))_{st}, \tag{5.21}$$

under the evident map. Since a strict quadratic form on Λ which vanishes on Λ_{sc} factors through $\pi_1(\mathrm{G})$, the fiber of (5.21) is also the full subgroupoid of the fiber of:

$$\underline{\Gamma}_e(\mathrm{BT}, \mathrm{B}^4\mathrm{A}(1)) \to \underline{\Gamma}_e(\mathrm{BT}_{\mathrm{sc}}, \mathrm{B}^4\mathrm{A}(1)), \tag{5.22}$$

characterized by the property of Q factoring through $\pi_1(\mathrm{G})$.

Since $\mathrm{B}(\mathrm{T}/\mathrm{T}_{\mathrm{sc}})$ is the quotient of BT by $\mathrm{BT}_{\mathrm{sc}}$, we may identify pointed sections of $\mathrm{B}^4\mathrm{A}(1)$ over $\mathrm{B}(\mathrm{T}/\mathrm{T}_{\mathrm{sc}})$ as a cosimplicial limit:

$$\begin{array}{ccc} & \lim_{[n]}(\check{\Lambda} \oplus \check{\Lambda}_{\mathrm{sc}}^{\oplus n}) \otimes \mathrm{B}^2\mathrm{A} & \cong \underline{\mathrm{Maps}}_{\mathbb{Z}}(\pi_1(\mathrm{G}), \mathrm{B}^2\mathrm{A}) \\ & \downarrow & \\ \underline{\Gamma}_e(\mathrm{B}(\mathrm{T}/\mathrm{T}_{\mathrm{sc}}), \mathrm{B}^4\mathrm{A}(1)) \cong & \lim_{[n]} \underline{\Gamma}_e(\mathrm{BT} \times \mathrm{BT}_{\mathrm{sc}}^{\times n}, \mathrm{B}^4\mathrm{A}(1)) & \\ & \downarrow & \\ & \lim_{[n]} \mathrm{Sym}^2(\check{\Lambda} \oplus \check{\Lambda}_{\mathrm{sc}}^{\oplus n}) \otimes \mathrm{A}(-1) & \cong \mathrm{Quad}(\pi_1(\mathrm{G}), \mathrm{A}(-1)) \end{array} \tag{5.23}$$

where the vertical triangle is that of (4.13). The identification of the top limit follows from the fact that the complex $[\check{\Lambda} \xrightarrow{d} \check{\Lambda} \oplus \check{\Lambda}_{\mathrm{sc}} \xrightarrow{d} \cdots]$ is equivalent to its subcomplex $[\check{\Lambda} \to \check{\Lambda}_{\mathrm{sc}}]$ of nondegenerate cochains, which is in turn isomorphic to the $\mathbb{Z}$-linear dual of $\pi_1(\mathrm{G})$. The identification of the bottom limit follows the fact that a quadratic form on Λ whose two pullbacks to $\Lambda \oplus \Lambda_{\mathrm{sc}}$ is precisely one that factors through $\pi_1(\mathrm{G})$.

The vertical sequence in (5.23) is a cofiber sequence because $\underline{\mathrm{Maps}}_{\mathbb{Z}}(\pi_1\mathrm{G}, \mathrm{B}^2\mathrm{A})$ has vanishing H^1. Comparing it with the canonical map from $\underline{\Gamma}_e(\mathrm{B}(\mathrm{T}/\mathrm{T}_{\mathrm{sc}}), \mathrm{B}^4\mathrm{A}(1))$ to the fiber of (5.22) yields the desired conclusion. ☐

Remark 5.3.3. The proof of Proposition 5.3.2 yields another statement. For any sheaf of finitely generated abelian groups Γ, there is a cofiber sequence:

$$\underline{\mathrm{Maps}}_{\mathbb{Z}}(\Gamma, \mathrm{B}^2\mathrm{A}) \to \underline{\Gamma}_e(\mathrm{B}(\Gamma \otimes \mathbb{G}_m), \mathrm{B}^4\mathrm{A}(1)) \to \mathrm{Quad}(\Gamma, \mathrm{A}(-1)). \tag{5.24}$$

The cofiber sequence in (4.13) is recovered as the special case where Γ is torsion-free.

For $\Gamma = \pi_1(\mathrm{G})$, we note that (5.20) and (5.24) combine into the cofiber sequence (5.9), which can be informally summarized as follows: the étale metaplectic covers with vanishing quadratic form on Λ not only desends to $\pi_1(\mathrm{G}) \otimes \mathbb{G}_m$, but is also of "abelian nature" there.

5.3.4. Suppose that $\mathrm{P} \subset \mathrm{G}$ is a parabolic subgroup with unipotent radical N_{P} and Levi quotient M. Since $\mathrm{B}(\mathrm{N}_{\mathrm{P}}) \to \mathrm{S}$ has vanishing étale cohomology in degrees ≥ 1, étale metaplectic covers of P are equivalent to those of M. Thus, restriction along P defines a functor:

$$\underline{\Gamma}_e(\mathrm{BG}, \mathrm{B}^4\mathrm{A}(1)) \to \underline{\Gamma}_e(\mathrm{BM}, \mathrm{B}^4\mathrm{A}(1)), \quad \mu \mapsto \mu_{\mathrm{M}}.$$

In the study of Eisenstein series for covering groups, the following question arises naturally[4]: what are the "interesting" metaplectic covers of G whose restriction to P descends to $\pi_1(\mathrm{M}) \otimes \mathbb{G}_m$?

Corollary 5.3.5. *Suppose that there exists an étale metaplectic cover μ of* G *which does not descend to $\pi_1(\mathrm{G}) \otimes \mathbb{G}_m$, but μ_{M} descends to $\pi_1(\mathrm{M}) \otimes \mathbb{G}_m$. Then at least one of the following statements hold:*

(1) $\mathrm{A}(-1)$ *has* 2*-torsion and* G_{sc} *contains a factor of type* B_n, C_n, *or* F_4*;*

[4] I thank Laurent Clozel for posing this question to the author.

(2) $\mathrm{A}(-1)$ *has* 3*-torsion and* G_{sc} *contains a factor of type* G_2.

Proof. By Proposition 5.3.2, the descent properties of μ, respectively μ_{M}, are of étale local nature on S. Combined with Proposition 5.1.11, we see that they are controlled by the vanishing of the associated quadratic form Q on simple coroots.

For Q to vanish on the simple coroots contained in M but not on all simple coroots of G, the corresponding Dynkin diagram must be non-simply laced and some $\mathrm{Q}(\alpha) \neq 0$ ($\alpha \in \Delta$) must be annihilated by the ratio of the square length of the long and short coroots. ☐

Remark 5.3.6. For $\mathrm{G} = \mathrm{Sp}_{2n}$ and $\mathrm{A} = \{\pm 1\}$ over a local field F of characteristic $\neq 2$, the metaplectic double cover of $\mathrm{Sp}_{2n}(\mathrm{F})$ is defined by the étale metaplectic cover corresponding to the $\mathbb{Z}/2$-valued quadratic form Q taking value 1 on a short coroot.

It follows from a classical calculation that when restricted to the Siegel parabolic, the metaplectic cover descends along $\det : \mathrm{GL}_n(\mathrm{F}) \to \mathrm{F}^\times$, see [RR93, Corollary 5.5].

Proposition 5.3.2 lifts this statement to the level of geometry: it asserts that the corresponding étale metaplectic cover already descends to $\mathbb{G}_m$. The resulting cover of $\mathbb{G}_m$ may be identified by restriction along the unique short simple coroot α of Sp_{2n}:

$$\begin{array}{ccccc} \mathbb{G}_m & \xrightarrow{e_n} & \mathrm{GL}_n & \xrightarrow{\det} & \mathbb{G}_m \\ \downarrow & & \downarrow & & \\ \mathrm{SL}_2 & \xrightarrow{f_\alpha} & \mathrm{Sp}_{2n} & & \end{array}$$

where e_n is given by the action on the last basis vector, f_α is the map corresponding to α, and $\mathbb{G}_m \subset \mathrm{SL}_2$, $\mathrm{GL}_n \subset \mathrm{Sp}_{2n}$ are the Levi subgroups. Thus the induced cover of $\mathbb{G}_m$ is defined by the cocycle $\mathbb{Z} \otimes \mathbb{Z} \to \mathbb{Z}/2$, $1 \otimes 1 \mapsto 1$ by Lemma 5.1.19.

We also note that the integral parametrization of metaplectic covers, by central extension by $\underline{\mathrm{K}}_2$ for example, does not see this phenomenon.

Remark 5.3.7. For S the spectrum of a local field F, it is possible to describe covering groups of G(F) coming from étale metaplectic covers of $\pi_1(\mathrm{G}) \otimes \mathbb{G}_m$ with the help of a *z-extension*, i.e. a central extension:

$$1 \to \mathrm{T}_2 \to \mathrm{G}' \to \mathrm{G} \to 1$$

of reductive groups satisfying the following properties:

(1) the map $\mathrm{G}' \to \mathrm{G}$ induces the simply connected cover $\mathrm{G}_{\mathrm{sc}} \to \mathrm{G}_{\mathrm{der}}$ upon taking derived subgroups;
(2) T_2 is a quasi-trivial torus.

Letting $\mathrm{T}_1 := \mathrm{G}'/\mathrm{G}'_{\mathrm{der}}$, we have an identification $\mathrm{T}_1/\mathrm{T}_2 \cong \pi_1(\mathrm{G}) \otimes \mathbb{G}_m$ of monoidal stacks. Thus any étale metaplectic cover μ of $\pi_1(\mathrm{G}) \otimes \mathbb{G}_m$ defines one for T_1, with equivariance structure against the T_2-action. Upon evaluation at Spec(F) (see §2.1), we obtain a central extension of topological groups equipped with a splitting:

$$\begin{array}{ccccccccc} & & & & & & \mathrm{T}_2(\mathrm{F}) & & \\ & & & & & \swarrow & \downarrow & & \\ 1 & \longrightarrow & \mathrm{H}_0(\mathrm{F}, \mathrm{A}) & \longrightarrow & \tilde{\mathrm{T}}_1 & \longrightarrow & \mathrm{T}_1(\mathrm{F}) & \longrightarrow & 1 \end{array}$$

whose image is *central* in $\tilde{T}_1$. Indeed, the centrality assertion follows from the equivariance structure $\tilde{T}_1 \times T_2(F) \cong \mathrm{act}^* \tilde{T}_1$ (for $\mathrm{act} : T_1(F) \times T_2(F) \to T_1(F)$ the action map) being a group isomorphism.

Pulling back the extension $\tilde{T}_1$ along $G'(F) \to T_1(F)$ and taking the quotient by $T_2(F)$, we obtain a central extension $\tilde{G}$ of $G(F)$ by $H_0(F, A)$, using the vanishing of $H^1(F, T_2)$. This $\tilde{G}$ is the covering group associated to the pullback of μ along $G \to \pi_1(G) \otimes \mathbb{G}_m$.

5.4. Metaplectic center.

5.4.1. Suppose that $G \to S$ is a reductive group scheme. Let Z_G denote its center and $Z \subset Z_G$ its maximal torus. Equivalently, Z is the radical torus of G. We write Λ_Z for its sheaf of cocharacters.

The map $Z \times G \to G$, $(z, g) \mapsto zg$ being a group homomorphism, it induces an action of the monoidal stack $B(Z)$ on BG.

5.4.2. Given an étale metaplectic cover μ of G with associated quadratic form Q, we define an isogeny $Z^\sharp \to Z$ as follows. The symmetric form b associated to Q induces a pairing:

$$\Lambda_Z \otimes \pi_1(G) \to A(-1), \quad \lambda_1, \lambda_2 \mapsto b(\lambda_1, \lambda_2) \tag{5.25}$$

as Λ_Z pairs trivially with Λ_{sc} according to the formula (5.4). The kernel of this pairing is a sublattice $\Lambda_{Z^\sharp} \subset \Lambda_Z$ with finite cokernel, so it defines the isogeny $Z^\sharp \to Z$.

Let $\mu_{Z^\sharp}$ denote the restriction of μ along $B(Z) \to BG$. By Proposition 4.6.2, $\mu_{Z^\sharp}$ has the canonical structure of an $\mathbb{E}_1$-monidal (in fact $\mathbb{E}_\infty$-monoidal) morphism $B(Z^\sharp) \to B^4A(1)$.

The goal of this section is to establish the following result, whose proof gives a geometric interpretation of the pairing (5.25).

Proposition 5.4.3. *With respect to the* $B(Z^\sharp)$*-action on* BG*, the pointed morphism* $\mu : BG \to B^4A(1)$ *has a canonical* $B(Z^\sharp)$*-equivariance structure against the* $\mathbb{E}_1$*-monoidal character* $\mu^\sharp : B(Z^\sharp) \to B^4A(1)$.

5.4.4. In concrete terms, the assertion of Proposition 5.4.3 means that along the simplicial diagram encoding the $B(Z^\sharp)$-action on BG:

$$\cdots \; B(Z^\sharp)^{\times 2} \times BG \underset{\mathrm{pr}}{\overset{m\times \mathrm{id}}{\underset{\mathrm{id}\times\mathrm{act}}{\rightrightarrows}}} B(Z^\sharp) \times BG \underset{\mathrm{pr}}{\overset{\mathrm{act}}{\rightrightarrows}} BG,$$

we have an isomorphism of rigidified sections of $B^4A(1)$ over $B(Z^\sharp) \times BG$:

$$\mathrm{act}^*(\mu) \cong \mu_{Z^\sharp} \boxtimes \mu, \tag{5.26}$$

and a 2-isomorphism witnessing the commutativity of:

$$\begin{array}{ccc} (m \times \mathrm{id})^* \mathrm{act}^*(\mu) & \xrightarrow{\cong} & (\mu_{Z^\sharp} \boxtimes \mu_{Z^\sharp}) \boxtimes \mu \\ \downarrow \cong & & \downarrow \cong \\ (\mathrm{id} \times \mathrm{act}^*) \mathrm{act}^*(\mu) & \xrightarrow{\cong} & \mu_{Z^\sharp} \boxtimes (\mu_{Z^\sharp} \boxtimes \mu) \end{array} \tag{5.27}$$

subject to the natural cocycle condition over $B(Z^\sharp)^{\times 3} \times BG$.

Remark 5.4.5. If S is the spectrum of a local field F, then μ defines a topological covering group $\tilde{G}$ of $G(F)$ (or $G(F)^0$ if F is real). Denote by $\tilde{Z}$ its restriction to $Z^\sharp(F)$. The assertion of Proposition 5.4.3 implies that the image of $\tilde{Z}$ is central in $\tilde{G}$, using the same observation as in Remark 5.3.7.

5.4.6. In order to prove Proposition 5.4.3, we first observe that for any torus $\mathrm{T} \to \mathrm{S}$ with sheaf of cocharacters Λ_{T}, the groupoid of bi-rigidified morphisms:

$$\mathrm{BT} \times \mathrm{BG} \to \mathrm{B}^4\mathrm{A}(1)$$

is equivalent to discrete abelian group of pairings $\Lambda_{\mathrm{T}} \times \pi_1(\mathrm{G}) \to \mathrm{A}(-1)$.

Indeed, this assertion follows from the same proof as Lemma 4.4.3, where we use the fiber sequence (5.9) to identify the neutral component of $\underline{\Gamma}_e(\mathrm{BG}, \mathrm{B}^4\mathrm{A}(1))$.

Proof of Proposition 5.4.3. We observe that $\mathrm{act}^*(\mu)\otimes(\mu_{\mathrm{Z}^\sharp}\boxtimes\mu)^{\otimes -1}$ admits a natural structure as a bi-rigidified morphism $\mathrm{B}(\mathrm{Z}^\sharp) \times \mathrm{BG} \to \mathrm{B}^4\mathrm{A}(1)$. In order to construct the isomorphism (5.26), it suffices to show that the corresponding pairing:

$$\Lambda_{\mathrm{Z}^\sharp} \otimes \pi_1(\mathrm{G}) \to \mathrm{A}(-1) \tag{5.28}$$

vanishes. The statement being of étale local nature on S, we may split G as well as fix a Borel subgroup containing a split maximal torus $\mathrm{T}_1 \subset \mathrm{B}_1 \subset \mathrm{G}$. Then T_1 is isomorphic to the universal Cartan of G and the pairing (5.28) may be calculated after restriction along $\Lambda_{\mathrm{T}_1} \to \pi_1(\mathrm{G})$. Proposition 4.7.3 then shows that (5.28) is the restriction of (5.25) along $\Lambda_{\mathrm{Z}^\sharp} \to \Lambda_{\mathrm{Z}}$, which vanishes by the definition of $\Lambda_{\mathrm{Z}^\sharp}$.

To construct the 2-isomorphism (5.27), we note that all four terms are naturally isomorphic over $e \times \mathrm{BG}$, $\mathrm{B}(\mathrm{Z}^\sharp)^{\times 2} \times e$, and these isomorphisms are compatible over $e \times e$. By the discreteness of bi-rigidified morphisms $\mathrm{B}(\mathrm{Z}^\sharp)^{\times 2} \times \mathrm{BG} \to \mathrm{B}^4\mathrm{A}(1)$, any isomorphism:

$$(m \times \mathrm{id})^*\mathrm{act}^*(\mu) \cong \mu_{\mathrm{Z}^\sharp} \boxtimes (\mu_{\mathrm{Z}^\sharp} \boxtimes \mu)$$

compatible with the given ones, if exists, is necessarily unique. The same consideration yields the cocycle condition over $\mathrm{B}(\mathrm{Z}^\sharp)^{\times 3} \times \mathrm{BG}$. ☐

5.5. The $(\mathrm{T}_{\mathrm{ad}}, \mathrm{T}_{\mathrm{sc}})$-commutator.

5.5.1. Suppose that $\mathrm{G} \to \mathrm{S}$ is a reductive group scheme. Write G_{ad} for the quotient $\mathrm{G}/\mathrm{Z}_{\mathrm{G}}$ and T_{ad} its universal Cartan with sheaf of cocharacters Λ_{ad}. The map $\mathrm{G} \to \mathrm{G}_{\mathrm{ad}}$ induces a natural map $\Lambda \to \Lambda_{\mathrm{ad}}$, which is generally not surjective.

On the other hand, the conjugation action of G on itself factors through G_{ad}. By the functoriality of the simply connected cover, we also obtain a G_{ad}-action on G_{sc}.

5.5.2. Suppose that μ is an étale metaplectic cover of G with associated quadaratic form Q and symmetric form b. Since Q is strict, the restriction of b to $\Lambda \otimes \Lambda_{\mathrm{sc}}$ extends to a bilinear form according to the formula:

$$\Lambda_{\mathrm{ad}} \otimes \Lambda_{\mathrm{sc}} \to \mathrm{A}(-1), \quad \lambda, \alpha \mapsto \mathrm{Q}(\alpha)\langle \check{\alpha}, \lambda \rangle, \tag{5.29}$$

where α is any simple coroot. Here, the bracket denotes the natural duality pairing between Λ_{ad} and the root lattice.

The goal of this section is interpret (5.29) as a kind of "commutator pairing" between T_{ad} and T_{sc}, arising from G_{ad}-action on G_{sc}.

We write $\mu_{\mathrm{G}_{\mathrm{sc}}}$ for the restriction of μ to G_{sc}, viewed as an $\mathbb{E}_1$-monoidal morphism $\mathrm{G}_{\mathrm{sc}} \to \mathrm{B}^3\mathrm{A}(1)$.

5.5.3. We assume the existence of a Borel subgroup $\mathrm{B} \subset \mathrm{G}$, which we fix from now on. The induced Borel subgroup of G_{ad} is written as B_{ad}. The B_{ad}-action on G_{sc} induces two commutative diagrams:

$$\begin{array}{ccccc} \mathrm{B_{ad}} \times \mathrm{G_{sc}} & \longleftarrow & \mathrm{B_{ad}} \times \mathrm{B_{sc}} & \longrightarrow & \mathrm{T_{ad}} \times \mathrm{T_{sc}} \\ {\scriptstyle \mathrm{pr}}\downdownarrows{\scriptstyle \mathrm{act}} & & {\scriptstyle \mathrm{pr}}\downdownarrows{\scriptstyle \mathrm{act}} & & {\scriptstyle \mathrm{pr}}\downdownarrows{\scriptstyle \mathrm{act}} \\ \mathrm{G_{sc}} & \longleftarrow & \mathrm{B_{sc}} & \longrightarrow & \mathrm{T_{sc}} \end{array} \tag{5.30}$$

Since $\mathrm{G_{sc}}$ is simply connected, $\underline{\Gamma}_e(\mathrm{BG_{sc}}, \mathrm{B}^4\mathrm{A}(1))$ is discrete and, under the functor of forgetting the $\mathbb{E}_1$-monoidal structure, isomorphic to $\underline{\Gamma}_e(\mathrm{G_{sc}}, \mathrm{B}^3\mathrm{A}(1))$. In particular, $\mu_{\mathrm{G_{sc}}}$ acquires an automatic $\mathrm{B_{ad}}$-equivariance structure. The isomorphism $\mathrm{pr}^*(\mu_{\mathrm{G_{sc}}}) \cong \mathrm{act}^*(\mu_{\mathrm{G_{sc}}})$ may be restricted along (5.30) to give an isomorphism over $\mathrm{T_{ad}} \times \mathrm{T_{sc}}$:

$$\mathrm{pr}^*(\mathrm{res}_{\mathrm{B_{sc}}}(\mu_{\mathrm{G_{sc}}})) \cong \mathrm{act}^*(\mathrm{res}_{\mathrm{B_{sc}}}(\mu_{\mathrm{G_{sc}}})). \tag{5.31}$$

On the other hand, $\mathrm{act} = \mathrm{pr}$ as maps $\mathrm{T_{ad}} \times \mathrm{T_{sc}} \to \mathrm{T_{sc}}$. Hence (5.31) may be viewed as an automorphism of the trivial object in the groupoid $\underline{\Gamma}_{e,e}(\mathrm{T_{ad}} \times \mathrm{T_{sc}}, \mathrm{B}^3\mathrm{A}(1))$, i.e. a section of $\underline{\Gamma}_{e,e}(\mathrm{T_{ad}} \times \mathrm{T_{sc}}, \mathrm{B}^2\mathrm{A}(1))$. Finally, a variant of Lemma 4.4.3 (with the same proof) gives an equivalence:

$$\underline{\Gamma}_{e,e}(\mathrm{T_{ad}} \times \mathrm{T_{sc}}, \mathrm{B}^2\mathrm{A}(1)) \cong \check{\Lambda}_{\mathrm{ad}} \otimes \check{\Lambda}_{\mathrm{sc}} \otimes \mathrm{A}(-1). \tag{5.32}$$

Proposition 5.5.4. *The section of* (5.32) *defined by the isomorphism* (5.31) *agrees with the pairing* (5.29)*.*

Proof. The assertion is of étale local nature on S, so we may assume that G splits. Furthermore, we fix a splitting of the surjection $\mathrm{B} \to \mathrm{T}$ and view T (resp. $\mathrm{T_{sc}}$) as a subgroup of G (resp. $\mathrm{G_{sc}}$). The functor $\mathrm{res}_{\mathrm{B_{sc}}}$ may be identified with the restriction $\mathrm{res}_{\mathrm{T_{sc}}}$ along $\mathrm{T_{sc}} \subset \mathrm{G_{sc}}$.

With this set-up, the isomorphism (5.31), evaluated at $t \in \mathrm{T_{ad}}$, has a concrete interpretation as the following loop in $\underline{\mathrm{Maps}}_{\mathbb{E}_1}(\mathrm{T_{sc}}, \mathrm{B}^3\mathrm{A}(1))$:

$$\begin{array}{ccc} \mathrm{res}_{\mathrm{T_{sc}}}(\mathrm{int}_t^*\mu_{\mathrm{G_{sc}}}) & \xrightarrow{\cong} & \mathrm{res}_{\mathrm{T_{sc}}}(\mu_{\mathrm{G_{sc}}}) \\ {\scriptstyle \cong}\uparrow & & \downarrow{\scriptstyle \cong} \\ \mathrm{int}_t^*\mathrm{res}_{\mathrm{T_{sc}}}(\mu_{\mathrm{G_{sc}}}) & \xleftarrow[\cong]{} & \mathrm{res}_{\mathrm{T_{sc}}}(\mu_{\mathrm{G_{sc}}}) \end{array} \tag{5.33}$$

Here, the top isomorphism is induced from $\mathrm{int}_t^*(\mu_{\mathrm{G_{sc}}}) \cong \mu_{\mathrm{G_{sc}}}$ over $\mathrm{G_{sc}}$, whereas the bottom isomorphism is due to the triviality of the $\mathrm{T_{ad}}$-action on $\mathrm{T_{sc}}$.

To identify this loop, we may choose a simple coroot α of G, viewed as a morphism $\mathbb{G}_m \to \mathrm{T_{sc}}$. It extends to an inclusion $f_\alpha : \mathrm{SL}_2 \subset \mathrm{G_{sc}}$ preserved by the $\mathrm{T_{ad}}$-action. For each $\lambda \in \Lambda_{\mathrm{T_{ad}}}$, viewed as a morphism $\mathbb{G}_m \to \mathrm{T_{ad}}$, we find an induced $\mathbb{G}_m$-action on SL_2 which is the $\langle \check{\alpha}, \lambda \rangle$-multiple of the action of the adjoint torus of SL_2 on SL_2. Hence the pullback of (5.33) along α is given by the same diagram applied to $f_\alpha^*(\mu_{\mathrm{G_{sc}}})$ and $t^{\langle \check{\alpha}, \lambda \rangle} \in \mathbb{G}_m$, viewed as a section of the adjoint torus of SL_2.

On the other hand, the étale metaplectic cover $f_\alpha^*(\mu_{\mathrm{G_{sc}}})$ of SL_2 is classfied by $\mathrm{Q}(\alpha) \in \mathrm{A}(-1)$. By construction, the loop (5.33) applied to $t \in \mathbb{G}_m$ is naturally the restriction along $t \times \mathbb{G}_m$ of a bi-rigidified morphism:

$$\mathbb{G}_m \times \mathbb{G}_m \to \mathrm{B}^2\mathrm{A}(1). \tag{5.34}$$

It remains to show that (5.34) corresponds to $\mathrm{Q}(\alpha) \in \mathrm{A}(-1)$ under Lemma 4.4.3. However, we know from Lemma 5.2.4 that when applied to $t^2 \in \mathbb{G}_m$, (5.33) corresponds to the commutator $b(\alpha, \alpha) = 2\mathrm{Q}(\alpha)$. Thus (5.34) corresponds to a square root of $2\mathrm{Q}(\alpha)$.

Finally, a square root of $2a$ defined functorially for all finite abelian groups A equipped with an element $a \in \mathrm{A}$ must equal a, e.g. by the integrality of 2-adic integers. □

5.6. The case $\mathrm{S} = \mathrm{Spec}(\mathbb{R})$.

5.6.1. We study the case $\mathrm{S} = \mathrm{Spec}(\mathbb{R})$ as an extended example. The étale metaplectic covers of G define topological covers of a subgroup $\mathrm{G}(\mathbb{R})^0 \subset \mathrm{G}(\mathbb{R})$ by the construction of §2.1.6.

In this section, we give a criterion for when $\mathrm{G}(\mathbb{R})^0 = \mathrm{G}(\mathbb{R})$ in terms of the classification of étale metaplectic covers.

5.6.2. Let us begin with $\mathrm{G} = \mathbb{G}_m$. The splitting of Remark 4.2.8 defines a $\mathbb{Z}$-linear morphism:

$$\underline{\Gamma}_e(\mathrm{B}\mathbb{G}_m, \mathrm{B}^4\mathrm{A}(1)) \to \mathrm{B}^2\mathrm{A}. \tag{5.35}$$

On the other hand, every pointed morphism $\mu : \mathrm{B}\mathbb{G}_m \to \mathrm{B}^4\mathrm{A}(1)$ defines an $\mathbb{E}_1$-monoidal morphism $\mathbb{G}_m \to \mathrm{B}^3\mathrm{A}(1)$, hence a map $\mathbb{G}_m(\mathbb{R}) \to \mathrm{H}^3(\mathbb{R}, \mathrm{A}(1))$.

Denote its value at $(-1) \in \mathbb{G}_m(\mathbb{R})$ by $\mathrm{sgn}(\mu)$.

Lemma 5.6.3. *The element* $\mathrm{sgn}(\mu)$ *equals the class of the image of* μ *under* (5.35)*, followed by the cup product with the restriction of the Kummer class to* $(-1) \in \mathbb{G}_m(\mathbb{R})$*:*

$$\cup (-1)^*[\Psi] : \mathrm{H}^2(\mathbb{R}, \mathrm{A}) \to \mathrm{H}^3(\mathbb{R}, \mathrm{A}(1)). \tag{5.36}$$

Proof. If μ is the image of the splitting $\mathrm{A}(-1) \to \underline{\Gamma}_e(\mathrm{B}\mathbb{G}_m, \mathrm{B}^4\mathrm{A}(1))$ of Remark 4.2.8, viewing elements of the source as $\mathrm{A}(-1)$-valued quadratic forms on $\mathbb{Z}$, then $\mathrm{sgn}(\mu) = 0$. Indeed, $\mathrm{sgn}(\mu)$ depends only on the underlying pointed morphism of the associated $\mathbb{E}_1$-monoidal morphism $\mathbb{G}_m \to \mathrm{B}^3\mathrm{A}(1)$, which is trivial in this case.

It remains to show that for a pointed morphism $\mu : \mathrm{B}\mathbb{G}_m \to \mathrm{B}^4\mathrm{A}(1)$ defined by a section t of $\mathrm{A}[2]$, the element $\mathrm{sgn}(\mu)$ is the image of t under (5.36). In this case, we have $\mu \cong \Psi^*(t)$.

Evaluation at $(-1) \in \mathbb{G}_m(\mathbb{R})$ corresponds to pulling back along $(-1) : \mathrm{Spec}(\mathbb{R}) \to \mathbb{G}_m$. Replacing the Yoneda product by the cup product, we obtain an equality of cohomology classes:

$$\mathrm{sgn}(\mu) = [t] \cup (-1)^*[\Psi] \in \mathrm{H}^3(\mathbb{R}, \mathrm{A}(1)),$$

which is what we wanted to prove. □

Remark 5.6.4. If A is the constant sheaf $\{\pm 1\}$, then (5.36) is the unique isomorphism between two copies of $\mathbb{Z}/2$. Indeed, the cohomology ring $\mathrm{H}^*(\mathbb{R}, \{\pm 1\})$ is isomorphic to $\mathbb{F}_2[x]$ with a degree-1 generator $x = (-1)^*[\Psi]$.

5.6.5. Let G be a reductive group scheme over $\mathbb{R}$ together with a maximal torus T. Write $\mathrm{T}_0 \subset \mathrm{T}$ for its maximal split subtorus. Given a pointed morphism $\mu : \mathrm{BG} \to \mathrm{B}^4\mathrm{A}(1)$, we obtain a homomorphism:

$$\Lambda_{\mathrm{T}_0} \to \mathrm{H}^3(\mathbb{R}, \mathrm{A}(1)), \quad \lambda \mapsto \mathrm{sgn}(\lambda^*\mu). \tag{5.37}$$

where Λ_{T_0} denotes the group of cocharacters of T_0.

On the other hand, recall from §2.1.6 that the $\mathbb{E}_1$-monoidal morphism $\mathrm{G} \to \mathrm{B}^3\mathrm{A}(1)$ associated to μ defines a morphism of topological groups $\mathrm{G}(\mathbb{R}) \to \mathrm{H}^3(\mathbb{R}, \mathrm{A}(1))$, whose kernel is $\mathrm{G}(\mathbb{R})^0$.

Lemma 5.6.6. *The map* (5.37) *is zero if and only if* $\mathrm{G}(\mathbb{R})^0 = \mathrm{G}(\mathbb{R})$*.*

Proof. The "only if" direction requires a proof. Since the morphism $G(\mathbb{R}) \to H^3(\mathbb{R}, A(1))$ is continuous ([Del96, Lemme 2.10]), it factors through $\pi_0(G(\mathbb{R}))$.

According to [BT65, Théorème 14.4], the map $\pi_0(T_0(\mathbb{R})) \to \pi_0(G(\mathbb{R}))$ is surjective. Hence, it suffices to show that the induced map of abelian groups:

$$\pi_0(T_0(\mathbb{R})) \to H^3(\mathbb{R}, A(1))$$

vanishes identically. Now, $\pi_0(T_0(\mathbb{R}))$ is a direct sum of copies of $\mathbb{Z}/2$, and the hypothesis shows that its restriction to each copy of $\mathbb{Z}/2$ vanishes. □

Remark 5.6.7. Using Lemma 5.6.6, we see that $G(\mathbb{R})^0 = G(\mathbb{R})$ for all étale metaplectic covers μ if T does not contain a split subtorus.

6. Metaplectic dual data

In this section, we define the metaplectic dual data of a reductive group scheme $G \to S$ equipped with an étale metaplectic cover μ. In the split case, the definition we give in §6.1 relies on the materials of §4 and §5. The construction is functorial in the pair (G, μ) and the nonsplit case follows from étale descent.

Metaplectic dual data are first defined in the function field context for split reductive groups by Gaitsgory–Lysenko [GL18] and take the form of a triple $(H, \mathscr{G}_{Z_H}, \epsilon)$. Our definition mimics theirs, although it is purely group-theoretic and is valid in general. Over a local or global field, our construction is also closely related to Weissman's metaplectic L-group [Wei18], with which we eventually compare in §6.4.

The main construction of this section is in §6.3: we obtain the category of representations of the metaplectic L-group *without* the need to construct the group itself. It is sufficient for defining L-parameters, which are certain tensor functors (Definition 6.3.12). This Tannakian perspective is heavily inspired by [AGK⁺20], and I believe that it adds some transparency, especially in the metaplectic context.

6.0.1. Throughout this section, we fix a scheme S. We shall need a "field of coefficients" which is typically $\mathbb{C}$ or $\overline{\mathbb{Q}}_\ell$. For concreteness, we focus on the latter case.

More precisely, let ℓ be a prime invertible on S and fix an algebraic closure $\mathbb{Q}_\ell \subset \overline{\mathbb{Q}}_\ell$. Let $E \subset \overline{\mathbb{Q}}_\ell$ be a finite extension of $\mathbb{Q}_\ell$. The coefficient for metaplectic covers will be a subsheaf $A \subset \underline{E}^\times$ of finite abelian groups whose order is invertible on S.

6.1. Construction of (H, ν).

6.1.1. Let $G \to S$ be a split reductive group scheme and μ be a pointed morphism $B(G) \to B^4 A(1)$. We use the notations of §5.1.1 for group-theoretic notions.

We shall first construct a pair (H, ν) where H is a pinned reductive group scheme over $\mathrm{Spec}(\mathbb{Z})$ and ν is an $\mathbb{E}_\infty$-monoidal morphism:

$$\nu : \hat{Z}_H \to B^2(A), \tag{6.1}$$

where Z_H denotes the center of H and $\hat{Z}_H$ the abelian group of its characters, viewed as a constant étale sheaf of abelian groups over S.

Remark 6.1.2. Using the inclusion $A \subset \underline{E}^\times$, we may view ν as valued in $B^2(\underline{E}^\times)$, but we will still use the same notation for it.

6.1.3. Recall that to each pointed morphism $\mu : \mathrm{BG} \to \mathrm{B}^4\mathrm{A}(1)$, there is an associated strict quadratic form $\mathrm{Q} \in \mathrm{Quad}(\Lambda, \mathrm{A}(-1))_{\mathrm{st}}$, see §5.1.4.

The construction of H depends only on Q (see also [Lus93, 2.2.5]). Define $\Phi^\sharp \subset \Lambda^\sharp$ to be the subset $\{\mathrm{ord}(\mathrm{Q}(\alpha))\alpha \mid \alpha \in \Phi\}$, where $\mathrm{ord}(\mathrm{Q}(\alpha)) \in \mathbb{N}$ denotes the order of $\mathrm{Q}(\alpha)$.

Let $\check{\Lambda}^\sharp$ be the dual of $\Lambda^\sharp$. Since Q is strict, $\mathrm{ord}(\mathrm{Q}(\alpha))^{-1}\check{\alpha}$ takes integral values on $\Lambda^\sharp$. Hence we have a well-defined subset $\check{\Phi}^\sharp = \{\mathrm{ord}(\mathrm{Q}(\alpha))^{-1}\check{\alpha} \mid \check{\alpha} \in \check{\Phi}\}$ of $\check{\Lambda}^\sharp$. The bijection $\Phi \cong \check{\Phi}$ induces a bijection $\Phi^\sharp \cong \check{\Phi}^\sharp$. The systems of simple coroots and roots $\Delta^\sharp$, $\check{\Delta}^\sharp$ are determined by the corresponding elements of Δ, $\check{\Delta}$.

One verifies that $(\Delta^\sharp \subset \Phi^\sharp \subset \Lambda^\sharp, \check{\Delta}^\sharp \subset \check{\Phi}^\sharp \subset \check{\Lambda}^\sharp)$, $\Phi^\sharp \cong \check{\Phi}^\sharp$ defines a based reduced root datum. Let $\mathrm{G}^\sharp$ be the corresponding pinned reductive group over S. Define H to be the Langlands dual of $\mathrm{G}^\sharp$, viewed as a pinned split reductive group over $\mathrm{Spec}(\mathbb{Z})$. (This means that $\Lambda^\sharp_{\mathrm{T}}$ is the *character* lattice of the maximal torus of H, etc.)

6.1.4. To construct ν, we need some more notations. Regard the quadratic form Q as fixed and write $\underline{\Gamma}_e(\mathrm{BG}, \mathrm{B}^4\mathrm{A}(1))_{\mathrm{Q}}$ for the full subgroupoid of pointed morphisms $\mathrm{BG} \to \mathrm{B}^4\mathrm{A}(1)$ whose associated quadratic form is Q.

Composing the canonical morphisms (5.16) and (4.37) (applied to $\mathrm{T}^\sharp$) yields the following morphism:

$$\underline{\Gamma}_e(\mathrm{BG}, \mathrm{B}^4\mathrm{A}(1))_{\mathrm{Q}} \to \underline{\mathrm{Maps}}_{\mathbb{E}_\infty}(\Lambda^\sharp, \mathrm{B}^2\mathrm{A}). \tag{6.2}$$

Its naturality with respect to the map $\mathrm{G}_{\mathrm{sc}} \to \mathrm{G}$ shows that the following diagram commutes:

$$\begin{array}{ccc} \underline{\Gamma}_e(\mathrm{BG}, \mathrm{B}^4\mathrm{A}(1))_{\mathrm{Q}} & \longrightarrow & \underline{\mathrm{Maps}}_{\mathbb{E}_\infty}(\Lambda^\sharp, \mathrm{B}^2\mathrm{A}) \\ \downarrow & & \downarrow \\ \underline{\Gamma}_e(\mathrm{BG}_{\mathrm{sc}}, \mathrm{B}^4\mathrm{A}(1))_{\mathrm{Q}_{\mathrm{sc}}} & \to & \underline{\mathrm{Maps}}_{\mathbb{E}_\infty}(\Lambda^{\sharp,r}, \mathrm{B}^2\mathrm{A}) \end{array} \tag{6.3}$$

6.1.5. Let us construct a trivialization of the bottom horizontal functor in (6.3).

Construction. The question concerns only a simply connected group G_{sc} so we will write $\mathrm{G} = \mathrm{G}_{\mathrm{sc}}$ to ease the notation. We shall construct the trivialization étale locally on S *using a fixed pinning* $\mathrm{T} \subset \mathrm{B} \subset \mathrm{G}$, $\mathrm{N}_\alpha \cong \mathbb{G}_a$ $(\alpha \in \Delta)$ on G. Then we argue that the trivialization does not depend on the pinning, so it exists over S by étale descent.

Since G is simply connected, $\underline{\Gamma}_e(\mathrm{BG}, \mathrm{B}^4\mathrm{A}(1))_{\mathrm{Q}}$ is the singleton with element Q, corresponding to an étale metaplectic cover μ. We shall trivialize its image in $\underline{\mathrm{Maps}}_{\mathbb{E}_\infty}(\Lambda^{\sharp,r}, \mathrm{B}^2\mathrm{A})$. The lattice $\Lambda^{\sharp,r}$ has a globally defined basis with elements $\alpha^\sharp \in \Delta^\sharp$. It suffices to construct a trivialization of

$$(\alpha^\sharp)^*\mu_{\mathrm{T}} \in \underline{\Gamma}_e(\mathrm{B}\mathbb{G}_m, \mathrm{B}^4\mathrm{A}(1)),$$

where $\mu_{\mathrm{T}} \in \underline{\Gamma}_e(\mathrm{BT}, \mathrm{B}^4\mathrm{A}(1))$ is the restriction of G along $\mathrm{T} \subset \mathrm{G}$.

Let us write $d := \mathrm{ord}(\mathrm{Q}(\alpha))$, so $\alpha^\sharp = d \cdot \alpha$. Furthermore, we shall extend α to a morphism $f_\alpha : \mathrm{SL}_2 \to \mathrm{G}$ compatible with the pinnings. Recall the canonical quadratic structure on μ_{T}, constructed in Proposition 4.7.3. Iteratively applying it d times yields an isomorphism

$$(\alpha^\sharp)^*\mu_{\mathrm{T}} \cong d^2 \cdot \alpha^*\mu_{\mathrm{T}} + \binom{d}{2} \cdot b(\alpha, \alpha) \otimes (\Psi \cup \Psi) \tag{6.4}$$

where b is the symmetric form associated to Q. We claim that the two terms on the right-hand-side of (6.4) are both canonically trivial:

(1) for the first term, this is because $\alpha^*\mu_{\mathrm{T}}$ coincides with the restriction of $(f_\alpha)^*\mu$ along the diagonal torus $\mathbb{G}_m \subset \mathrm{SL}_2$. Since $(f_\alpha)^*\mu$ is classified by $\mathrm{Q}(\alpha) \in \mathrm{A}(-1)$, its dth multiple is canonically trivial;
(2) for the second term, this is because

$$\binom{d}{2} \cdot b(\alpha, \alpha) = (d-1) \cdot d \cdot \mathrm{Q}(\alpha) = 0.$$

Next, we shall argue that the trivialization of $(\alpha^\sharp)^*\mu_{\mathrm{T}}$ constructed above is independent of the pinning on G. Indeed, any other pinning $\mathrm{T}_1 \subset \mathrm{B}_1 \subset \mathrm{G}$, $\mathrm{N}_\alpha \cong \mathbb{G}_a$ ($\alpha \in \Delta$) is conjugate to the given one by a unique section g of G_{ad}. Let us record the commutative diagram

$$\begin{array}{ccccc} \mathbb{G}_m & \xrightarrow{\alpha} & \mathrm{T} & \subset & \mathrm{G} \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{int}_g} & & \downarrow{\scriptstyle \mathrm{int}_g} \\ \mathbb{G}_m & \xrightarrow{\mathrm{int}_g \circ \alpha} & \mathrm{T}_1 & \subset & \mathrm{G} \end{array}$$

where int_g denotes the conjugation action by g.

Since G is simply connected, we have a canonical isomorphism

$$\varphi_g : \mu \cong (\mathrm{int}_g)^*\mu, \tag{6.5}$$

which we shall use to identify the trivializations defined with respect to the two pinnings. More precisely, we need to show that the trivializations of $(\alpha^\sharp)^*\mu_{\mathrm{T}}$ and $(\mathrm{int}_g \circ \alpha^\sharp)^*\mu_{\mathrm{T}_1}$ constructed above are related by the following commutative diagram

$$\begin{array}{ccc} (\alpha^\sharp)^*\mu_{\mathrm{T}} & \cong & * \\ \downarrow{\scriptstyle (\alpha^\sharp)^*(\varphi_g|_{\mathrm{T}})} & & \downarrow{\scriptstyle \mathrm{id}} \\ (\mathrm{int}_g \circ \alpha^\sharp)^*\mu_{\mathrm{T}_1} & \cong & * \end{array} \tag{6.6}$$

where $\varphi_g|_{\mathrm{T}} : \mu_{\mathrm{T}} \cong (\mathrm{int}_g)^*\mu_{\mathrm{T}_1}$ is the restriction of (6.5) to BT.

Since the canonical quadratic structure is natural with respect to isomorphisms of tori, we may replace the left vertical arrow in (6.6) by the sum of two isomorphisms

$$d^2 \cdot \alpha^*(\varphi_g|_{\mathrm{T}}) : d^2 \cdot \alpha^*\mu_{\mathrm{T}} \cong d^2 \cdot (\mathrm{int}_g \circ \alpha)^*\mu_{\mathrm{T}_1},$$
$$\mathrm{id} : \binom{d}{2} \cdot b(\alpha,\alpha) \otimes (\Psi \cup \Psi) \cong \binom{d}{2} \cdot b(\alpha,\alpha) \otimes (\Psi \cup \Psi).$$

It remains to show that under the trivializations of $d^2 \cdot \alpha^*\mu_{\mathrm{T}}$ and $d^2 \cdot (\mathrm{int}_g \circ \alpha)^*\mu_{\mathrm{T}_1}$ given in (1) above, the isomorphism $d^2 \cdot \alpha^*(\varphi_g|_{\mathrm{T}})$ corresponds to the identity automorphism. For this, we extend $\alpha^*(\varphi_g|_{\mathrm{T}})$ to the isomorphism

$$(f_\alpha)^*\varphi_g : (f_\alpha)^*\mu \cong (\mathrm{int}_g \circ f_\alpha)^*\mu \tag{6.7}$$

of étale metaplectic covers of SL_2. We need to show that upon multiplying by d and trivializing both the source and the target, (6.7) corresponds to the identity automorphism. This follows immediately from the discreteness of $\underline{\Gamma}_e(\mathrm{BSL}_2, \mathrm{B}^4\mathrm{A}(1))$. ☐

6.1.6. Using the Construction of §6.1.5 and the identification $\hat{\mathrm{Z}}_{\mathrm{H}} \cong \Lambda^\sharp/\Lambda^{\sharp,r}$, we find a canonical morphism:

$$\underline{\Gamma}_e(\mathrm{BG}, \mathrm{B}^4\mathrm{A}(1))_{\mathrm{Q}} \to \underline{\mathrm{Maps}}_{\mathbb{E}_\infty}(\hat{\mathrm{Z}}_{\mathrm{H}}, \mathrm{B}^2\mathrm{A}). \tag{6.8}$$

We set ν to be the image of μ under (6.8). This concludes the construction of the pair (H, ν) alluded to in §6.1.1.

6.2. Splitting ν.

6.2.1. We keep the notations of the previous §6.1. Given a split reductive group G together with an étale metaplectic cover μ, we have constructed a pair (H, ν) from the combinatorial data associated to G and μ.

Following [GL18], it is possible to re-package the pair (H, ν) as a triple $(\mathrm{H}, \nu^0, \epsilon)$ by writing ν as the sum of a $\mathbb{Z}$-linear morphism $\hat{\mathrm{Z}}_{\mathrm{H}} \to \mathrm{B}^2\mathrm{A}$ and an $\mathbb{E}_\infty$-morphism with trivial underlying $\mathbb{E}_1$-monoidal structure.

Recall that $\underline{\mathrm{Maps}}_{\mathbb{E}_\infty}(\hat{\mathrm{Z}}_{\mathrm{H}}, \mathrm{B}^2\mathrm{A})$ fits into a fiber sequence:

$$\underline{\mathrm{Maps}}_{\mathbb{Z}}(\hat{\mathrm{Z}}_{\mathrm{H}}, \mathrm{B}^2\mathrm{A}) \to \underline{\mathrm{Maps}}_{\mathbb{E}_\infty}(\hat{\mathrm{Z}}_{\mathrm{H}}, \mathrm{B}^2\mathrm{A}) \to \underline{\mathrm{Maps}}_{\mathbb{Z}}(\hat{\mathrm{Z}}_{\mathrm{H}}/2, \mathrm{A}), \tag{6.9}$$

where the second map sends an $\mathbb{E}_\infty$-monoidal morphism $\hat{\mathrm{Z}}_{\mathrm{H}} \to \mathrm{B}^2\mathrm{A}$ to the map $\hat{\mathrm{Z}}_{\mathrm{H}}/2 \to \mathrm{A}$ defined by its commutativity constraint as in §4.6.5.

6.2.2. The fiber sequence (6.9) admits a canonical splitting.

Construction. Since A is a subsheaf of $\mathrm{E}^\times$, its subgroup $\mathrm{A}_{[2]}$ of 2-torsion elements belongs to $\{\pm 1\}$. There is nothing to construct if $\mathrm{A}_{[2]} = 0$ as the first map in (6.9) becomes an isomorphism, so let us assume $\mathrm{A}_{[2]} = \{\pm 1\}$.

In this case, any homomorphism $\hat{\mathrm{Z}}_{\mathrm{H}} \to \mathrm{A}_{[2]}$ is of the form $\lambda \mapsto (-1)^{\epsilon(\lambda)}$ for a homomorphism $\epsilon : \hat{\mathrm{Z}}_{\mathrm{H}} \to \mathbb{Z}/2$. We define an $\mathbb{E}_\infty$-monoidal morphism $\hat{\mathrm{Z}}_{\mathrm{H}} \to \mathrm{B}^2\mathrm{A}$ by the associated extension of sheaves of symmetric monoidal groupoids:

$$\mathrm{B}(\mathrm{A}) \to (\hat{\mathrm{Z}}_{\mathrm{H}})^\dagger \to \hat{\mathrm{Z}}_{\mathrm{H}},$$

where $(\hat{\mathrm{Z}}_{\mathrm{H}})^\dagger := \mathrm{B}(\mathrm{A}) \times \hat{\mathrm{Z}}_{\mathrm{H}}$ as a sheaf of monoidal groupoids, but with commutativity constraint defined by:

$$(-1)^{\epsilon(\lambda_1)\epsilon(\lambda_2)} : \lambda_1 + \lambda_2 \cong \lambda_2 + \lambda_1,$$

for any elements $\lambda_1, \lambda_2 \in \hat{\mathrm{Z}}_{\mathrm{H}}$, viewed as sections of $(\hat{\mathrm{Z}}_{\mathrm{H}})^\dagger$. □

Remark 6.2.3. The construction of §6.2.2 remains valid when $\hat{\mathrm{Z}}_{\mathrm{H}}$ is replaced by any sheaf of abelian groups. Recall that for the constant abelian group $\mathbb{Z}$, there is another splitting constructed in Remark 4.2.8 which may be restricted to $\mathrm{A}(-1)_{[2]} \subset \mathrm{A}(-1)$.

These two splittings are *a priori* different, since the splitting in Remark 4.2.8 produces $\mathbb{E}_\infty$-monoidal morphisms $\mathbb{Z} \to \mathrm{B}^2\mathrm{A}$ whose underlying $\mathbb{E}_1$-monoidal morphism are not trivialized. They do become isomorphic if a fourth root of unity is chosen.

6.2.4. Using the splitting of §6.2.2, we obtain from ν a $\mathbb{Z}$-linear morphism $\nu^0 : \hat{\mathrm{Z}}_{\mathrm{H}} \to \mathrm{B}^2\mathrm{A}$. Write $(\hat{\mathrm{Z}}_{\mathrm{H}})^\vee$ for the $\mathbb{Z}$-linear dual of $\hat{\mathrm{Z}}_{\mathrm{H}}$. It is a complex in degrees $[0,1]$:

$$\begin{aligned}(\hat{\mathrm{Z}}_{\mathrm{H}})^\vee &\cong [\check{\Lambda}^\sharp \to \check{\Lambda}^{\sharp,r}] \\ &\cong [\Lambda_{\mathrm{T}_{\mathrm{H}}} \to \Lambda_{\mathrm{T}_{\mathrm{H}_{\mathrm{ad}}}}],\end{aligned}$$

where $\Lambda_{\mathrm{T}_{\mathrm{H}}}$ denotes the character group of the maximal torus $\mathrm{T}_{\mathrm{H}} \subset \mathrm{H}$ and $\mathrm{T}_{\mathrm{H}_{\mathrm{ad}}}$ that of the adjoint group H_{ad}.

Hence ν^0 may also be viewed as a section of $(\hat{\mathrm{Z}}_{\mathrm{H}})^\vee \otimes \mathrm{A}[2]$. Inducing along the map $\mathrm{A} \subset \overline{\mathbb{Q}}_\ell^\times$ and using the identification $(\hat{\mathrm{Z}}_{\mathrm{H}})^\vee \otimes \overline{\mathbb{Q}}_\ell^\times \cong \mathrm{Z}_{\mathrm{H}}(\overline{\mathbb{Q}}_\ell)$, we obtain an étale $\mathrm{Z}_{\mathrm{H}}(\overline{\mathbb{Q}}_\ell)$-gerbe $\mathscr{G}_{\mathrm{Z}_{\mathrm{H}}(\overline{\mathbb{Q}}_\ell)}$.

Finally, we let ϵ be the image of ν along the second map in (6.9): it is also given by restricting the quadratic form Q to $\Lambda^\sharp$, which is linear and factors through $\hat{\mathrm{Z}}_{\mathrm{H}}$, with image in $\mathrm{A}(-1)_{[2]} \cong \mathrm{A}_{[2]}$ (Proposition 4.6.6).

Remark 6.2.5. If $S = X$ is a smooth algebraic cuver over a field k, our triple $(H, \mathscr{G}_{Z_H(\overline{\mathbb{Q}}_\ell)}, \epsilon)$ is closely related to the metaplectic dual data of [GL18]. However, we caution the reader that our $\mathscr{G}_{Z_H(\overline{\mathbb{Q}}_\ell)}$ differs slightly from the $Z_H(\overline{\mathbb{Q}}_\ell)$-gerbe defined in *op.cit.*.

More precisely, there is a canonical $Z_H(\overline{\mathbb{Q}}_\ell)$-gerbe $\omega^{\epsilon}_{X/k}$ on X defined by inducing the $\{\pm 1\}$-gerbe $\omega^{1/2}_{X/k}$ of square roots of the canonical sheaf along ϵ, viewed as a map $\{\pm 1\} \to Z_H(\overline{\mathbb{Q}}_\ell)$. Then the $Z_H(\overline{\mathbb{Q}}_\ell)$-gerbe of *op.cit.* is equivalent to our $\mathscr{G}_{Z_H(\overline{\mathbb{Q}}_\ell)}$ tensored with $\omega^{\epsilon}_{X/k}$.

6.3. The dual category $\underline{\mathrm{Rep}}_{H,\nu}$.

6.3.1. We use the term *tensor category* to refer to a symmetric monoidal E-linear abelian category, and *tensor functor* to refer to a symmetric monoidal E-linear additive functor between them.

Denote by Vect^f_E the tensor category of finite-dimensional E-vector spaces.

6.3.2. Consider the stack of tensor categories Lis_E over $S_{\text{ét}}$ whose value at $S_1 \to S$ is the category of lisse (i.e. locally constant constructible) sheaves of E-vector spaces. As usual, this category is bootstrapped from lisse $\mathscr{O}_E/\mathfrak{m}^n$-modules for $n \geq 1$:

$$\mathrm{Lis}_E(S_1) := (\lim_n \mathrm{Lis}_{\mathscr{O}_E/\mathfrak{m}^n}(S_1))[\frac{1}{\ell}],$$

where we invert ℓ on the Hom-modules.

The category $\mathrm{Lis}_E(S_1)$ is tensored over Vect^f_E. Its ind-completion $\mathrm{Ind}(\mathrm{Lis}_E(S_1))$ is tensored over the category Vect_E of all E-vector spaces. In particular, for any coalgebra in Vect_E, one may consider its comodules in the category $\mathrm{Ind}(\mathrm{Lis}_E(S_1))$.

Given an affine group scheme H over E, an H-*representation* in $\mathrm{Lis}_E(S_1)$ is an object $V \in \mathrm{Lis}_E(S_1)$ equipped with the structure of an $\mathscr{O}_H$-comodule. Denote the category they form by:

$$\underline{\mathrm{Rep}}_H(S_1) := \mathrm{Comod}_{\mathscr{O}_H}(\mathrm{Lis}_E(S_1)).$$

Then $\underline{\mathrm{Rep}}_H$ is itself a stack of tensor categories over $S_{\text{ét}}$.

6.3.3. Let Γ be a finitely generated abelian group and $\hat{\Gamma}$ be its Cartier dual group scheme over Spec(E). Then there is a canonical equivalence:

$$\underline{\mathrm{Rep}}_{\hat{\Gamma}}(S_1) \cong \bigoplus_{\lambda \in \Gamma} \mathrm{Lis}_E(S_1),$$

where the copy of $\mathrm{Lis}_E(S_1)$ corresponding to $\lambda \in \Gamma$ has $\hat{\Gamma}$-action of weight λ.

This is a classical fact if S_1 is the spectrum of a separably closed field ([ABD+66, I, Proposition 4.7.3]). The proof is unchanged in our setting: $\mathscr{O}_{\hat{\Gamma}} \cong E[\Gamma]$ and $V \in \underline{\mathrm{Rep}}_{\hat{\Gamma}}(S_1)$ decomposes according to the image of the coaction map $V \to V \otimes E[\Gamma] \cong \bigoplus_{\lambda \in \Gamma} V$.

6.3.4. Let H be a split reductive group scheme over Spec(E) with center Z_H. Then we have a direct sum decomposition via restriction to the Z_H-action:

$$\underline{\mathrm{Rep}}_H(S_1) \cong \bigoplus_{\lambda \in \hat{Z}_H} \underline{\mathrm{Rep}}^{\lambda}_H(S_1). \tag{6.10}$$

Indeed, this is because H-action fixes Z_H-weights. By varying S_1, we obtain a direct sum decomposition of the stack $\underline{\mathrm{Rep}}_H$.

6.3.5. The constant sheaf of abelian groups $\underline{\mathrm{E}}^\times$ acts multiplicatively on $\mathrm{id}_{\mathrm{Rep}_{\mathrm{H,S}}}$ (§A.1.1). In particular, given an $\mathbb{E}_\infty$-monoidal morphism $\nu : \hat{\mathrm{Z}}_{\mathrm{H}} \to \mathrm{B}^2(\underline{\mathrm{E}}^\times)$ over S, we obtain another stack of tensor categories $\underline{\mathrm{Rep}}_{\mathrm{H},\nu}$ by the twisting construction of §A.2.4.

It inherits a decomposition from (6.10):

$$\underline{\mathrm{Rep}}_{\mathrm{H},\nu} \cong \bigoplus_{\lambda\in\hat{\mathrm{Z}}_{\mathrm{H}}} \underline{\mathrm{Rep}}^{\lambda}_{\mathrm{H},\nu(\lambda)}, \tag{6.11}$$

where the summands are stacks of abelian categories $\underline{\mathrm{Rep}}^{\lambda}_{\mathrm{H}}$ twisted by $\nu(\lambda)$.

6.3.6. Let $\mathrm{G} \to \mathrm{S}$ be a reductive group scheme equipped with a pointed morphism $\mu : \mathrm{BG} \to \mathrm{B}^4\mathrm{A}(1)$. Over an étale cover $\mathrm{S}_1 \to \mathrm{S}$ splitting G, we obtain from §6.1 a pair (H, ν) where $\mathrm{H} \to \mathrm{Spec}(\mathbb{Z})$ is a pinned reductive group and $\nu : \hat{\mathrm{Z}}_{\mathrm{H}} \to \mathrm{B}^2(\mathrm{A})$ is an $\mathbb{E}_\infty$-monoidal morphism. Using the maps $\mathbb{Z} \to \mathrm{E}$, $\mathrm{A} \subset \underline{\mathrm{E}}^\times$, we may form a stack of tensor categories $\underline{\mathrm{Rep}}_{\mathrm{H},\nu}$ over S_1.

Since $\underline{\mathrm{Rep}}_{\mathrm{H},\nu}$ is functorially attached to the pair (G, μ) and stacks of tensor categories are étale local objects, we find a stack of tensor categories over S, to be denoted by the same notation:

$$(\mathrm{G}, \mu) \mapsto \underline{\mathrm{Rep}}_{\mathrm{H},\nu}. \tag{6.12}$$

Remark 6.3.7. The stack of tensor categories $\underline{\mathrm{Rep}}_{\mathrm{H},\nu}$ satisfies the following conditions:

(1) every section V of $\underline{\mathrm{Rep}}_{\mathrm{H},\nu}$ admits a dual;
(2) the étale sheaf of endomorphisms of the monoidal unit in $\underline{\mathrm{Rep}}_{\mathrm{H},\nu}$ is naturally isomorphic to $\underline{\mathrm{E}}$.

Indeed, both statements are of étale local nature, so we may assume that G is split and use the decomposition (6.11). For statement (1), it suffices to treat the case where V belongs to $\underline{\mathrm{Rep}}^{\lambda}_{\mathrm{H},\nu(\lambda)}$ for some $\lambda \in \hat{\mathrm{Z}}_{\mathrm{H}}$; the assertion then follows from the fact that every section of $\underline{\mathrm{Rep}}^{\lambda}_{\mathrm{H}}$ admits a dual in $\underline{\mathrm{Rep}}^{-\lambda}_{\mathrm{H}}$, and a trivialization of $\nu(\lambda)$ may be chosen étale locally. Statement (2) follows likewise from the fact that the étale sheaf of endomorphisms of the monoidal unit in $\underline{\mathrm{Rep}}_{\mathrm{H}}$ is naturally isomorphic to $\underline{\mathrm{E}}$.

In particular, when S is connected, $\underline{\mathrm{Rep}}_{\mathrm{H},\nu}(\mathrm{S})$ is a *catégorie tensorielle* in the sense of [Del90, §1.2]. If it admits a fiber functor, then we may identify $\underline{\mathrm{Rep}}_{\mathrm{H},\nu}(\mathrm{S})$ with the category of representations of an affine group scheme [Del90, Théorème 1.12].

Remark 6.3.8. Note that $\underline{\mathrm{Rep}}_{\mathrm{H},\nu}$ is naturally tensored over $\mathrm{Lis}_{\mathrm{E}}$. Étale locally on S, this $\mathrm{Lis}_{\mathrm{E}}$-action preserves the decomposition (6.11). Moreover, acting on the monoidal unit $\mathbf{1} \in \underline{\mathrm{Rep}}_{\mathrm{H},\nu}$ induces a *symmetric* monoidal functor:

$$\mathrm{Lis}_{\mathrm{E}} \to \underline{\mathrm{Rep}}_{\mathrm{H},\nu}. \tag{6.13}$$

It encodes the operation of viewing a lisse sheaf as a trivial ν-twisted H-representation.

Remark 6.3.9. For nonsplit G, it is possible to assign intrinsic meanings to H and ν in the notation (6.12) as follows:

(1) H is a locally constant sheaf (over S) of pinned reductive groups (over $\mathrm{Spec}(\mathbb{Z})$);
(2) $\hat{\mathrm{Z}}_{\mathrm{H}}$ is a locally constant sheaf of abelian groups and:

$$\nu : \hat{\mathrm{Z}}_{\mathrm{H}} \to \mathrm{B}^2(\mathrm{A})$$

is an $\mathbb{E}_\infty$-monoidal morphism.

There is always a faithful exact tensor functor $\underline{\mathrm{Rep}}_{\mathrm{H},\nu} \to \underline{\mathrm{Rep}}_{\mathrm{Z}_{\mathrm{H}},\nu}$.

6.3.10. Due to the commutativity constraint of ν, it is unreasonable to expect there to be fiber functors out of $\underline{\mathrm{Rep}}_{\mathrm{H},\nu}(\mathrm{S})$ in general. In order to define L-parameters, we replace ν by its $\mathbb{Z}$-linear component ν^0 in the formation of $\underline{\mathrm{Rep}}_{\mathrm{H},\nu}$.

Recall from §6.2.4 that $\nu^0 : \hat{\mathrm{Z}}_{\mathrm{H}} \to \mathrm{B}^2(\mathrm{A})$ is a $\mathbb{Z}$-linear morphism functorially attached to ν (hence to (G, μ)). Repeating the construction in §6.3.6 gives an assignment:

$$(\mathrm{G}, \mu) \mapsto \underline{\mathrm{Rep}}_{\mathrm{H},\nu^0}. \tag{6.14}$$

Thus $\underline{\mathrm{Rep}}_{\mathrm{H},\nu^0}$ is canonically equivalent to $\underline{\mathrm{Rep}}_{\mathrm{H},\nu}$ as stacks of *monoidal* categories, but the commutativity constraint is modified by the values of ϵ on Z_{H}-weights.

Note that $\underline{\mathrm{Rep}}_{\mathrm{H},\nu^0}$ is naturally tensored over $\mathrm{Lis}_{\mathrm{E}}$ like its sister $\underline{\mathrm{Rep}}_{\mathrm{H},\nu}$ (Remark 6.3.8).

6.3.11. A functor $\mathrm{F} : \mathscr{C}_1 \to \mathscr{C}_2$ of categories tensored over a monoidal category $\mathscr{A}$ is said to *commute* with the $\mathscr{A}$-action if it is equipped with a natural isomorphism:

$$\mathrm{F}(a \otimes c) \cong a \otimes \mathrm{F}(c), \quad a \in \mathscr{A}, c \in \mathscr{C}_1, \tag{6.15}$$

compatible with the monoidal structure of $\mathscr{A}$.

If $\mathrm{F} : \mathscr{C}_1 \to \mathscr{C}_2$ is a tensor functor between tensor categories and the $\mathscr{A}$-actions on them come from tensor functors $\varphi_1 : \mathscr{A} \to \mathscr{C}_1$, $\varphi_2 : \mathscr{A} \to \mathscr{C}_2$, then the datum (6.15) is equivalent to an isomorphism of tensor functors $\mathrm{F} \circ \varphi_1 \cong \varphi_2$.

Definition 6.3.12. A *Tannakian* L-*parameter* is a faithful exact tensor functor:

$$\underline{\mathrm{Rep}}_{\mathrm{H},\nu^0}(\mathrm{S}) \to \mathrm{Lis}_{\mathrm{E}}(\mathrm{S})$$

commuting with the $\mathrm{Lis}_{\mathrm{E}}(\mathrm{S})$-actions.

Remark 6.3.13. Let us illustrate the idea behind Tannakian L-parameters using the example of a split reductive group G over a field F with no metaplectic cover. We fix a geometric point $\bar{s} \to \mathrm{S} := \mathrm{Spec}(\mathrm{F})$.

In this case, the Langlands dual group is the pinned split reductive group $\check{\mathrm{G}} \to \mathrm{Spec}(\mathrm{E})$ and a classical L-parameter is a morphism $\pi_1(\mathrm{S}, \bar{s}) \to \check{\mathrm{G}}(\mathrm{E})$ of topological groups—the source is equipped with the profinite topology and the target the ℓ-adic topology—defined up to $\check{\mathrm{G}}(\mathrm{E})$-conjugation.

There is a natural functor between groupoids:

$$\mathrm{Hom}(\pi_1(\mathrm{S}, \bar{s}), \check{\mathrm{G}}(\mathrm{E}))/\check{\mathrm{G}}(\mathrm{E}) \to \mathrm{Hom}^{\otimes}(\mathrm{Rep}_{\check{\mathrm{G}}}, \mathrm{Lis}_{\mathrm{E}}(\mathrm{S})), \tag{6.16}$$

where the target consists of faithful exact tensor functors $\mathrm{Rep}_{\check{\mathrm{G}}} \to \mathrm{Lis}_{\mathrm{E}}(\mathrm{S})$, where $\mathrm{Lis}_{\mathrm{E}}(\mathrm{S})$ is identified with continuous $\pi_1(\mathrm{S}, \bar{s})$-representations on a finite-dimensional E-vector space.

Claim: (6.16) is fully faithful. Indeed, it suffices to prove that $\mathrm{Hom}(\pi_1(\mathrm{S}, \bar{s}), \check{\mathrm{G}}(\mathrm{E}))$ maps bijectively to the set of such functors $\mathrm{Rep}_{\check{\mathrm{G}}} \to \mathrm{Lis}_{\mathrm{E}}(\mathrm{S})$ equipped with a rigidification along $\bar{s}$, i.e. a commutative diagram

$$\begin{array}{ccc} \mathrm{Rep}_{\check{\mathrm{G}}} & \longrightarrow & \mathrm{Lis}_{\mathrm{E}}(\mathrm{S}) \\ & \underset{e^*}{\searrow} & \downarrow \bar{s}^* \\ & & \mathrm{Vect}^f_{\mathrm{E}} \end{array}$$

where e^* is the forgetful functor (pullback along $e : \mathrm{Spec}(\mathrm{E}) \to \mathrm{B}\check{\mathrm{G}}$).

This follows from the fact that continuous homomorphisms $\pi_1(\mathrm{S}, \bar{s}) \to \check{\mathrm{G}}(\mathrm{E})$ are identified with *algebraic* homomorphisms $\pi_1^{\mathrm{alg}}(\mathrm{S}, \bar{s}) \to \check{\mathrm{G}}$, where the pro-algebraic completion $\pi_1^{\mathrm{alg}}(\mathrm{S}, \bar{s})$ agrees with the automorphism group scheme of $\bar{s}^*$, c.f. [Pri09, §1.2].

The proof also shows that the essential image of (6.16) consists of functors $T : \mathrm{Rep}_{\check{G}} \to \mathrm{Lis}_E(S)$ such that $\bar{s}^* \circ T$ is isomorphic to e^* as tensor functors. The E-scheme parametrizing such isomorphisms $\bar{s}^* \circ T \cong e^*$ form a $\check{G}$-torsor over Spec(E), for which a section exists over a finite extension of E. In particular, taking colimit of (6.16) over finite extensions of $\mathbb{Q}_\ell$ defines an equivalence between:

(1) continuous homomorphisms $\pi_1(S, \bar{s}) \to \check{G}(\overline{\mathbb{Q}}_\ell)$ factoring through a finite extension of $\mathbb{Q}_\ell$, modulo $\check{G}(\overline{\mathbb{Q}}_\ell)$-conjugation; and
(2) faithful exact tensor functors $\mathrm{Rep}_{\check{G},\overline{\mathbb{Q}}_\ell} \to \mathrm{Lis}_{\overline{\mathbb{Q}}_\ell}(S)$ defined over a finite extension of $\mathbb{Q}_\ell$. (Here, $\mathrm{Rep}_{\check{G},\overline{\mathbb{Q}}_\ell}$ stands for the category of finite-dimensionl algebraic representations over $\overline{\mathbb{Q}}_\ell$; see [Del89, §4.6] for extending scalars of a Tannakian category.)

For G nonsplit and equipped with an étale metaplectic cover μ, the "variance along S" is encoded in the definition of $\underline{\mathrm{Rep}}_{H,\nu^0}$ together with its Lis_E-module structure. We shall make a similar comparison between Definition 6.3.12 and an L-parameter of classical flavor in the next subsection.

6.4. The classical L-group.

6.4.1. When S is the spectrum of a field or a discrete valuation ring, it is possible to obtain an L-group in the style of Langlands, i.e. an extension of the étale fundamental group of S by E-points of a split reductive group. The construction explained below paraphrases [Wei18, §5 & 19].

To be precise about our hypotheses, we shall consider a connected scheme S satisfying the following conditions:

(1) every reductive group scheme $G \to S$ splits over a finite étale cover of S;
(2) any gerbe banded by a sheaf of finite abelian groups is trivial over a finite étale cover of S (e.g. S is a $K(\pi, 1)$-scheme).

In what follows, we fix a geometric point $\bar{s} \to S$ and write $\pi_1(S, \bar{s})$ for the (pro-finite) étale fundamental group of S with base point $\bar{s}$.

In the construction of (6.20) below, we will need E to be large enough relative to (G, μ); see *loc.cit.* for the precise meaning of this condition.

6.4.2. Suppose that $G \to S$ is a reductive group scheme and $\mu : BG \to B^4 A(1)$ is a pointed morphism.

The base change $G_{\bar{s}} := G \times_S \bar{s}$ being split, we find a pinned reductive group scheme $H \to \mathrm{Spec}(E)$. Let T_H denote its maximal torus and $T_{H_{ad}}$ the induced maximal torus of the adjoint group H_{ad}.

Since $G \to S$ splits over a finite étale cover and the construction of H is functorial, we obtain a "continuous" $\pi_1(S, \bar{s})$-action on H preserving the pinning. Here, "continuity" means that the action factors through a finite quotient of $\pi_1(S, \bar{s})$.

In particular, the character group $\hat{Z}_H$ of the center of H is a $\pi_1(S, \bar{s})$-module. Its $\mathbb{Z}$-linear dual is the complex:

$$(\hat{Z}_H)^\vee \cong [\Lambda_{T_H} \to \Lambda_{T_{H_{ad}}}] \tag{6.17}$$

of $\pi_1(S, \bar{s})$-modules in degrees $[0, 1]$.

6.4.3. Recall the equivalence between étale sheaves of finite abelian groups on S and finite $\pi_1(S, \bar{s})$-modules, given by taking stalks $K \mapsto K_{\bar{s}}$ at $\bar{s}$.

Under this equivalence, the $\pi_1(\mathrm{S},\bar{s})$-module $\hat{\mathrm{Z}}_{\mathrm{H}}$ corresponds to the étale sheaf of finite abelian groups denoted by $\hat{\mathrm{Z}}_{\mathrm{H}}$ in Remark 6.3.9. The presentation (6.17) corresponds to a presentation of its dual $(\hat{\mathrm{Z}}_{\mathrm{H}})^{\vee}$ as a complex of étale sheaves.

Moreover, ν^0 may be viewed as a section of $(\hat{\mathrm{Z}}_{\mathrm{H}})^{\vee}\otimes \mathrm{A}[2]$ over S.

6.4.4. Fix a trivialization of ν^0 along $\bar{s}$. We shall construct a short exact sequence of topological groups:

$$1 \to \mathrm{H}(\mathrm{E}) \to {}^{\mathrm{L}}\mathrm{H}_{\mathrm{S}} \to \pi_1(\mathrm{S},\bar{s}) \to 1, \tag{6.18}$$

which is induced from a finite quotient of $\pi_1(\mathrm{S},\bar{s})$.

Construction. The passage from ν^0 to an extension of $\pi_1(\mathrm{S},\bar{s})$ by $\mathrm{Z}_{\mathrm{H}}(\mathrm{E})$ arises from the standard correspondence between étale cochains and Galois cochains.

More precisely, suppose that K is an étale sheaf of finite abelian groups on S. Consider the groupoid of extensions:

$$1 \to \mathrm{K}_{\bar{s}} \to \mathrm{E} \to \pi_1(\mathrm{S},\bar{s}) \to 1 \tag{6.19}$$

which are induced from a finite quotient of $\pi_1(\mathrm{S},\bar{s})$ and such that E-conjugation on $\mathrm{K}_{\bar{s}}$ factors through the $\pi_1(\mathrm{S},\bar{s})$-action.

By étale descent, there is a functor from such extensions to the groupoid of étale K-gerbes on S rigidified along $\bar{s}$. It is fully faithful with essential image being those étale K-gerbes trivialized over a *finite* étale cover of S. By our assumption on S in §6.4.1, this is in fact an equivalence of groupoids.

Since ν^0 is rigidified along $\bar{s}$, we obtain from this equivalence an extension of $\pi_1(\mathrm{S},\bar{s})$ by $\Lambda_{\mathrm{T}_{\mathrm{H}}}\otimes \mathrm{A}_{\bar{s}}$ whose induced extension by $\Lambda_{\mathrm{T}_{\mathrm{H}_{\mathrm{ad}}}}\otimes \mathrm{A}$ is equipped with a splitting.

Suppose E is sufficiently large so that $\Lambda_{\mathrm{T}_{\mathrm{H}}}\otimes \mathrm{E}^{\times} \to \Lambda_{\mathrm{T}_{\mathrm{H}_{\mathrm{ad}}}}\otimes \mathrm{E}^{\times}$ is *surjective.* Using the inclusion $\mathrm{A}\subset \underline{\mathrm{E}}^{\times}$, we obtain an extension of topological groups:

$$1 \to \mathrm{Z}_{\mathrm{H}}(\mathrm{E}) \to {}^{\mathrm{L}}(\mathrm{Z}_{\mathrm{H}})_{\mathrm{S}} \to \pi_1(\mathrm{S},\bar{s}) \to 1, \tag{6.20}$$

which is induced from a finite quotient of $\pi_1(\mathrm{S},\bar{s})$ and such that the ${}^{\mathrm{L}}(\mathrm{Z}_{\mathrm{H}})_{\mathrm{S}}$-conjugation action on $\mathrm{Z}_{\mathrm{H}}(\mathrm{E})$ factors through the natural $\pi_1(\mathrm{S},\bar{s})$-action.

Finally, the extension (6.18) is formed out of (6.20) by inducing along the $\pi_1(\mathrm{S},\bar{s})$-equivariant map $\mathrm{Z}_{\mathrm{H}}(\mathrm{E})\subset \mathrm{H}(\mathrm{E})$, namely:

$${}^{\mathrm{L}}\mathrm{H}_{\mathrm{S}} \coloneqq (\mathrm{H}(\mathrm{E}) \rtimes {}^{\mathrm{L}}(\mathrm{Z}_{\mathrm{H}})_{\mathrm{S}})/\mathrm{Z}_{\mathrm{H}}(\mathrm{E}), \tag{6.21}$$

where the ${}^{\mathrm{L}}(\mathrm{Z}_{\mathrm{H}})_{\mathrm{S}}$-action on $\mathrm{H}(\mathrm{E})$ factors through $\pi_1(\mathrm{S},\bar{s})$ and the embedding of $\mathrm{Z}_{\mathrm{H}}(\mathrm{E})$ is the anti-diagonal one. □

Remark 6.4.5. The extension (6.18) may be viewed as the L-group of (G,μ). Contrary to the L-group of reductive group schemes, it has no *canonical* semi-direct product structure.

If μ is obtained from a central extension of G by $\underline{\mathrm{K}}_2$ (see §2.3), then our definition of ${}^{\mathrm{L}}\mathrm{H}_{\mathrm{S}}$ agrees with Weissman's "second twist". When S is the spectrum a local or global field, incorporating the "first twist" amounts to replacing (6.20) by its Baer sum with the extension of $\pi_1(\mathrm{S},\bar{s})$ by $\mathrm{Z}_{\mathrm{H}}(\mathrm{E})$ given by inducing the meta-Galois group:

$$1 \to \{\pm 1\} \to \tilde{\pi}_1(\mathrm{S},\bar{s}) \to \pi_1(\mathrm{S},\bar{s}) \to 1$$

along the map $\epsilon : \{\pm 1\} \to \mathrm{Z}_{\mathrm{H}}(\mathrm{E})$ of §6.2.4.

Remark 6.4.6. Suppose that μ is the pullback along the map $G \to \pi_1(G) \otimes \mathbb{G}_m$, or equivalently its associated quadratic form vanishes, see Proposition 5.1.11. In this case, the construction of ν^0 simplifies significantly and so does the construction of the L-group.

Indeed, since $Q = 0$, the group scheme H is the usual Langlands dual $\check{G}$, equipped with the natural $\pi_1(S, \bar{s})$-action. The étale sheaf $\hat{Z}_H$ corresponding to the character group of its center $Z_{\check{G}}$ is naturally isomorphic to $\pi_1(G)$.

On the other hand, Proposition 5.1.11 defines a $\mathbb{Z}$-linear morphism $\pi_1(G) \to B^2A$, hence a $\mathbb{Z}$-linear morphism $\hat{Z}_H \to B^2(E^\times)$. If E is sufficiently large (e.g. $E = \overline{\mathbb{Q}}_\ell$), we obtain an extension of $\pi_1(S, \bar{s})$ by $Z_{\check{G}}(E)$ as in (6.20), and thus the L-group by the formation (6.21).[5]

6.4.7. The extension (6.18) gives an alternative description of global sections of the stack of tensor categories $\underline{\mathrm{Rep}}_{H,\nu^0}$.

To be precise, we write $\mathrm{Rep}^{\mathrm{alg}}_{{}^L H_S}$ for the category of continuous ${}^L H_S$-representations on finite-dimensional E-vector spaces, such that the action is algebraic on H(E), i.e. the induced H(E)-action comes from an algebraic H-representation.

If ${}^L H_S = H(E) \times \pi_1(S, \bar{s})$, then $\mathrm{Rep}^{\mathrm{alg}}_{{}^L H_S}$ is identified with the category of H-representations on lisse sheaves over S, i.e. the category $\underline{\mathrm{Rep}}_H(S)$ introduced in §6.3.2.

Proposition 6.4.8. *There is an equivalence of tensor categories:*

$$\underline{\mathrm{Rep}}_{H,\nu^0}(S) \cong \mathrm{Rep}^{\mathrm{alg}}_{{}^L H_S}. \tag{6.22}$$

Proof. Fix a finite étale Galois cover $S_1 \to S$ with structure group Γ and a lift $\bar{s}_1$ of $\bar{s}$ such that the following statements hold:

(1) the $\pi_1(S, \bar{s})$-action on H factors through Γ;
(2) the restriction of ν^0 to S_1, viewed as a section of the constant étale sheaf $(\hat{Z}_H)^\vee \otimes A[2]$, is trivial.

We furthermore fix a trivialization of ν^0 over S_1 extending the given trivialization over $\bar{s}_1$.

This trivialization induces an equivalence of tensor categories:

$$\underline{\mathrm{Rep}}_{H,\nu^0}(S_1) \cong \mathrm{Rep}^{\mathrm{alg}}_{H(E) \times \pi_1(S_1, \bar{s}_1)}, \tag{6.23}$$

as both sides are equivalent to H-representations on lisse sheaves over S_1. We shall obtain (6.22) as the equivalence of Γ-equivariant objects of (6.23), for naturally defined Γ-actions as endofunctors on both tensor categories.

The Γ-action on $\underline{\mathrm{Rep}}_{H,\nu^0}(S_1)$ comes from the descent data for ν^0 and the Γ-action on H. The fact that $\underline{\mathrm{Rep}}_{H,\nu^0}(S)$ is identified with Γ-equivariant objects in $\underline{\mathrm{Rep}}_{H,\nu^0}(S_1)$ follows from the fact that $\underline{\mathrm{Rep}}_{H,\nu^0}$ is an étale stack.

The Γ-action on $\mathrm{Rep}^{\mathrm{alg}}_{H(E) \times \pi_1(S_1, \bar{s}_1)}$ comes from the short exact sequence:

$$1 \to H(E) \times \pi_1(S_1, \bar{s}_1) \to {}^L H_S \to \Gamma \to 1.$$

Indeed, ${}^L H_S$-conjugation on $\mathrm{Rep}^{\mathrm{alg}}_{H(E) \times \pi_1(S_1, \bar{s}_1)}$ factors through Γ since inner automorphisms induce the identity functor on the category of representations. It follows from general principles that $\mathrm{Rep}^{\mathrm{alg}}_{{}^L H_S}$ is identified with Γ-equivariant objects in $\mathrm{Rep}^{\mathrm{alg}}_{H(E) \times \pi_1(S_1, \bar{s}_1)}$.

We omit the verification that (6.23) is compatible with these two Γ-actions, which follows routinely from the construction of ${}^L H_S$. □

[5] This construction is made independently by Tasho Kaletha [Kal22, §2].

Remark 6.4.9. Under the tensor equivalence (6.22), restriction along $\bar{s} \to \mathrm{S}$ on the left-hand-side corresponds to restriction along $\mathrm{H}(\mathrm{E}) \subset {}^{\mathrm{L}}\mathrm{H}_{\mathrm{S}}$ on the right-hand-side. Namely, we have a commutative diagram:

$$\begin{array}{ccc} \underline{\mathrm{Rep}}_{\mathrm{H},\nu^0}(\mathrm{S}) & \xrightarrow{\cong} & \mathrm{Rep}^{\mathrm{alg}}_{{}^{\mathrm{L}}\mathrm{H}_{\mathrm{S}}} \\ \big\downarrow{\scriptstyle \bar{s}\to\mathrm{S}} & & \big\downarrow{\scriptstyle \mathrm{H}(\mathrm{E})\subset{}^{\mathrm{L}}\mathrm{H}_{\mathrm{S}}} \\ \underline{\mathrm{Rep}}_{\mathrm{H},\nu^0}(\bar{s}) & \xrightarrow{\cong} & \mathrm{Rep}_{\mathrm{H}} \end{array}$$

where the bottom equivalence comes from the rigidification of ν^0 along $\bar{s}$.

Under the equivalence (6.22), the $\mathrm{Lis}_{\mathrm{E}}(\mathrm{S})$-action on $\underline{\mathrm{Rep}}_{\mathrm{H},\nu^0}(\mathrm{S})$ corresponds to tensoring with ${}^{\mathrm{L}}\mathrm{H}_{\mathrm{S}}$-representations restricted along the map ${}^{\mathrm{L}}\mathrm{H}_{\mathrm{S}} \to \pi_1(\mathrm{S}, \bar{s})$.

6.4.10. Write $\mathrm{Hom}_{/\pi_1(\mathrm{S},\bar{s})}(\pi_1(\mathrm{S},\bar{s}), {}^{\mathrm{L}}\mathrm{H}_{\mathrm{S}})$ for the set of continuous sections of (6.18):

$$\sigma : \pi_1(\mathrm{S}, \bar{s}) \to {}^{\mathrm{L}}\mathrm{H}_{\mathrm{S}}.$$

From σ, we obtain a faithful exact tensor functor $\mathrm{T} := \sigma^*$ rigidified along $\bar{s}$:

$$\begin{array}{ccc} \mathrm{Rep}^{\mathrm{alg}}_{{}^{\mathrm{L}}\mathrm{H}_{\mathrm{S}}} & \xrightarrow{\mathrm{T}} & \mathrm{Lis}_{\mathrm{E}}(\mathrm{S}) \\ & {\scriptstyle e^*}\searrow & \big\downarrow{\scriptstyle \bar{s}^*} \\ & & \mathrm{Vect}^f_{\mathrm{E}} \end{array} \tag{6.24}$$

where e^* stands for the tautological forgetful functor.

The fact that σ is a section implies that T commutes with natural $\mathrm{Lis}_{\mathrm{E}}(\mathrm{S})$-actions and the isomorphism $\bar{s}^* \circ \mathrm{T} \cong e^*$ exhibited in (6.24) is compatible with $\mathrm{Lis}_{\mathrm{E}}(\mathrm{S})$-actions (where $\mathrm{Lis}_{\mathrm{E}}(\mathrm{S})$ acts on $\mathrm{Vect}^f_{\mathrm{E}}$ via $\bar{s}^*$), i.e. T is $\mathrm{Lis}_{\mathrm{E}}(\mathrm{S})$-*linearly rigidified* along $\bar{s}$.

Under the equivalence of Proposition 6.4.8, we obtain a Tannakian L-parameter T which is $\mathrm{Lis}_{\mathrm{E}}(\mathrm{S})$-linearly rigidified along $\bar{s}$, see Definition 6.3.12.

The H(E)-conjugation action on $\mathrm{Hom}_{/\pi_1(\mathrm{S},\bar{s})}(\pi_1(\mathrm{S},\bar{s}), {}^{\mathrm{L}}\mathrm{H}_{\mathrm{S}})$ corresponds to modifying the isomorphism $\bar{s}^* \circ \mathrm{T} \cong e^*$ by a $\mathrm{Lis}_{\mathrm{E}}(\mathrm{S})$-linear automorphism of e^*. In particular, we obtain a functor of groupoids:

$$\mathrm{Hom}_{/\pi_1(\mathrm{S},\bar{s})}(\pi_1(\mathrm{S},\bar{s}), {}^{\mathrm{L}}\mathrm{H}_{\mathrm{S}})/\mathrm{H}(\mathrm{E}) \to \mathrm{Hom}^{\otimes}_{\mathrm{Lis}_{\mathrm{E}}(\mathrm{S})}(\underline{\mathrm{Rep}}_{\mathrm{H},\nu^0}(\mathrm{S}), \mathrm{Lis}_{\mathrm{E}}(\mathrm{S})), \tag{6.25}$$

where the target is the groupoid of Tannakian L-parameters.

The following fact is the "$\mathrm{Lis}_{\mathrm{E}}(\mathrm{S})$-linear version" of Remark 6.3.13.

Proposition 6.4.11. *The functor* (6.25) *is fully faithful. It induces an equivalence upon taking colimit over finite extensions* $\mathbb{Q}_\ell \subset \mathrm{E}$ *contained in* $\overline{\mathbb{Q}}_\ell$.

Proof. The assertions follow from the statements below:

(1) $\mathrm{Hom}_{/\pi_1(\mathrm{S},\bar{s})}(\pi_1(\mathrm{S},\bar{s}), {}^{\mathrm{L}}\mathrm{H}_{\mathrm{S}})$ maps bijectively to $\mathrm{Lis}_{\mathrm{E}}(\mathrm{S})$-linearly rigidified Tannakian L-parameters;
(2) $\mathrm{Lis}_{\mathrm{E}}(\mathrm{S})$-linear rigidifications of a Tannakian L-parameter along $\bar{s}$ form an H-torsor over Spec(E). (In particular, it splits over a finite extension of E.)

To prove these statements, it is convenient to use an algebraic version of the L-group (6.18). Namely, there is a short exact sequence of affine group schemes over Spec(E):

$$1 \to \mathrm{H} \to {}^{\mathrm{L}}\mathrm{H}^{\mathrm{alg}}_{\mathrm{S}} \to \pi_1^{\mathrm{alg}}(\mathrm{S}, \bar{s}) \to 1, \tag{6.26}$$

where ${}^{\mathrm{L}}\mathrm{H}_{\mathrm{S}}^{\mathrm{alg}}$ is the automorphism group scheme of the fiber functor:

$$\mathrm{Rep}^{\mathrm{alg}}_{{}^{\mathrm{L}}\mathrm{H}_{\mathrm{S}}} \to \mathrm{Rep}_{\mathrm{H}} \xrightarrow{e^*} \mathrm{Vect}^f_{\mathrm{E}}. \tag{6.27}$$

In particular, $\mathrm{Rep}^{\mathrm{alg}}_{{}^{\mathrm{L}}\mathrm{H}_{\mathrm{S}}}$ (resp. $\mathrm{Lis}_{\mathrm{E}}(\mathrm{S})$) is recovered as the category of finite-dimensional algebraic representations of ${}^{\mathrm{L}}\mathrm{H}_{\mathrm{S}}^{\mathrm{alg}}$ (resp. $\pi_1^{\mathrm{alg}}(\mathrm{S},\bar{s})$); see Remark 6.3.7.

The E-points of (6.26) receive continuous maps from (6.18), exhibiting ${}^{\mathrm{L}}\mathrm{H}_{\mathrm{S}}$ as the pullback of ${}^{\mathrm{L}}\mathrm{H}_{\mathrm{S}}^{\mathrm{alg}}(\mathrm{E})$ along $\pi_1(\mathrm{S},\bar{s}) \to \pi_1^{\mathrm{alg}}(\mathrm{S},\bar{s})(\mathrm{E})$. By the universal property of $\pi_1^{\mathrm{alg}}(\mathrm{S},\bar{s})$, continuous sections of ${}^{\mathrm{L}}\mathrm{H}_{\mathrm{S}} \to \pi_1(\mathrm{S},\bar{s})$ are in bijection with algebraic sections of ${}^{\mathrm{L}}\mathrm{H}_{\mathrm{S}}^{\mathrm{alg}} \to \pi_1^{\mathrm{alg}}(\mathrm{S},\bar{s})$. Under Tannakian duality, the latter are in bijecion with faithful exact tensor functors:

$$\mathrm{T} : \mathrm{Rep}^{\mathrm{alg}}_{{}^{\mathrm{L}}\mathrm{H}_{\mathrm{S}}} \to \mathrm{Lis}_{\mathrm{E}}(\mathrm{S}) \tag{6.28}$$

equipped with a $\mathrm{Lis}_{\mathrm{E}}(\mathrm{S})$-linear rigidification along $\bar{s}$.

Given a faithful exact tensor functor (6.28), the scheme of $\mathrm{Lis}_{\mathrm{E}}(\mathrm{S})$-linear rigidifications of $\bar{s}^* \circ \mathrm{T}$ is a torsor under the group scheme of $\mathrm{Lis}_{\mathrm{E}}(\mathrm{S})$-linear automorphisms of the fiber functor (6.27). The latter is equivalent to the group scheme of automorphisms of the fiber functor $e^* : \mathrm{Rep}_{\mathrm{H}} \to \mathrm{Vect}^f_{\mathrm{E}}$, i.e. H. □

7. The De Rham context

The structure theory of étale metaplectic covers developed in sections §3–5 is closely related to "quantum parameters", or "levels" of affine Kac–Moody Lie algebras. Indeed, it is a classical observation that the level κ of an affine Kac–Moody Lie algebra $\hat{\mathfrak{g}}^\kappa$ over $\mathbb{C}$ has a natural intepretation as a class in $\mathrm{H}^4_{\mathrm{dR}}(\mathrm{BG},\mathbb{C})$. Naïvely, one would then take $\Gamma_{\mathrm{dR},e}(\mathrm{BG},\mathbb{C}[4])$ to be the space of quantum parameters.

When the base is a smooth, proper curve X, a good notion of quantum parameters is supposed to induce rings of twisted differential operators on the moduli stack of G-bundles on X. For this to happen, we need to replace $\Gamma_{\mathrm{dR},e}(\mathrm{BG},\mathbb{C}[4])$ by those de Rham cochains which "belong to the Hodge filtration $\mathrm{F}^{\geq 2}$."

In §7.1, we turn this idea into a precise definition and show that it recovers the usual notion of quantum parameters. In §7.2, we compare the space of quantum parameters with integral metaplectic covers as studied in §2.3.

7.1. Quantum parameters.

7.1.1. In this section, we work over a field k of characteristic zero and denote by $\mathrm{Sm}_{/k}$ the category of smooth k-schemes.

For each integer $p \geq 0$ and $\mathrm{S} \in \mathrm{Sm}_{/k}$, there is a complex of étale sheaves of k-vector spaces concentrated in cohomological degrees $\geq p$:

$$\Omega_{\mathrm{S}}^{\geq p} := [\Omega_{\mathrm{S}}^p \xrightarrow{d} \Omega_{\mathrm{S}}^{p+1} \xrightarrow{d} \cdots],$$

where each Ω_{S}^n denotes the sheaf of nth differential forms on S relative to k.

The association $\mathrm{S} \mapsto \Gamma(\mathrm{S},\Omega_{\mathrm{S}}^{\geq p})$ is an étale sheaf of k-module spectra on $\mathrm{Sm}_{/k}$, with functoriality defined by pulling back differential forms.

Let $\mathrm{S} \mapsto \mathrm{QCoh}(\mathrm{S})$ denote the functor assigning the stable ∞-category of quasi-coherent $\mathscr{O}_{\mathrm{S}}$-modules to an affine k-scheme S. It extends to the category of algebraic stacks over k, by the operation of right Kan extension.

7.1.2. Fix $S \in Sm_{/k}$. Let $G \to S$ be a reductive group scheme. Denote by $\mathfrak{g}$ the Lie algebra of G, viewed as a locally free sheaf of $\mathscr{O}_S$-modules equipped with a G-action.

The ∞-category QCoh(BG) is identified with the ∞-category of quasi-coherent $\mathscr{O}_S$-modules equipped with a G-action (i.e. a comodule structure over $\mathscr{O}_G$). Under this identification, the cotangent complex $L_{BG/S}$ corresponds to $\mathfrak{g}^*[-1]$, where $\mathfrak{g}^*$ denotes the $\mathscr{O}_S$-linear dual of $\mathfrak{g}$ equipped with the co-adjoint G-action.

7.1.3. Define:

$$\underline{\Gamma}(BG, \Omega^{\geq p}_{BG}) \coloneqq \lim_{[n]} \underline{\Gamma}(G^{\times n}, \Omega^{\geq p}_{G^{\times n}})$$

as an étale sheaf of k-module spectra on S.

We also write $\underline{\Gamma}_e(BG, \Omega^{\geq p}_{BG})$ for the rigidified version, i.e. the fiber of the morphism of complexes $e^* : \underline{\Gamma}(BG, \Omega^{\geq p}_{BG}) \to \Omega^{\geq p}_S$.

We define *quantum parameters* to be sections of $\tau^{\leq 0}\underline{\Gamma}_e(BG, \Omega^{\geq 2}_{BG}[4])$, i.e. rigidified section of $\Omega^{\geq 2}_{BG}[4]$ over BG. The name is justified in Remark 7.1.9 below.

7.1.4. Denote by $\Omega^{p,cl}_S \subset \Omega^p_S$ the subsheaf of closed p-forms. We shall encounter the following complex in degrees $[-1, 0]$:

$$[\Omega^1_S \xrightarrow{d} \Omega^{2,cl}_S]. \tag{7.1}$$

As a sheaf of Picard groupoids, sections of (7.1) consist of Ω^1_S-torsors whose induced $\Omega^{2,cl}_S$-torsor is equipped with a trivialization. Such objects are equivalent to rings of twisted differential operators on S, see [BB93, Lemma 2.1.6].

The following calculation is an analogue of Proposition 5.1.11.

Proposition 7.1.5. *There is a canonical triangle of complexes of sheaves of k-vector spaces:*

$$(\mathfrak{g}^*)^G \otimes [\Omega^1_S \xrightarrow{d} \Omega^{2,cl}_S] \to \tau^{\leq 0}\underline{\Gamma}_e(BG, \Omega^{\geq 2}_{BG}[4]) \to \mathrm{Sym}^2(\mathfrak{g}^*)^G. \tag{7.2}$$

Remark 7.1.6. The invariants $(\mathfrak{g}^*)^G$, $\mathrm{Sym}^2(\mathfrak{g}^*)^G$ are *a priori* derived, i.e. given as the image of the corresponding objects under $\pi_* : QCoh(BG) \to QCoh(S)$. They are in fact concentrated in cohomological degree 0.

To prove this assertion, we may perform an étale base change on S and assume that G is split. In particular, $G = G_0 \times S$ for a reductive group G_0 over k. Then $\mathfrak{g}^*$ (resp. $\mathrm{Sym}^2(\mathfrak{g}^*)$) is the pullback of the quasi-coherent sheaf $\mathfrak{g}_0^*$ (resp. $\mathrm{Sym}^2(\mathfrak{g}_0^*)$) over BG_0. So it remains to show that the direct image along $BG_0 \to \mathrm{Spec}(k)$ is of cohomological amplitude ≤ 0 and commutes with arbitrary base change.

Identifying $QCoh(BG_0)$ with the derived category of algebraic G_0-representations over k, the first statement follows from the fact that reductive groups over a field of characteristic zero are linearly reductive, and the second statement follows from the first by [HLP14, Corollary B.16].

In particular, writing $\mathfrak{g}_{ab}$ for the Lie algebra of the maximal quotient torus $G_{ab} \coloneqq G/G_{der}$, the natural map $\mathfrak{g} \to \mathfrak{g}_{ab}$ defines an isomorphism $\mathfrak{g}^*_{ab} \cong (\mathfrak{g}^*)^G$.

7.1.7. Using flat descent of cotangent complexes, we see that $\underline{\Gamma}(BG, \Omega^{\geq 0}_{BG})$ admits a filtration with associated graded pieces:

$$\mathrm{Gr}^p\underline{\Gamma}(BG, \Omega^{\geq 0}_{BG}) \cong \underline{\Gamma}(BG, (\wedge^p L_{BG})[-p]),$$

see [KP22, Corollary 1.1.6]. This is the "Hodge filtration" of BG, studied in detail in *op.cit.*.

The proof of Proposition 7.1.5 will follow from this filtration, combined with a calculation of the Hodge cohomology of BG given below.

Lemma 7.1.8. *There is a canonical triangle of complexes of sheaves of k-vector spaces:*

$$(\mathfrak{g}^*)^{\mathrm{G}} \otimes \Omega_{\mathrm{S}}^{p-1}[1] \to \tau^{\le 0}\underline{\Gamma}_e(\mathrm{BG}, (\wedge^p \mathrm{L}_{\mathrm{BG}})[2]) \to \mathrm{Sym}^2(\mathfrak{g}^*)^{\mathrm{G}} \otimes \Omega_{\mathrm{S}}^{p-2}. \tag{7.3}$$

Proof. Denote by $\pi : \mathrm{BG} \to \mathrm{S}$ the projection map. The canonical triangle of cotangent complexes, combined with the identification $\mathrm{L}_{\mathrm{BG/S}} \cong \mathfrak{g}^*[-1]$, yields a triangle in $\mathrm{QCoh}(\mathrm{BG})$:

$$\pi^*\Omega_{\mathrm{S}} \to \mathrm{L}_{\mathrm{BG}} \to \mathfrak{g}^*[-1]. \tag{7.4}$$

Therefore, $\wedge^p \mathrm{L}_{\mathrm{BG}}$ admits a filtration with associated graded pieces $\mathrm{Sym}^n(\mathfrak{g}^*)[-n] \otimes \pi^*\Omega_{\mathrm{S}}^{p-n}$ for $0 \le n \le p$. In particular, the filtrants for $n \ge 3$ do not contribute to the connective truncation $\tau^{\le 0}\underline{\Gamma}(\mathrm{BG}, (\wedge^p \mathrm{L}_{\mathrm{BG}})[2])$.

Next, we compute by the projection formula:

$$\begin{aligned} \pi_*(\mathrm{Sym}^n(\mathfrak{g}^*)[-n] \otimes \pi^*\Omega_{\mathrm{S}}^{p-n}) &\cong \pi_*(\mathrm{Sym}^n(\mathfrak{g}^*))[-n] \otimes \Omega_{\mathrm{S}}^{p-n} \\ &\cong \mathrm{Sym}^n(\mathfrak{g}^*)^{\mathrm{G}}[-n] \otimes \Omega_{\mathrm{S}}^{p-n} \end{aligned}$$

The filtrant for $n = 0$ defines the canonical map $\Omega_{\mathrm{S}}^p[2] \to \tau^{\le 0}\underline{\Gamma}(\mathrm{BG}, (\wedge^p \mathrm{L}_{\mathrm{BG}})[2])$, for which e^* is a splitting. Hence the complex of rigidified sections fits into a triangle (7.3), as desired. □

Proof of Proposition 7.1.5. The complex $\underline{\Gamma}_e(\mathrm{BG}, \Omega_{\mathrm{BG}}^{\ge 2}[4])$ admits a filtration whose associated graded pieces are $\underline{\Gamma}_e(\mathrm{BG}, (\wedge^p \mathrm{L}_{\mathrm{BG}})[4-p])$ for $p \ge 2$. It follows from Lemma 7.1.8 that the connective truncations:

$$\tau^{\le 0}\underline{\Gamma}_e(\mathrm{BG}, (\wedge^p \mathrm{L}_{\mathrm{BG}})[4-p]) = 0, \quad \text{for } p \ge 4.$$

Therefore, $\tau^{\le 0}\underline{\Gamma}_e(\mathrm{BG}, \Omega_{\mathrm{BG}}^{\ge 2}[4])$ is the connective fiber of a map of complexes, corresponding to the above filtration in degrees $p = 2, 3$:

$$\begin{array}{ccccc} & & \tau^{\le 0}\underline{\Gamma}_e(\mathrm{BG}, \Omega_{\mathrm{BG}}^{\ge 2}[4]) & & \\ & & \downarrow & & \\ (\mathfrak{g}^*)^{\mathrm{G}} \otimes \Omega_{\mathrm{S}}^1[1] & \to & \tau^{\le 0}\underline{\Gamma}_e(\mathrm{BG}, (\wedge^2 \mathrm{L}_{\mathrm{BG}})[2]) & \to & \mathrm{Sym}^2(\mathfrak{g}^*)^{\mathrm{G}} \otimes \mathscr{O}_{\mathrm{S}} \\ \downarrow d & & \downarrow & & \downarrow d \\ (\mathfrak{g}^*)^{\mathrm{G}} \otimes \Omega_{\mathrm{S}}^2[1] & \to & \tau^{\le 0}\underline{\Gamma}_e(\mathrm{BG}, (\wedge^3 \mathrm{L}_{\mathrm{BG}})[2]) & \to & \mathrm{Sym}^2(\mathfrak{g}^*)^{\mathrm{G}} \otimes \Omega_{\mathrm{S}} \end{array}$$

Here, the rows are split triangles from Lemma 7.1.8 and the outer vertical arrows are induced by the differentials on Ω_{S}^n ($n = 0, 1$), as follows from the description of the Hodge filtration on a smooth scheme. The desired split triangle thus follows. □

Remark 7.1.9. If G is defined over k, then the triangles (7.2) and (7.3) are canonically split. Indeed, the hypothesis implies that (7.4) is canonically split, from which we deduce the splittings of the other triangles by following their constructions.

If $\mathrm{S} = \mathrm{X}$ is an algebraic curve, then sections of $\tau^{\le 0}\underline{\Gamma}_e(\mathrm{BG}, \Omega_{\mathrm{BG}}^{\ge 2}[4])$ are given by a pair (κ, E) where κ is a G-invariant symmetric form on $\mathfrak{g}$ and E is an extension of quasi-coherent $\mathscr{O}_{\mathrm{X}}$-modules:

$$0 \to \Omega_{\mathrm{X}} \to \mathrm{E} \to \mathfrak{g}_{\mathrm{ab}} \otimes \mathscr{O}_{\mathrm{X}} \to 0.$$

Such pairs are indeed the "quantum parameters" which appear in the geometric Langlands program, see [Zha23].

7.2. Comparison with integral metaplectic covers.

7.2.1. We continue to work over a field of characteristic zero. Recall the complex $\mathbb{Z}(n)$ of étale sheaves of $\mathbb{Z}$-modules from §2.3.3. Write $\mathbb{Q}(n) := \mathbb{Z}(n) \otimes \mathbb{Q}$. There is a canonical $\mathbb{Q}$-linear map:

$$\mathbb{Q}(n)[n] \to \Omega^n_{\mathrm{S}} \tag{7.5}$$

natural in $\mathrm{S} \in \mathrm{Sm}_{/k}$.

To define (7.5), we may assume $k = \bar{k}$ by étale descent. Viewing $\Omega^n : \mathrm{S} \mapsto \mathrm{H}^0(\mathrm{S}, \Omega^n_{\mathrm{S}})$ as an étale sheaf of k-vector spaces on $\mathrm{Sm}_{/k}$, it has a subsheaf $\Omega^n_{\log}$ of differential forms of logarithmic singularity along D, for any smooth compactification $\mathrm{S} \subset \bar{\mathrm{S}}$ with a normal crossing boundary divisor D.

The degeneration of the Hodge-de Rham spectral sequence for logarithmic forms [Del71, Corollaire 3.2.13(ii)] gives a natural isomorphism:

$$\mathrm{H}^0(\mathrm{S}, \Omega^n_{\log}) \cong \mathrm{F}^{\geq n}\mathrm{H}^n_{\mathrm{dR}}(\mathrm{S}, k),$$

where $\mathrm{F}^{\geq n}$ denotes the Hodge filtration. In particular, $\Omega^n_{\log}$ is an $\mathbb{A}^1$-invariant étale sheaf of k-vector spaces on $\mathrm{Sm}_{/k}$.

The morphism:

$$\mathbb{G}_m^{\times n} \to \Omega^n_{\log}, \quad f_1, \cdots, f_n \mapsto d\log(f_1) \wedge \cdots d\log(f_n)$$

induces a morphism $\mathbb{Z}(n)[n] \to \Omega^n_{\log}$ of complexes using the $\mathbb{A}^1$-invariance of the target, as in §2.3.5. Embedding the target in Ω^n and using its $\mathbb{Q}$-linear structure, we obtain the morphism (7.5).

By construction, the composition of (7.5) with $d : \Omega^n_{\mathrm{S}} \to \Omega^{n+1}_{\mathrm{S}}$ vanishes. Since $\mathbb{Q}(n)$ is concentrated in cohomological degrees $\leq n$, it lifts to a morphism:

$$\mathbb{Q}(n) \to \Omega^{\geq n}_{\mathrm{S}}. \tag{7.6}$$

7.2.2. The morphism (7.6) allows us to relate integral metaplectic covers to de Rham ones. Namely, it induces a functor:

$$\begin{aligned} \tau^{\leq 0}\underline{\Gamma}_e(\mathrm{BG}, \mathbb{Z}(2)[4]) \otimes \mathbb{Q} &\cong \tau^{\leq 0}\underline{\Gamma}_e(\mathrm{BG}, \mathbb{Q}(2)[4]) \\ &\to \tau^{\leq 0}\underline{\Gamma}_e(\mathrm{BG}, \Omega^{\geq 2}_{\mathrm{BG}}[4]). \end{aligned} \tag{7.7}$$

7.2.3. Let us suppose that G contains a maximal torus T with sheaf of cocharacters Λ. Theorem 2.3.8, together with [BD01, Theorem 7.2], shows that $\tau^{\leq 0}\underline{\Gamma}_e(\mathrm{BG}, \mathbb{Z}(2)[4])$ fits into a canonical triangle:

$$\underline{\mathrm{Hom}}(\pi_1(\mathrm{G}), \mathbb{Z}(1)[2]) \to \tau^{\leq 0}\underline{\Gamma}_e(\mathrm{BG}, \mathbb{Z}(2)[4]) \to \mathrm{Sym}^2(\check{\Lambda})^{\mathrm{W}}, \tag{7.8}$$

where W denotes the Weyl group $\mathrm{N}_{\mathrm{G}}(\mathrm{T})/\mathrm{T}$.

The morphism (7.7) then induces a morphism of triangles:

$$\begin{array}{ccccc} \underline{\mathrm{Hom}}(\pi_1(\mathrm{G}), \mathbb{Q}(1)[2]) & \to & \tau^{\leq 0}\underline{\Gamma}_e(\mathrm{BG}, \mathbb{Q}(2)[4]) & \to & \mathrm{Sym}^2(\check{\Lambda})^{\mathrm{W}} \otimes \mathbb{Q} \\ \downarrow{\scriptstyle (1)} & & \downarrow & & \downarrow{\scriptstyle (2)} \\ (\mathfrak{g}^*)^{\mathrm{G}} \otimes [\Omega^1_{\mathrm{S}} \xrightarrow{d} \Omega^{2,\mathrm{cl}}_{\mathrm{S}}] & \longrightarrow & \tau^{\leq 0}\underline{\Gamma}_e(\mathrm{BG}, \Omega^{\geq 2}_{\mathrm{BG}}[4]) & \longrightarrow & \mathrm{Sym}^2(\mathfrak{g}^*)^{\mathrm{G}} \end{array} \tag{7.9}$$

where the bottom triangle comes from Proposition 7.1.5.

Remark 7.2.4. The two outer morphisms of (7.9) have the following descriptions:

(1) Writing $\pi_1(\mathrm{G}) = \Lambda/\Lambda_{\mathrm{sc}}$ for Λ_{sc} the sheaf of cocharacters of the induced maximal torus of G_{sc}, we have:

$$\begin{aligned}\underline{\mathrm{Hom}}(\pi_1(\mathrm{G}), \mathbb{Q}(1)[2]) &\cong [\check{\Lambda} \otimes \mathbb{Q} \to \check{\Lambda}_{\mathrm{sc}} \otimes \mathbb{Q}] \otimes \mathbb{G}_m[1] \\ &\cong [\check{\Lambda}_{\mathrm{ab}} \otimes \mathbb{Q}] \otimes \mathbb{G}_m[1],\end{aligned}$$

where Λ_{ab} denotes the sheaf of cocharacters of G_{ab}.

This complex admits a natural map to $(\mathfrak{g}^*)^{\mathrm{G}} \otimes [\Omega^1_{\mathrm{S}} \xrightarrow{d} \Omega^{2,\mathrm{cl}}_{\mathrm{S}}]$, induced from the maps $\check{\Lambda}_{\mathrm{ab}} \otimes \mathbb{Q} \to \mathfrak{g}^*_{\mathrm{ab}} \cong (\mathfrak{g}^*)^{\mathrm{G}}$ and $d\log : \mathbb{G}_m[1] \to [\Omega^1_{\mathrm{S}} \xrightarrow{d} \Omega^{2,\mathrm{cl}}_{\mathrm{S}}]$.

(2) Writing $\mathfrak{t}$ for the Lie algebra of T, we have:

$$\mathrm{Sym}^2(\check{\Lambda})^{\mathrm{W}} \otimes \mathbb{Q} \to \mathrm{Sym}^2(\mathfrak{t}^*)^{\mathrm{W}} \cong \mathrm{Sym}^2(\mathfrak{g}^*)^{\mathrm{G}},$$

where the second isomorphism comes from Chevalley's theorem.

Remark 7.2.5. The following diagram summarizes the relationship among cochains on BG in motivic, étale, and de Rham cohomological contexts:

$$\begin{array}{ccc}
\tau^{\le 0}\underline{\Gamma}_e(\mathrm{BG}, \mathbb{Z}(2)[4]) & & \\
\downarrow & & \\
\tau^{\le 0}\underline{\Gamma}_e(\mathrm{BG}, \mathbb{Q}(2)[4]) & \longrightarrow & \tau^{\le 0}\underline{\Gamma}_e(\mathrm{BG}, \Omega^{\ge 2}_{\mathrm{BG}}[4]) \\
\downarrow & & \\
\tau^{\le 0}\underline{\Gamma}_e(\mathrm{BG}, \mathbb{Q}/\mathbb{Z}(2)[4]) \cong \mathrm{colim}_{\mathrm{N}}\, \underline{\Gamma}_e(\mathrm{BG}, \mathrm{B}^4\mu_{\mathrm{N}}^{\otimes 2}) & &
\end{array}$$

Here, the bottom isomorphism appeals to [MVW06, Theorem 10.3].

If the ground field k is algebraically closed and coincides with the coefficient field, there is a canonical character $\mathbb{Q}/\mathbb{Z}(1) \to k^\times$ defined by the inclusion $\mu(k) \subset k^\times$. Hence, any object of the bottom groupoid defines a metaplectic L-group as in §6. Furthermore, any object of $\tau^{\le 0}\underline{\Gamma}_e(\mathrm{BG}, \mathbb{Q}(2)[4])$ defines a pair of quantum parameters for G and its Langlands dual group $\check{\mathrm{G}}$, by [Zha23]. In this context, one may formulate compatibility statements between the quantum and the metaplectic Langlands programs. They are, however, beyond the scope of the present article.

Appendix A. Twisting construction

Given a sheaf of groups A, an A-torsor P, and a sheaf of sets X equipped with an A-action, we obtain another sheaf of sets $\mathrm{X}_{\mathrm{P}} := \mathrm{P} \times^{\mathrm{A}} \mathrm{X}$: the twist of X by P. When A is abelian, two A-torsors P_1, P_2 define a third one $\mathrm{P}_1 \otimes \mathrm{P}_2$. When X has a monoid structure, we obtain a multiplication map:

$$\mathrm{X}_{\mathrm{P}_1} \times \mathrm{X}_{\mathrm{P}_2} \to \mathrm{X}_{\mathrm{P}_1 \otimes \mathrm{P}_2}, \tag{A.1}$$

if the equality $(a_1 a_2) \cdot (x_1 x_2) = (a_1 \cdot x_1)(a_2 \cdot x_2)$ holds for all $a_1, a_2 \in \mathrm{A}$ and $x_1, x_2 \in \mathrm{X}$.

Let us involve another piece of structure: X is now a sheaf of E-algebras equipped with a grading $\mathrm{X} = \bigoplus_{\lambda \in \Gamma} \mathrm{X}_\lambda$ by some abelian group Γ, such that $1 \in \mathrm{X}_0$ and $x_1 x_2$ has grading $\lambda_1 + \lambda_2$ if x_1, x_2 have gradings λ_1, λ_2. Then any multiplicative A-torsor P on Γ defines a new sheaf of E-algebras $\mathrm{X}_{\mathrm{P}} := \bigoplus_{\lambda \in \Gamma} (\mathrm{X}_\lambda)_{\mathrm{P}(\lambda)}$ with multiplicative rule (A.1). Here, $\mathrm{P}(\lambda)$ denotes the A-torsor $\mathrm{P} \times_\Gamma \{\lambda\}$.

The goal of this section is to explain an analogous construction where X is replaced by a Γ-graded stack of tensor categories $\mathscr{C}$. We explain the meaning of an A-action on $\mathrm{id}_{\mathscr{C}}$ and how to form a twisted stack of tensor categories $\mathscr{C}_\nu$ for an $\mathbb{E}_\infty$-functor $\nu : \Gamma \to \mathrm{B}^{(2)}(\mathrm{A})$.

Remark A.0.1. There are analogous constructions when $\mathscr{C}$ is monoidal (resp. braided monoidal) and $\nu : \Gamma \to \mathrm{B}^{(2)}(\mathrm{A})$ is $\mathbb{E}_1$-monoidal (resp. $\mathbb{E}_2$-monoidal). The result $\mathscr{C}_\nu$ is then a sheaf of monoidal (resp. braided monoidal) categories.

A.1. Actions.

A.1.1. Suppose that A is an abelian group. Let $\mathscr{C}$ be a category. We say that A *acts on* $\mathrm{id}_{\mathscr{C}}$ if there is a group homomorphism $\mathrm{A} \to \mathrm{Aut}(\mathrm{id}_{\mathscr{C}})$. Here, $\mathrm{id}_{\mathscr{C}}$ denotes the identity functor viewed as an object of the category of endofunctors of $\mathscr{C}$. Concretely, an A-action on $\mathrm{id}_{\mathscr{C}}$ means that to each $a \in \mathrm{A}$ and $c \in \mathscr{C}$, there is an isomorphism $a_c : c \cong c$. They satisfy:

(1) 1_c is the identity for all $c \in \mathscr{C}$;
(2) $(a_1 a_2)_c = (a_1)_c \circ (a_2)_c$ for all $a_1, a_2 \in \mathrm{A}$ and $c \in \mathscr{C}$;
(3) $a_{c_2} \circ f = f \circ a_{c_1}$ for all $a \in \mathrm{A}$ and $f : c_1 \to c_2$ in $\mathscr{C}$.

Suppose that $\mathscr{C}$ is a symmetric monoidal category. We say that an A-action on $\mathrm{id}_{\mathscr{C}}$ is *multiplicative* if the following condition is satisfied:

(4) $(a_1)_{c_1} \otimes (a_2)_{c_2} = (a_1 a_2)_{c_1 \otimes c_2}$ for all $a_1, a_2 \in \mathrm{A}$ and $c_1, c_2 \in \mathscr{C}$.

A.1.2. Denote by $\mathrm{B}(\mathrm{A})$ the groupoid with a single object $*$ and $\mathrm{Aut}(*) = \mathrm{A}$ (i.e. the Bar construction in Spc). It has a symmetric monoidal structure defined by the group structure of A. The notion of an A-action on $\mathrm{id}_{\mathscr{C}}$ is really a description of a $\mathrm{B}(\mathrm{A})$-action on $\mathscr{C}$, in the following sense: it defines a groupoid object $[n] \mapsto \mathscr{C}^{[n]}$ in the 2-category of categories covering the groupoid object $[n] \mapsto \mathrm{B}(\mathrm{A})^{\times [n]} := \mathrm{B}(\mathrm{A})^{\times n}$, together with an isomorphism $\mathscr{C}^{[0]} \cong \mathscr{C}$, such that the following diagram is Cartesian for both $i = 0$, 1:

$$\begin{array}{ccc} \mathscr{C}^{[1]} & \xrightarrow{\delta^i} & \mathscr{C}^{[0]} \\ \downarrow & & \downarrow \\ \mathrm{B}(\mathrm{A}) & \longrightarrow & * \end{array}$$

The groupoid object $[n] \mapsto \mathscr{C}^{[n]}$ is explicitly constructed by $[n] \mapsto \mathrm{B}(\mathrm{A})^{\times n} \times \mathscr{C}$. One of the boundary maps, say δ^0, passes to projection onto $\mathscr{C}$. The other one δ^1 is given by:

$$\mathrm{act} : \mathrm{B}(\mathrm{A}) \times \mathscr{C} \to \mathscr{C}, \quad (a, f) \mapsto af := a_{c_2} \circ f = f \circ a_{c_1}. \tag{A.2}$$

(The formula in (A.2) describes what act does to morphisms $a \in \mathrm{A}$ and $f : c_1 \to c_2$.) The higher boundary maps are compositions of actions and projections. The degeneracy maps are insertions along $* \in \mathrm{B}(\mathrm{A})$. Conditions (1) and (2) of §A.1.1 ensure that the simplicial object is well defined. Taking geometric realization of the morphism $\mathscr{C}^{[n]} \to \mathrm{B}(\mathrm{A})^{[n]}$, we obtain a functor of 2-categories (see §1.2.5):

$$\mathscr{C}^{[-1]} \to \mathrm{B}^{(2)}(\mathrm{A}). \tag{A.3}$$

A.1.3. If $\mathscr{C}$ is a symmetric monoidal category and the A-action on $\mathscr{C}$ is multiplicative, then (A.2) is itself a functor of symmetric monoidal categories. The isomorphism between

$\mathrm{act}(*, c_1) \otimes \mathrm{act}(*, c_2)$ and $\mathrm{act}(*, c_1 \otimes c_2)$ is the obvious one. However, it demands a commutative diagram:

$$\begin{array}{ccc} c_1 \otimes d_1 & \xrightarrow{\cong} & c_1 \otimes d_1 \\ \Big\downarrow {\scriptstyle (a_1 f)\otimes(a_2 g)} & & \Big\downarrow {\scriptstyle (a_1 a_2)\otimes(fg)} \\ c_2 \otimes d_2 & \xrightarrow{\cong} & c_2 \otimes d_2 \end{array}$$

This follows from conditions (3) and (4) of §A.1.1. It follows that $[n] \to \mathscr{C}^{[n]}$ is a simplicial object in the 2-category of symmetric monoidal categories. Furthermore, the morphism $\mathscr{C}^{[n]} \to \mathrm{B}(\mathrm{A})^{[n]}$ is a morphism of such. It follows that (A.3) lifts to a functor of $\mathbb{E}_\infty$-monoidal 2-categories. (We have used the fact that forgetting the $\mathbb{E}_\infty$-structure commutes with sifted colimits, see the proof of Lemma 1.1.6.)

A.1.4. The above constructions carry sheaf-theoretic meaning. Fix a site $\mathscr{X}$ and let A be a sheaf of abelian groups, $\mathscr{C}$ be a stack of categories. Then an A-action on $\mathrm{id}_{\mathscr{C}}$ defines a morphism of simplicial stacks of categories $\mathscr{C}^{[n]} \to \mathrm{B}(\mathrm{A})^{[n]}$. Here, B(A) denotes the Bar construction of A in the ∞-category of Spc-valued sheaves. By taking the geometric realization, we obtain a morphism of sheaves of 2-categories:

$$\mathscr{C}^{[-1]} \to \mathrm{B}^{(2)}(\mathrm{A}). \tag{A.4}$$

If $\mathscr{C}$ carries a symmetric monoidal structure and the A-action is multiplicative, then (A.4) lifts to a morphism of sheaves of $\mathbb{E}_\infty$-monoidal 2-categories.

A.2. How $(\mathscr{C}, \nu)$ defines $\mathscr{C}_\nu$.

A.2.1. We continue to fix a site $\mathscr{X}$ and let A be a sheaf of abelian groups, $\mathscr{C}$ be a stack of categories. Suppose that A acts on $\mathrm{id}_{\mathscr{C}}$. Given any section $\mathscr{G}$ of $\mathrm{B}^{(2)}(\mathrm{A})$ over x, the fiber product of (A.4) with $\mathscr{G} : x \to \mathrm{B}^{(2)}(\mathrm{A})$ defines a stack of categories over x. We denote it by $\mathscr{C}_{\mathscr{G}}$ and view it as the "$\mathscr{G}$-twist of $\mathscr{C}$."

Note that for any $x_1 \to x$ such that the pullback $\mathscr{G}_{x_1}$ of $\mathscr{G}$ is trivialized, i.e., factors as $x_1 \to * \to \mathrm{B}^{(2)}(\mathrm{A})$, the pullback of $\mathscr{C}_{\mathscr{G}}$ to x_1 is isomorphic to the pullback of $\mathscr{C}$.

A.2.2. If E is a ring and $\mathscr{C}$ is an E-linear category, the same structure is inherited by $\mathscr{C}_{\mathscr{G}}$. Indeed, we may let $\mathscr{S} \subset \mathrm{Hom}(-, x)$ be the covering sieve consisting of morphisms $x_1 \to x$ such that $\mathscr{G}_{x_1}$ is trivial. Then we have:

$$\mathscr{C}_{\mathscr{G}}(x) \cong \lim_{(x_1 \to x) \in \mathscr{S}} \mathscr{C}_{\mathscr{G}}(x_1).$$

On the right-hand-side, $\mathscr{C}_{\mathscr{G}}(x_1)$ has an E-linear structure by choosing *any* trivialization of $\mathscr{G}_{x_1}$ and transport the E-linear structure from $\mathscr{C}(x_1)$. Two distinct trivializations of $\mathscr{G}_{x_1}$ differ by a section of B(A), which locally defines an automorphism of $\mathscr{C}(x_1)$ by the action map (A.2). Since this action map is E-linear on the Hom-sets, the category $\mathscr{C}_{\mathscr{G}}(x_1)$ acquires an E-linear structure independently of the trivialization of $\mathscr{G}_{x_1}$. The same structure then passes to $\mathscr{C}_{\mathscr{G}}(x)$.

A.2.3. Let us now suppose that $\mathscr{C}$ is a stack of symmetric monoidal E-linear additive categories, together with a decomposition $\mathscr{C} = \bigoplus_{\lambda \in \Gamma} \mathscr{C}_\lambda$ for an abelian group Γ such that:

1. (1) $\mathbf{1}_{\mathscr{C}} \in \mathscr{C}_0$;
2. (2) $c_1 \otimes c_2 \in \mathscr{C}_{\lambda_1 + \lambda_2}$ if $c_1 \in \mathscr{C}_{\lambda_1}$ and $c_2 \in \mathscr{C}_{\lambda_2}$.

These data may be packaged differently: let $\mathscr{C}^{\sqcup}$ denote the stack of categories over Γ whose fiber at $\lambda \in \Gamma$ is $\mathscr{C}_\lambda$. Then $\mathscr{C}^{\sqcup}$ admits a symmetric monoidal structure such that $\mathscr{C}^{\sqcup} \to \Gamma$ is a symmetric monoidal functor. (Here, Γ is viewed as a discrete category whose symmetric monoidal structure comes from the group operations.)

Conversely, given a symmetric monoidal functor $\mathscr{D} \to \Gamma$ where $\mathscr{D}$ is a stack of symmetric monoidal E-linear categories whose *fibers* over Γ are additive, we obtain a stack of symmetric monoidal E-linear additive categories $\mathscr{D}^{\oplus} := \bigoplus_{\lambda\in\Gamma} \mathscr{D}_\lambda$ by taking direct sum of the fibers.

A.2.4. Let $\mathscr{C}$ be as in §A.2.3. Suppose that A acts multiplicatively on $\mathrm{id}_{\mathscr{C}}$. Then it induces a multiplicative action on $\mathrm{id}_{\mathscr{C}^{\sqcup}}$. It also acts trivially on id_Γ and the functor $\mathscr{C}^{\sqcup} \to \Gamma$ is tautologically compatible with the actions. The construction of (A.4) is functorial in $\mathscr{C}$. In particular, the symmetric monoidal functor $\mathscr{C}^{\sqcup} \to \Gamma$ yields an $\mathbb{E}_\infty$-monoidal functor:

$$\mathscr{C}^{\sqcup,[-1]} \to \Gamma \times \mathrm{B}^{(2)}(\mathrm{A}). \tag{A.5}$$

Suppose that $\nu : \Gamma \to \mathrm{B}^{(2)}(\mathrm{A})$ is an $\mathbb{E}_\infty$-monoidal morphism. Taking fiber product of (A.5) with $(\mathrm{id}_\Gamma, \nu)$ yields an $\mathbb{E}_\infty$-monoidal functor $\mathscr{C}^{\sqcup}_\nu \to \Gamma$.

Finally, we apply the construction of §A.2.3 to obtain $\mathscr{C}_\nu := (\mathscr{C}^{\sqcup}_\nu)^{\oplus}$, which is a stack of symmetric monoidal E-linear additive categories equipped with a compatible Γ-grading. This is the "ν-twist" of $\mathscr{C}$.

Remark A.2.5. Let us give an informal account of $\mathscr{C}_\nu$. It admits a Γ-grading:

$$\mathscr{C}_\nu \cong \bigoplus_{\lambda\in\Gamma} (\mathscr{C}_\lambda)_{\nu(\lambda)},$$

where $(\mathscr{C}_\lambda)_{\nu(\lambda)}$ is the $\nu(\lambda)$-twist of $\mathscr{C}_\lambda$ in the sense of §A.2.1. The monoidal operation on $\mathscr{C}_\nu$ is given as follows: for $\lambda_1, \lambda_2 \in \Gamma$, we have $(\mathscr{C}_{\lambda_1})_{\nu(\lambda_1)} \times (\mathscr{C}_{\lambda_2})_{\nu(\lambda_2)} \to (\mathscr{C}_{\lambda_1+\lambda_2})_{\nu(\lambda_1+\lambda_2)}$ coming from the monoidal operation on $\mathscr{C}$ and the monoidal structure on ν. The symmetric monoidal structure is likewise induced from those of $\mathscr{C}$ and ν.

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