SPECTRAL DECOMPOSITION OF GENUINE CUSP FORMS OVER GLOBAL FUNCTION FIELDS

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Abstract. We prove a version of the twisted geometric Satake equivalence and extend the Langlands parametrization of V. Lafforgue to certain covers of reductive groups.

This is an updated version of the same-named article in *Compos. Math.* **160**, 1194–1260, containing several corrections and improvements.

Contents

Date: April 3, 2026.

Introduction

In the seminal work [Laf18], V. Lafforgue constructed the Langlands parametrization of cusp forms for reductive groups over global function fields.

The goal of this article is to extend this parametrization to a large class of covers of reductive groups, following the strategy indicated in [Laf18, §14]. The main new ingredient is a version of the twisted geometric Satake equivalence, extending earlier works of Finkelberg, Lysenko, Reich, and Gaitsgory (*cf.* [FL10, Rei12, GL18]).

The class of covers we treat includes the ones considered by Brylinski and Deligne in [BD01], so our result contributes to the Langlands–Weissman program for Brylinski–Deligne covers, as formulated by Weissman, Gan, and Gao (*cf.* [Wei18, GG18]). We refer the reader to [GGW18] for a survey of this program.

0.1. Spectral decomposition.

0.1.1. Let F be a global field of positive characteristic p. Denote by $\mathbb{A}_{\mathrm{F}}$ the topological ring of adèles of F. Let $\ell \neq p$ be a prime and fix an algebraic closure $\overline{\mathbf{Q}}_\ell$ of $\mathbf{Q}_\ell$, with a chosen half-integer Tate twist $\overline{\mathbf{Q}}_\ell(\frac{1}{2})$.

Our group-theoretic input is a pair (G, μ), where G is a split reductive group over F and μ is a "parameter" for covers of (the adèlic points of) G.

The precise meaning of μ will be explained in §0.1.5. For now, let us take for granted that μ gives rise to a topological central extension

$$1 \to \mathrm{A} \to \widetilde{\mathrm{G}} \to \mathrm{G}(\mathbb{A}_{\mathrm{F}}) \to 1, \tag{0.1}$$

for some finite abelian group A, equipped with a canonical splitting over $\mathrm{G}(\mathrm{F})$.

Fix an injective character $\zeta : \mathrm{A} \hookrightarrow \overline{\mathbf{Q}}_\ell^\times$. The Langlands–Weissman program, in its global function field incarnation, studies the $\overline{\mathbf{Q}}_\ell$-vector space of ζ*-genuine automorphic forms* on $\widetilde{\mathrm{G}}$, *i.e.* locally constant functions $f : \mathrm{G}(\mathrm{F})\backslash\widetilde{\mathrm{G}} \to \overline{\mathbf{Q}}_\ell$ which are A-equivariant against ζ.

0.1.2. As in the case for reductive groups, ζ-genuine automorphic forms admit a notion of cuspidality, defined by the vanishing of constant terms.

Furthermore, one may constrain the action of the connected part Z° of the center of G, by demanding our ζ-genuine automorphic forms to be invariant under the action of a cocompact lattice Ξ in $\mathrm{Z}^\sharp(\mathrm{F})\backslash\widetilde{\mathrm{Z}}^\sharp$, for a torus $\mathrm{Z}^\sharp$ isogenous to Z° with induced cover $\widetilde{\mathrm{Z}}^\sharp$. The torus $\mathrm{Z}^\sharp$ appears because the induced cover of Z° may *not* be abelian, while $\widetilde{\mathrm{Z}}^\sharp$ is abelian and its image in $\widetilde{\mathrm{G}}$ is central by construction.

Imposing these two conditions leads us to the $\overline{\mathbf{Q}}_\ell$-vector space

$$\mathrm{Fun}_{\mathrm{cusp},\zeta}(\mathrm{G}(\mathrm{F})\backslash\widetilde{\mathrm{G}}/\Xi, \overline{\mathbf{Q}}_\ell). \tag{0.2}$$

The main goal of this article is to decompose (0.2) according to spectral data.

0.1.3. Let us now describe the spectral data involved. For this, we fix an algebraic closure $\bar{\mathrm{F}}$ of F and denote by $\mathrm{Gal}_{\mathrm{F}}$ the Galois group of $\bar{\mathrm{F}}/\mathrm{F}$.

To the pair (G, μ), one may attach certain combinatorial data. The "classical" part of these data consists of the root data of G together with a certain quadratic form. Using them, one defines the "metaplectic dual group" H: This is the dual group found in [Lus93, FL10, McN12, GL18], for increasingly general kinds of (G, μ).

Contrary to the case of reductive groups, the combinatorial package associated to (G, μ) also includes certain 2-categorical data. Following Weissman's approach in [Wei18], we obtain from them an extension of topological groups

$$1 \to \mathrm{H}(\overline{\mathbf{Q}}_\ell) \to {}^{\mathrm{L}}\mathrm{H}_{\mathrm{F}} \to \mathrm{Gal}_{\mathrm{F}} \to 1, \tag{0.3}$$

which we shall refer to as the L-*group* of (G, μ). Then we proceed as in the case of reductive groups, defining an L-*parameter* to be an $\mathrm{H}(\overline{\mathbf{Q}}_\ell)$-conjugacy class of continuous sections $\sigma : \mathrm{Gal}_{\mathrm{F}} \to {}^{\mathrm{L}}\mathrm{H}_{\mathrm{F}}$ of the surjection in (0.3).

Let us mention that the L-group (0.3) is *not* the only way to package the dual data of (G, μ). In fact, in the main body of the text, we will consider a refinement of (0.3), which we call "metaplectic dual data" following Gaitsgory and Lysenko (*cf.* [GL18]). For the moment, however, let us use (0.3) to formulate our main result.

Theorem A. *There is a canonical decomposition*

$$\mathrm{Fun}_{\mathrm{cusp},\zeta}(\mathrm{G}(\mathrm{F})\backslash\widetilde{\mathrm{G}}/\Xi, \overline{\mathbf{Q}}_\ell) \simeq \bigoplus_{[\sigma]} \mathbf{H}_{[\sigma]}, \tag{0.4}$$

indexed by L-*parameters* $[\sigma]$.

0.1.4. Theorem A, as stated above, is nearly meaningless. A precise version of it appears as Theorem 8.1.6 in the main body of the text. It includes the compatibility with Hecke action expected of the Langlands parametrization, as well as other information on the L-parameters $[\sigma]$ appearing in the index set of (0.4).

The main body of the text also treats the case where G is not necessarily split. In that case, the left-hand-side of (0.4) must be enlarged to account for different forms of G.

Theorem A is an addendum to a long series of works on the "automorphic-to-Galois" direction of the Langlands program for global function fields, due to Drinfeld, L. Lafforgue, and V. Lafforgue (*cf.* [Dri77, Dri87a, Dri87b, Laf02, Laf18]). Contrary to these highly original works, the main ideas of the present text have already been laid out in [Laf18] and [GL18]. The only contribution from the author is a new treatment of the twisted geometric Satake equivalence, discussed in §0.3 below.

0.1.5 *The meaning of* μ**.** Let us now say what μ stands for. This is the only place where our formulation of the Langlands parametrization differs from the usual one found in the Langlands–Weissman program.

Namely, in the latter context, one parametrizes covers of $\mathrm{G}(\mathbb{A}_{\mathrm{F}})$ by pairs (E, N), where E is a Brylinski–Deligne cover of G and $\mathrm{N} \geq 1$ is an integer such that $\mu_{\mathrm{N}}(\mathrm{F})$ has cardinality N (*cf.* [Wei18]). However, this formalism leads to technical difficulties in the proof of the twisted geometric Satake equivalence.[1]

Fortunately, an alternative parametrization of covers found by Deligne (*cf.* [Del96]) allows us to circumvent these difficulties. Formulated in modern language, a parameter of [Del96] is a morphism of pointed (higher) étale stacks

$$\mathsf{B}_{\mathrm{F}}\mathrm{G} \to \mathsf{B}^4_{\mathrm{F}}\mathrm{A}(1), \tag{0.5}$$

where B_{F} denotes the deloop functor for étale stacks over F, and A(1) is the Tate twist of the finite abelian group A. In this text, we let μ be a morphism (0.5), which we shall refer to as an (A-valued) *étale level*.[2]

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[1] See §C.1 for a more detailed discussion.

[2] The terminology is borrowed from the quantum geometric Langlands program (which is in turn motivated by 2D conformal field theory), where an analogous notion in de Rham cohomology is refered to as a "level". In [Zha22], an étale level is called an (A-valued) étale metaplectic cover.

The idea of parametrizing covering groups by étale levels entered the Langlands program through Gaitsgory and Lysenko's work [GL18].

Remark 0.1.6. In [Del96, §2], étale levels are described by cocycles. One may wonder why we choose not to use this explicit description.

The reason is that in order to define metaplectic dual data, it is necessary to work with additional structures on étale levels, which are natural from the point of view of higher algebra but become quite cumbersome when written in terms of cocycles. For this reason, we systematically adopt the tools of higher algebra (*cf.* [Lur09, Lur17]) in this text.

0.2. Cohomology of Shtukas.

0.2.1. Let us sketch the proof of Theorem A, essentially following [Laf18, §14]. We begin by formulating an integral version of the problem.

Let X be a smooth, proper, geometrically connected curve over a finite field k, with field of fractions F. Let $\mathrm{D} \subset \mathrm{X}$ be a k-finite closed subscheme and $\mathring{\mathrm{X}}$ be its complement. We shall assume that our étale level comes from a morphism of pointed étale stacks over $\mathring{\mathrm{X}}$:

$$\mathsf{B}_{\mathring{\mathrm{X}}}\mathrm{G} \to \mathsf{B}^4_{\mathring{\mathrm{X}}}\mathrm{A}(1).$$

where $\mathsf{B}_{\mathring{\mathrm{X}}}$ stands for the deloop functor for étale stacks over $\mathring{\mathrm{X}}$.

Denote by $\mathrm{Bun}_{\mathrm{G},\mathrm{D}}$ the moduli stack of G-bundles over X trivialized along D. The étale level μ determines an (étale) A-gerbe $\mathcal{G}_{\mathrm{Bun}_{\mathrm{G},\mathrm{D}}}$ over $\mathrm{Bun}_{\mathrm{G},\mathrm{D}}$, which "categorifies" (0.1) in the following sense: A procedure akin to taking the trace of Frobenius yields an A-torsor

$$\widetilde{\mathrm{Bun}}_{\mathrm{G},\mathrm{D}} \to \mathrm{Bun}_{\mathrm{G},\mathrm{D}}(\mathsf{k}), \tag{0.6}$$

whose pullback along the uniformization map $\mathrm{G}(\mathbb{A}_{\mathrm{F}}) \to \mathrm{Bun}_{\mathrm{G},\mathrm{D}}(\mathsf{k})$ recovers $\widetilde{\mathrm{G}}$.

Instead of ζ-genuine functions on $\mathrm{G}(\mathrm{F})\backslash\widetilde{\mathrm{G}}$, we shall study those defined on $\widetilde{\mathrm{Bun}}_{\mathrm{G},\mathrm{D}}$.

0.2.2. Let us introduce, somewhat informally, two more geometric objects: the local Hecke stack and the moduli stack of Shtukas of Drinfeld and Varshavsky (*cf.* [Dri87b, Var04]). The formal definitions are provided in §1.1.3 and §8.2.2.

For a finite set I, the local Hecke stack $\mathrm{Hec}_{\mathrm{G},\mathrm{I}}$ parametrizes a point x^{I} of $\mathring{\mathrm{X}}^{\mathrm{I}}$, together with a modification at x^{I} of G-bundles over the formal disk $\mathrm{D}_{x^{\mathrm{I}}}$, represented as

$$\mathrm{P}^0 \overset{x^{\mathrm{I}}}{\sim} \mathrm{P}^1.$$

The moduli stack of Shtukas $\mathrm{Sht}^{\mathrm{I}}_{\mathrm{G},\mathrm{D}}$ parametrizes a point x^{I} of $\mathring{\mathrm{X}}^{\mathrm{I}}$, together with a modification at x^{I} of G-bundles over X equipped with trivializations along D:

$$(\mathrm{P}^0, \phi^0) \overset{x^{\mathrm{I}}}{\sim} (\mathrm{P}^1, \phi^1), \tag{0.7}$$

as well as an isomorphism $(\mathrm{P}^1, \phi^1) \simeq ({}^\tau\mathrm{P}^0, {}^\tau\phi^0)$, where ${}^\tau$ denotes the Frobenius twist.

Restricting (0.7) to $\mathrm{D}_{x^{\mathrm{I}}}$ defines a morphism of stacks over $\mathring{\mathrm{X}}^{\mathrm{I}}$:

$$r : \mathrm{Sht}^{\mathrm{I}}_{\mathrm{G},\mathrm{D}} \to \mathrm{Hec}_{\mathrm{G},\mathrm{I}}. \tag{0.8}$$

0.2.3. In addition to $\mathcal{G}_{\mathrm{Bun}_{\mathrm{G},\mathrm{D}}}$, the étale level μ defines an A-gerbe $\mathcal{G}_{\mathrm{Hec}_{\mathrm{G},\mathrm{I}}}$ over $\mathrm{Hec}_{\mathrm{G},\mathrm{I}}$. The key observation is that the pullback of $\mathcal{G}_{\mathrm{Hec}_{\mathrm{G},\mathrm{I}}}$ along (0.8) is *canonically* trivial.

This observation has the following practical consequence: Given a "$(\mathcal{G}_{\mathrm{Hec}_{\mathrm{G},\mathrm{I}}}, \zeta)$-twisted constructible complex" $\mathcal{A}$ of $\overline{\mathbf{Q}}_\ell$-sheaves over $\mathrm{Hec}_{\mathrm{G},\mathrm{I}}$, a notion we define in Appendix B, its pullback $r^*\mathcal{A}$ is *untwisted*, *i.e.* a usual constructible complex of $\overline{\mathbf{Q}}_\ell$-sheaves. In particular, one may take the (compactly supported) cohomology of $r^*\mathcal{A}$ along the structural morphism $\mathrm{Sht}^{\mathrm{I}}_{\mathrm{G,D}} \to \mathring{\mathrm{X}}^{\mathrm{I}}$ to obtain "cohomology of Shtukas with coefficients in $\mathcal{A}$."[3] For the kind of objects $\mathcal{A}$ we consider, $r^*\mathcal{A}$ will furthermore be perverse relative to $\mathring{\mathrm{X}}^{\mathrm{I}}$.

In the special case $\mathrm{I} = \varnothing$, the moduli stack $\mathrm{Sht}^{\varnothing}_{\mathrm{G,D}}$ coincides with the discrete stack $\mathrm{Bun}_{\mathrm{G,D}}(\mathsf{k})$ and $r^*\mathcal{G}_{\mathrm{Hec}_{\mathrm{G},\varnothing}}$ is the trivial A-gerbe. However, the canonical trivialization on $r^*\mathcal{G}_{\mathrm{Hec}_{\mathrm{G},\varnothing}}$ is *not* the identity map. Rather, it corresponds to the A-torsor (0.6). The cohomology of $\mathrm{Sht}^{\varnothing}_{\mathrm{G,D}}$ with coefficients in $\overline{\mathbf{Q}}_\ell$, taken in the above sense, thus encodes ζ-genuine functions on $\widetilde{\mathrm{Bun}}_{\mathrm{G,D}}$.

0.2.4. The source of coefficients on $\mathrm{Sht}^{\mathrm{I}}_{\mathrm{G,D}}$ is provided by the twisted geometric Satake equivalence, explained in more detail in §0.3 below.

For now, it suffices to say that this equivalence supplies a family of functors

$$\mathsf{Rep}(({}^{\mathrm{L}}\mathrm{H}_{\mathring{\mathrm{X}}})^{\mathrm{I}}) \to \mathsf{D}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G,I}}) \tag{0.9}$$

natural in the finite set I. Here, the source is an appropriately defined category of representation of I-copies of an integral form ${}^{\mathrm{L}}\mathrm{H}_{\mathring{\mathrm{X}}}$ of the L-group (0.3), and the target is the derived category of $(\mathcal{G}_{\mathrm{Hec}_{\mathrm{G,I}}}, \zeta)$-twisted constructible complexes of $\overline{\mathbf{Q}}_\ell$-sheaves over $\mathrm{Hec}_{\mathrm{G,I}}$.

In fact, we shall also need a version of (0.9) for *iterated* Hecke stacks, which allows us to endow the cohomology of Shtukas with equivariance structure with respect to the partial Frobenius endomorphisms of $\mathring{\mathrm{X}}^{\mathrm{I}}$.

0.2.5. With the above preparation, we are in a position to essentially repeat V. Lafforgue's construction of the spectral decomposition. This is done in §8 of the article. (It involves some confusing computations with gerbes, but no serious difficulties.)

We emphasize that all of the above is already explained in [Laf18, §14], with one minor difference: The gerbes considered there are the gerbes parametrizing Nth roots of unity of line bundles (where $\mathrm{N} \mid q-1$, for q the cardinality of k). The gerbes we consider are more general, so we explain in §7 how the relevant pieces of structure on them arise.

One may summarize the situation as follows: In generalizing [Laf18] to covers, the only new feature is that we consider more general coefficients in the cohomology of Shtukas.

0.3. Geometric Satake equivalence.

0.3.1. Let us now discuss the twisted geometric Satake equivalence, whose proof occupies the bulk of this article.

For this result, we may take X to be a smooth curve over an arbitrary field k.[4] Let G be a split reductive group equipped with an A-valued étale level μ defined over X. To μ, one associates an A-gerbe $\mathcal{G}_{\mathrm{Hec}_{\mathrm{G}}}$ over the local Hecke stack $\mathrm{Hec}_{\mathrm{G}}$. (As opposed in §0.2.2, we assume for simplicity that $\mathrm{I} = \{1\}$ here and omit it from the notation.) We fix an injective character $\zeta : \mathrm{A} \hookrightarrow \overline{\mathbf{Q}}_\ell$ as in §0.1.1.

In this context, we may consider the full subcategory

$$\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G}}) \subset \mathsf{D}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G}})$$

consisting of objects whose pullback to the affine Grassmannian Gr_{G} are perverse and universally locally acyclic (ULA) relative to X.

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[3] More precisely, we shall take compactly supported cohomology along $\mathrm{Sht}^{\mathrm{I}}_{\mathrm{G,D}}/\Xi \to \mathring{\mathrm{X}}^{\mathrm{I}}$, which is a relative ind-Deligne–Mumford stack. The result is then an ind-constructible complex over $\mathring{\mathrm{X}}^{\mathrm{I}}$.

[4] For the application to global function fields, X will be the possibly open curve $\mathring{\mathrm{X}}$ of §0.2.1.

Using several pieces of canonical structure on $\mathcal{G}_{\mathrm{Hec}_{\mathrm{G}}}$, one may endow $\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G}})$ with a symmetric monoidal structure. Furthermore, one "tweaks" the commutativity constraint on $\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G}})$ using the sum of positive roots as for the usual Satake category (*cf.* [MV07, §6]). This results in a symmetric monoidal category

$$^{+}\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G}}). \tag{0.10}$$

Remark 0.3.2. The above construction of the twisted Satake category (0.10) is due to Gaitsgory and Lysenko (*cf.* [GL18]), based on Reich's work [Rei12]. It differs from Finkelberg and Lysenko's approach to the twisted geometric Satake equivalence (*cf.* [Lys06, FL10, Lys14]) by removing factorization line bundles over Gr_{G} from the picture.

Since [GL18] is more focused on statements than proofs, we shall supply a self-contained construction of the symmetric monoidal category (0.10) in §1.

0.3.3. Let us now turn to the spectral side.

To the pair (G,μ), we shall attach another pair (H,ν), where H is the metaplectic dual group (*cf.* §0.1.3) and ν is a morphism of étale sheaves of $\mathbb{E}_\infty$-monoids over X:

$$\nu : \hat{\mathrm{Z}}_{\mathrm{H}} \to \mathsf{B}^2_{\mathrm{X}}\mathrm{A}. \tag{0.11}$$

Here, $\hat{\mathrm{Z}}_{\mathrm{H}}$ is the character group of the center Z_{H} of H, viewed as a constant étale sheaf.

The pair (H,ν) is our version of the *metaplectic dual data.* Contrary to the same-named notion introduced in [GL18], our construction of (H,ν) is of group-theoretic nature and remains valid over an arbitrary base scheme S over which A has invertible order.[5]

0.3.4. To formulate our version of the twisted geometric Satake equivalence, we need to add to (0.11) a term having to do with ϑ-characteristics over X.

More precisely, we shall define another $\mathbb{E}_\infty$-monoidal morphism

$$\nu + \vartheta : \hat{\mathrm{Z}}_{\mathrm{H}} \to \mathsf{B}^2_{\mathrm{X}}\mathrm{A}. \tag{0.12}$$

which coincides with ν whenever a ϑ-characteristic over X is chosen (*cf.* §2.1.21).

This modification is closely related to a phenomenon known in the Langlands–Weissman program: The L-group arises from the Baer sum of two extensions of the Galois group by $\mathrm{Z}_{\mathrm{H}}(\overline{\mathbf{Q}}_\ell)$ (*cf.* [Wei18]). The addition of the ϑ-term in (0.12) corresponds to the twist by Weissman's meta-Galois group (also known as "the first twist").

In fact, we shall prove a precise result to this effect: Our Theorem 7.4.6 asserts that in the function field context, Weissman's meta-Galois group is the fundamental group of the $\{\pm 1\}$-gerbe of ϑ-characteristics.

0.3.5. Let us write $\mathsf{Rep}_{\mathrm{H}}$ for the symmetric monoidal category of H-representations on $\overline{\mathbf{Q}}_\ell$-local systems over X. It is of étale local nature over X and admits a $\hat{\mathrm{Z}}_{\mathrm{H}}$-grading defined by the action of Z_{H}.

Using a procedure explained in §A.2, we may use the $\mathbb{E}_\infty$-monoidal morphism (0.12) to "twist" $\mathsf{Rep}_{\mathrm{H}}$, obtaining a new symmetric monoidal category

$$\mathsf{Rep}_{\mathrm{H},\nu+\vartheta}. \tag{0.13}$$

This is a version of the representation category of the L-group. More precisely, the integral form of the L-group ${}^{\mathrm{L}}\mathrm{H}_{\mathrm{X}}$ is defined so that $\mathsf{Rep}({}^{\mathrm{L}}\mathrm{H}_{\mathrm{X}})$ is *monoidally* equivalent to

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[5] Our original motivation for giving this construction is that it is uniform for number fields and function fields (*cf.* [Zha22]). In the present context, it also has the practical consequence of removing an assumption on the characteristic of k from [GL18] (*cf.* the discussion in §C.3).

(0.13). However, we caution the reader that this equivalence is generally *incompatible* with the commutativity constraints.

From the perspective of the twisted geometric Satake equivalence, it is the symmetric monoidal category (0.13) which appears naturally on the dual side.

Theorem B. *There is a canonical equivalence of symmetric monoidal categories*

$$^{+}\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G}}) \simeq \mathsf{Rep}_{\mathrm{H},\nu+\vartheta}. \tag{0.14}$$

0.3.6. A more general version of Theorem B appears as Theorem 2.3.2. In the same subsection, we state various properties of our Satake equivalence which are needed for applications to global function fields.

Let us point out that the statement of Theorem 2.3.2 has already appeared in [GL18, §9], at least if we assume that our metaplectic dual data coincide with those of [GL18]. However, *op.cit.* does not provide a proof of this statement. By inspecting the existing literature [FL10, Lys14, Rei12], we have identified at least one place where new ideas are needed for the proof: the construction of the fiber functor.[6]

Let us briefly explain the difficulty involved.

0.3.7 *The trouble with the fiber functor***.** First, we note that the A-gerbe $\mathcal{G}_{\mathrm{Hec}_{\mathrm{G}}}$ is nontrivial along the fibers over X in general, so there is no analogue of the global cohomology functor in the twisted setting. As a substitute, we consider the constant term functor

$$^{+}\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G}}) \to \mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{T}}), \tag{0.15}$$

where T is the universal Cartan of G.[7] Let us also take for granted the twisted geometric Satake equivalence for T, which is a symmetric monoidal equivalence

$$\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{T}}) \simeq \mathsf{Rep}_{\mathrm{T}_{\mathrm{H}},\nu+\vartheta}, \tag{0.16}$$

for T_{H} the canonical maximal torus of H.

As Reich points out, $\mathsf{Rep}_{\mathrm{T}_{\mathrm{H}},\nu+\vartheta}$ may *not* admit a fiber functor due to the $(\nu+\vartheta)$-twist, and therefore one cannot directly apply the Tannakian formalism to $^{+}\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G}})$ (*cf.* [Rei12, §V.1]). However, one may try to resolve the issue as follows: By "untwisting" both (0.15) and the equivalence (0.16), one obtains the fiber functor as the composition of symmetric monoidal functors

$$\begin{aligned} ^{+}\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G}})_{-(\nu+\vartheta)} &\to \mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{T}})_{-(\nu+\vartheta)} \\ &\simeq (\mathsf{Rep}_{\mathrm{T}_{\mathrm{H}},\nu+\vartheta})_{-(\nu+\vartheta)} \simeq \mathsf{Rep}_{\mathrm{T}_{\mathrm{H}}} \to \mathsf{Lis}(\mathrm{X}), \end{aligned}$$

where $\mathsf{Lis}(\mathrm{X})$ denotes the category of $\overline{\mathbf{Q}}_{\ell}$-local systems on X. It is then possible to apply the Tannakian formalism to recognize $^{+}\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G}})_{-(\nu+\vartheta)}$ as $\mathsf{Rep}_{\mathrm{H}}$, and ultimately deduce (0.14) from this equivalence by restoring the twist.

0.3.8. To realize this strategy, however, one must construct a $\hat{\mathrm{Z}}_{\mathrm{H}}$-grading on the twisted Satake category $^{+}\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G}})$, compatible with its symmetric monoidal structure.

With respect to the natural maps

$$\hat{\mathrm{T}}_{\mathrm{H}} \twoheadrightarrow \hat{\mathrm{Z}}_{\mathrm{H}} \to \pi_1\mathrm{G},$$

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[6] We provide a detailed discussion on the status of the existing literature in §C.2-C.3.

[7] One needs to choose a Borel subgroup $\mathrm{B} \subset \mathrm{G}$ to obtain an étale level μ_{T} for T. However, the resulting category $\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{T}})$, as well as the functor (0.15), is independent of the choice of B.

this $\hat{\mathrm{Z}}_{\mathrm{H}}$-grading is supposed to be a *refinement* of the $\pi_1\mathrm{G}$-grading defined by the connected components of $\mathrm{Hec}_{\mathrm{G}}$, as well as a *coarsening* of the $\hat{\mathrm{T}}_{\mathrm{H}}$-grading that one may obtain from the semisimplicity of the (pointwise) Satake category. Note, however, that the $\hat{\mathrm{T}}_{\mathrm{H}}$-grading is *incompatible* with the symmetric monoidal structure, so it is insufficient for the construcion of the desired $\hat{\mathrm{Z}}_{\mathrm{H}}$-grading.

Let us make this discussion more concrete by considering a classical example.

Example 0.3.9. Take $\mathrm{G} = \mathrm{SL}_2$ and $\mathrm{A} = \{\pm 1\}$, under the assumption $\operatorname{char} \mathsf{k} \neq 2$. As for the étale level μ, we shall take the canonical lift of the mod 2 étale Chern class

$$c_2 \bmod 2 \in \mathrm{H}^4_{\text{ét}}(\mathsf{B}\mathrm{SL}_2, \{\pm 1\}^{\otimes 2}).$$

The existence of this canonical lift is justified by the calculation of the reduced cohomology of $\mathsf{B}\mathrm{SL}_2$ in degrees ≤ 4 (*cf.* [Del96]). It induces the metaplectic double cover of $\mathrm{SL}_2(\mathrm{F})$, for F a local field of characteristic $\neq 2$.

For this choice of μ, the dual group H is identified with SL_2, endowed with its canonical pinning. In particular, we have $\hat{\mathrm{T}}_{\mathrm{H}} \simeq \mathbf{Z}$ and $\hat{\mathrm{Z}}_{\mathrm{H}} \simeq \mathbf{Z}/2$, while $\pi_1\mathrm{G}$ vanishes.

Furthermore, if we fix a ϑ-characteristic over X, then $\mathsf{Rep}_{\mathrm{T}_{\mathrm{H}},\nu+\vartheta}$ is canonically identified with the symmetric monoidal category of $\mathbf{Z}$-graded $\overline{\mathbf{Q}}_\ell$-local systems over X, whose commutativity constraint is given by *the sign rule*. In particular, the functor of summing up its $\mathbf{Z}$-graded components is *not* symmetric monoidal. The discussion of §0.3.7 is thus necessary already in this classical example.

0.3.10 *The $\hat{\mathrm{Z}}_{\mathrm{H}}$-grading***.** Let us now explain the geometric origin of the $\hat{\mathrm{Z}}_{\mathrm{H}}$-grading on the twisted Satake category, established in §5 of the present article.[8]

To do so, we shall view objects of ${}^{+}\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G}})$ as $(\mathcal{G}_{\mathrm{Gr}_{\mathrm{G}}}, \zeta)$-twisted perverse sheaves over Gr_{G} (relative to X) equivariant with respect to the arc group $\mathrm{L}^+\mathrm{G}$. Here, $\mathcal{G}_{\mathrm{Gr}_{\mathrm{G}}}$ is the pullback of $\mathcal{G}_{\mathrm{Hec}_{\mathrm{G}}}$ to Gr_{G}, viewed as an $\mathrm{L}^+\mathrm{G}$-equivariant A-gerbe.

The key observation is that on each connected component $\mathrm{Gr}^{\theta}_{\mathrm{G}}$ of Gr_{G}, corresponding to $\theta \in \pi_1\mathrm{G}$, the A-gerbe $\mathcal{G}_{\mathrm{Gr}_{\mathrm{G}}}$ admits *a family* of $\mathrm{L}^+\mathrm{G}_{\mathrm{ad}}$-equivariance structures parametrized by lifts of θ to $\hat{\mathrm{Z}}_{\mathrm{H}}$, for G_{ad} the adjoint form of G. Distinct lifts of θ *always* give rise to distinct $\mathrm{L}^+\mathrm{G}_{\mathrm{ad}}$-equivariance structures.

The $\hat{\mathrm{Z}}_{\mathrm{H}}$-grading on ${}^{+}\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G}})$ is now defined as follows: For any $\xi \in \hat{\mathrm{Z}}_{\mathrm{H}}$ with image θ in $\pi_1\mathrm{G}$, the ξ-graded component of ${}^{+}\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G}})$ consists of $\mathrm{L}^+\mathrm{G}_{\mathrm{ad}}$-equivariant $(\mathcal{G}_{\mathrm{Gr}_{\mathrm{G}}}, \zeta)$-twisted perverse sheaves over $\mathrm{Gr}^{\theta}_{\mathrm{G}}$, with respect to the $\mathrm{L}^+\mathrm{G}_{\mathrm{ad}}$-equivariance structure of $\mathcal{G}_{\mathrm{Gr}_{\mathrm{G}}}$ *corresponding to* ξ. It is straightforward to verify that the $\hat{\mathrm{Z}}_{\mathrm{H}}$-grading on ${}^{+}\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G}})$ obtained this way is compatible with the symmetric monoidal structure.

The fact that, under the Satake equivalence (0.14), the natural $\hat{\mathrm{Z}}_{\mathrm{H}}$-grading on $\mathsf{Rep}_{\mathrm{H},\nu+\vartheta}$ corresponds to this $\hat{\mathrm{Z}}_{\mathrm{H}}$-grading on ${}^{+}\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G}})$ seems interesting in itself.

0.3.11. A few other components of our proof of the twisted geometric Satake equivalence have not appeared in the existing literature. They are in:

1. the proof of semisimplicity of the pointwise twisted Satake category;
2. the construction of the equivalence for tori;
3. the canonical identification of the Tannaka dual group with H.

Let us discuss them in turn.

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[8] An earlier version of this article contains a less direct construction of the $\hat{\mathrm{Z}}_{\mathrm{H}}$-grading. The present construction makes use of the "canonical quadratic structure" established more recently in [SZ25].

0.3.12 *Semisimplicity***.** Point (1) is the assertion that the twisted Satake category associated to a geometric point x of X is semisimple (*cf.* Corollary 3.3.7).

As in the untwisted setting, this assertion follows from the parity vanishing of intersection cohomology sheaves on Schubert varieties (*cf.* [Gai01, proof of Proposition 1 & §A.7]). However, the proof of parity vanishing there does *not* apply in the twisted setting.[9]

That being said, at least for G simple and simply connected, the required parity vanishing is known by Lusztig's work [Lus94, §5], predating all works on the twisted geometric Satake equivalence. In §3 of this article, we establish the parity vanishing following the argument of [LY20, Proposition 3.12].

I thank Gurbir Dhillon and Michael Finkelberg for the references [Lus94, LY20]. It must also be mentioned that the recent preprint [DLYZ25] establishes *significantly stronger* results concerning $\mathsf{D}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},x})$ than those in this article.

Another proof of semisimplicity, avoiding parity vanishing altogether, is suggested by Gaitsgory and reproduced in §C.4 of this article.

0.3.13 *Equivalence for tori***.** Unlike [GL18], our metaplectic dual data are *not* defined using the factorization structure of the affine Grassmannian. As a result, the twisted geometric Satake equivalence for tori is less tautological for us.

We perform the necessary calculation in §4. The results there are in turn used to relate Weissman's meta-Galois group to ϑ-characteristics (*cf.* Theorem 7.4.6).

0.3.14 *Canonicity of* **H.** We not only compute the root data of the Tannaka dual group in this text, but also construct its isomorphism with H. This is needed for treating covers of not necessarily split reductive groups.

In the untwisted setting, such an isomorphism is constructed in [FS24, VI.11]. There are some intricacies involved in the twisted setting: Essentially, if we view (0.11) as an $\mathbb{E}_\infty$-monoidal morphism $\hat{\mathrm{T}}_{\mathrm{H}} \to \mathsf{B}^2_{\mathrm{X}}\mathrm{A}$ trivialized over the root lattice of H, then the canonicity of (0.14) dictates which trivialization we must take. In this text, we specify the trivialization which makes (0.14) canonical.

0.4. Convention.

0.4.1. Let us note some convention and notation used throughout this article.

0.4.2. Denote by Spc the ∞-category of ∞-groupoids, whose objects are referred to as *spaces*. Denote by Spc_* the ∞-category of pointed spaces.

Denote by Sptr the ∞-category of spectra and by $\Omega^\infty : \mathsf{Sptr} \to \mathsf{Spc}_*$ the forgetful functor. For any (classical) ring R, we write R-Mod for the ∞-category of HR-module spectra.

Since the restriction of Ω^∞ to *connective* HR-module spectra is conservative, we may view a connective HR-module spectrum as a space with additional structure, and refer to it simply as an R-*linear space*. Similarly, we often call a morphim of connective HR-module spectra R-*linear* to distinguish it from a morphism of the underlying spaces.

0.4.3. Given a site $\mathcal{C}$ and an ∞-category $\mathcal{O}$ admitting finite limits, we write $\mathsf{Shv}(\mathcal{C}, \mathcal{O})$ for the ∞-category of $\mathcal{O}$-valued sheaves over $\mathcal{C}$.

We shall extensively use the iterated Bar construction for the ∞-category $\mathsf{Shv}(\mathcal{C}, \mathsf{Spc})$ endowed with the Cartesian symmetric monoidal structure (*cf.* [Lur17, §5.2.3]). For any

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[9] In an earlier version of this paper, it is *incorrectly* asserted that the semisimplicity of the twisted Satake category can be proved as in the untwisted case. This error is pointed out by Gaitsgory.

integer $n \geq 0$, we view the nth iterated Bar construction as a functor

$$\mathsf{B}^n : \mathsf{Shv}(\mathcal{C}, \mathsf{Alg}_{\mathbb{E}_n}(\mathsf{Spc})) \to \mathsf{Shv}(\mathcal{C}, \mathsf{Spc}_*).$$

We shall also refer to B as the *deloop* functor.

For a Sptr-valued sheaf $\mathcal{A}$ over $\mathcal{C}$, there is a canonical isomorphism in $\mathsf{Shv}(\mathcal{C}, \mathsf{Spc}_*)$:

$$\mathsf{B}^n \mathcal{A} \simeq \Omega^\infty(\mathcal{A}[n]).$$

In other words, the value of $\mathsf{B}^n\mathcal{A}$ at any $c \in \mathcal{C}$ coincides with $\Omega^\infty\Gamma(c, \mathcal{A}[n])$.

0.4.4. Given a site $\mathcal{C}$ and a sheaf of abelian groups $\mathcal{A}$ over $\mathcal{C}$, we refer to a section of $\mathsf{B}^2\mathcal{A}$ at $c \in \mathcal{C}$ as an *$\mathcal{A}$-gerbe* over c.

Contrary to most literature on banded gerbes, we denote the symmetric monoidal product on A-gerbes *additively*. We find this convention more natural in view of its relation to the $\mathbf{Z}$-linear structure on $\mathsf{B}^2\mathcal{A}$.

0.4.5. Denote by Ring the category of (unital, commutative) rings. A *prestack* is a functor of ∞-categories $\mathsf{Ring} \to \mathsf{Spc}$.

By an *indscheme*, we shall mean a prestack Z which admits a presentation $\mathrm{Z} \simeq \mathrm{colim}_\alpha \mathrm{Z}_\alpha$ as a filtered colimit, where each Z_α is a quasi-compact quasi-separated scheme and each transition map $\mathrm{Z}_\alpha \to \mathrm{Z}_{\alpha'}$ is a closed immersion.

Given a ring R (respectively a scheme S), an R-*prestack* is a prestack over $\operatorname{Spec}\mathrm{R}$ (respectively over S). We employ the same convention for R-*schemes*, R-*indschemes*, *etc.*

0.5. Acknowledgements.

0.5.1. This article, as well as its prequel [Zha22], grew out of my attempt to fill in the proof of the twisted geometric Satake equivalence formulated in [GL18]. I benefited tremedously from conversations with Dennis Gaitsgory on this topic.

In addition, I owe a large part of my understanding of covering groups to conversations with Wee Teck Gan, Sergey Lysenko, and Sam Raskin. I also thank Claudius Heyer for illuminating conversations during the preparation of this article.

I thank Kęstutis Česnavičius, Aron Heleodoro, and Cong Xue for offering me the opportunities to present parts of this work.

0.5.2. Since the first version of this article appeared, several errors were found and improvements were discovered, both by myself and by the community. This article is essentially rewritten to take all of them into account.

In this process, I received help from many mathematicians: Lin Chen, Gurbir Dhillon, Tianyi Feng, Tony Feng, Dennis Gaitsgory, Naoki Imai, and Luozi Shi. I am deeply grateful to all of them.

Part 1. Geometric Satake equivalence

1. THE SATAKE CATEGORY

In this section, we construct the twisted Satake category. We begin by constructing a gerbe $\mathcal{G}_{\mathrm{Hec}_{\mathrm{G}}}$ over the local Hecke stack $\mathrm{Hec}_{\mathrm{G}}$ in §1.1. This gerbe has a natural factorization structure compatible with the convolution structure of $\mathrm{Hec}_{\mathrm{G}}$. This is precisely formulated and constructed in §1.2. Using $\mathcal{G}_{\mathrm{Hec}_{\mathrm{G}}}$, we construct the twisted Satake category in §1.3, subject to some technical results whose proofs are supplied in §1.4. In §1.5, we define constant term functors and study their basic properties.

Our construction of the twisted Satake category is largely parallel to [GL18], except that we offer a more formal treatment of factorization.

1.1. The local A-gerbe.

1.1.1 *Context.* Let k be a field and X be a smooth curve over k, *i.e.* X is a quasi-compact and separated k-scheme which is smooth of relative dimension 1.

Let G be a smooth affine group X-scheme. Let A be a finite abelian group whose order is invertible in k. Let μ be an A-valued *étale level* of G, *i.e.* a morphism of pointed étale X-stacks:

$$\mu : \mathsf{B}_{\mathrm{X}}\mathrm{G} \to \mathsf{B}^4_{\mathrm{X}}\mathrm{A}(1), \tag{1.1}$$

where B_{X} denotes the Bar construction for sheaves on the big étale site of X (*cf.* §0.4.3).

1.1.2. Denote by Ran the k-prestack sending a k-algebra R to the set of finite subsets of X(R). It is called the *Ran space* of X. For an R-point $\underline{x}$ of Ran, we write $\mathrm{D}_{\underline{x}}$ for the formal completion of $\mathrm{X}_{\mathrm{R}} := \operatorname{Spec}\mathrm{R} \times \mathrm{X}$ along the union of the graphs $\Gamma_{\underline{x}} := \bigcup \Gamma_{x^i}$ over $x^i \in \underline{x}$.

We shall view $\mathrm{D}_{\underline{x}}$ as an affine scheme and refer to it as the *formal disk* around $\underline{x}$. The *formal punctured disk* around $\underline{x}$ is defined to be the affine scheme $\overset{\circ}{\mathrm{D}}_{\underline{x}} := \mathrm{D}_{\underline{x}} \smallsetminus \Gamma_{\underline{x}}$.

1.1.3. Denote by $\mathrm{L}^+\mathrm{G}$ (respectively $\mathrm{L}^+\mathrm{G}$) the group Ran-prestack whose R-points are pairs $(\underline{x}, g)$, where $\underline{x}$ is an R-point of Ran and $g : \mathrm{D}_{\underline{x}} \to \mathrm{G}$ (respectively $g : \overset{\circ}{\mathrm{D}}_{\underline{x}} \to \mathrm{G}$) is a morphism of affine X-schemes.

Denote by $\mathrm{Hec}_{\mathrm{G}}$ the Ran-prestack whose R-points are quadruples $(\underline{x}, \mathrm{P}^0, \mathrm{P}^1, \alpha)$, where $\underline{x}$ is an R-point of Ran, P^0, P^1 are G-bundles over $\mathrm{D}_{\underline{x}}$, and α is an isomorphism of their restrictions to $\overset{\circ}{\mathrm{D}}_{\underline{x}}$. We shall represent an R-point of $\mathrm{Hec}_{\mathrm{G}}$ by the expression

$$\mathrm{P}^0 \overset{\underline{x}}{\sim} \mathrm{P}^1, \tag{1.2}$$

which we refer to as a *modification* of G-bundles over $\mathrm{D}_{\underline{x}}$ along $\underline{x}$.

Denote by Gr_{G} the Ran-prestack whose R-points are modifications (1.2), where P^0 is the trivial G-bundle.

The Ran-prestacks $\mathrm{L}^+\mathrm{G}$, LG, $\mathrm{Hec}_{\mathrm{G}}$, Gr_{G} are called the *arc group*, the *loop group*, the *local Hecke stack*, and the *affine Grassmannian*, respectively. For any morphism $\underline{x} : \mathrm{S} \to \mathrm{Ran}$ of k-prestacks, we write $\mathrm{L}^+_{\underline{x}}\mathrm{G}$, $\mathrm{L}_{\underline{x}}\mathrm{G}$, $\mathrm{Hec}_{\mathrm{G},\underline{x}}$, $\mathrm{Gr}_{\mathrm{G},\underline{x}}$ for their base changes along $\underline{x}$.

1.1.4. We recall the well-known representability statements for the above Ran-prestacks, referring to [Zhu17, §3.1] for details.

The arc group $\mathrm{L}^+\mathrm{G}$ is an affine group Ran-scheme (*i.e.* the structural morphism $\mathrm{L}^+\mathrm{G} \to \mathrm{Ran}$ is representable in affine group schemes). The loop group LG is an ind-affine group Ran-indscheme.

The affine Grassmannian Gr_{G} is a Ran-indscheme of ind-finite type, and the map $\mathrm{LG} \to \mathrm{Gr}_{\mathrm{G}}$, sending $(\underline{x}, g)$ to the modification of the trivial G-bundle given by multiplication by g, realizes Gr_{G} as a quotient of étale Ran-stacks[10]:

$$\mathrm{LG}/\mathrm{L}^{+}\mathrm{G} \simeq \mathrm{Gr}_{\mathrm{G}}.$$

Likewise, $\mathrm{Hec}_{\mathrm{G}}$ is identified as a quotient of étale Ran-stacks:

$$\mathrm{Hec}_{\mathrm{G}} \simeq \mathrm{L}^{+}\mathrm{G}\backslash\mathrm{LG}/\mathrm{L}^{+}\mathrm{G}. \tag{1.3}$$

Remark 1.1.5. Although $\mathrm{Hec}_{\mathrm{G}}$ is not of ind-finite type over Ran, it can be approximated by finite type algebraic stacks as follows: For an R-point $\underline{x}$ of Ran, there is an indscheme presentation $\mathrm{Gr}_{\mathrm{G},\underline{x}} \simeq \operatorname{colim}_{\alpha} \mathrm{Z}_{\alpha}$, where each Z_{α} is an $\mathrm{L}^{+}_{\underline{x}}\mathrm{G}$-stable R-scheme of finite type, on which the $\mathrm{L}^{+}_{\underline{x}}\mathrm{G}$-action factors through a finite type quotient H_{β}, so we have

$$\mathrm{Hec}_{\mathrm{G},\underline{x}} \simeq \operatorname*{colim}_{\alpha} \lim_{\beta \gg \alpha} \mathrm{H}_{\beta}\backslash\mathrm{Z}_{\alpha}, \tag{1.4}$$

where $\beta \gg \alpha$ means that β ranges over a cofinal subset of indices depending on α. In other words, $\mathrm{Hec}_{\mathrm{G},\underline{x}}$ is an ind-pro-algebraic stack.

1.1.6 *Convolution***.** The Ran-prestack $\mathrm{Hec}_{\mathrm{G}}$ occurs as the 1-simplices of a groupoid Ran-prestack $\mathrm{Hec}_{\mathrm{G}}^{[\cdot]}$. Indeed, for each integer $n \geq 0$, we define an R-point of $\mathrm{Hec}_{\mathrm{G}}^{[n]}$ to be a chain of modifications

$$\mathrm{P}^{0} \overset{x}{\sim} \mathrm{P}^{1} \overset{x}{\sim} \cdots \overset{x}{\sim} \mathrm{P}^{n}. \tag{1.5}$$

Given a morphism $f : [m] \to [n]$ in the simplicial category, the induced morphism $\mathrm{Hec}_{\mathrm{G}}^{[n]} \to \mathrm{Hec}_{\mathrm{G}}^{[m]}$ sends (1.5) to the chain of modifications

$$\mathrm{P}^{f^{-1}(0)} \overset{x}{\sim} \mathrm{P}^{f^{-1}(1)} \overset{x}{\sim} \cdots \overset{x}{\sim} \mathrm{P}^{f^{-1}(n)},$$

where each modification is the composition of the corresponding segment in (1.5). The resulting simplicial Ran-prestack $\mathrm{Hec}_{\mathrm{G}}^{[\cdot]}$ is a groupoid (*i.e.* it obeys Segal's axioms, *cf.* [Lur09, Definition 6.1.2.7]).

We refer to this groupoid structure on $\mathrm{Hec}_{\mathrm{G}}$ as the *convolution structure.* Note that $\mathrm{Hec}_{\mathrm{G}}^{[\cdot]}$ is the Čech nerve of $\mathrm{L}^{+}(\mathsf{BG}) \to \mathrm{L}(\mathsf{BG})$, where $\mathrm{L}^{+}(\mathsf{BG})$ (respectively $\mathrm{L}(\mathsf{BG})$) parametrizes morphisms $\mathrm{D}_{\underline{x}} \to \mathsf{BG}$ (respectively $\mathring{\mathrm{D}}_{\underline{x}} \to \mathsf{BG}$) with $\underline{x} \in \mathrm{Ran}(\mathrm{R})$.

1.1.7 *The* A*-gerbe* $\mathcal{G}_{\mathrm{Hec}_{\mathrm{G}}}$**.** The goal of this subsection is to construct an A-gerbe $\mathcal{G}_{\mathrm{Hec}_{\mathrm{G}}}$ over $\mathrm{Hec}_{\mathrm{G}}$ using the étale level μ, *cf.* (1.1).

Indeed, in §1.1.11 below, we shall construct a morphism of spaces

$$\int_{\mathring{\mathrm{D}}} : \mathrm{Maps}_{*}(\mathsf{B}_{\mathrm{X}}\mathrm{G}, \mathsf{B}^{4}_{\mathrm{X}}\mathrm{A}(1)) \to \mathrm{Maps}(\mathrm{Hec}_{\mathrm{G}}, \mathsf{B}^{2}\mathrm{A}), \tag{1.6}$$

where $\mathrm{Maps}_{*}(\cdot,\cdot)$ denotes the mapping space of pointed X-stacks. Then we will set $\mathcal{G}_{\mathrm{Hec}_{\mathrm{G}}}$ to be the image of μ under (1.6).

This requires the formalism of trace maps, which we now turn to.

1.1.8 *The local trace map***.** Given an R-point $\underline{x}$ of Ran, we shall construct a morphism in the pro-category of $\mathrm{H}\mathbf{Z}$-module spectra

$$\mathrm{tr}_{\underline{x}} : \Gamma(\mathrm{D}_{\underline{x}} \bmod \mathring{\mathrm{D}}_{\underline{x}}, \hat{\mathbf{Z}}(1)[2]) \to \Gamma(\operatorname{Spec}\mathrm{R}, \hat{\mathbf{Z}}), \tag{1.7}$$

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[10] As above, this means that for a fixed R-point $\underline{x}$ of Ran, the base change $\mathrm{Gr}_{\mathrm{G},\underline{x}}$ is the quotient $\mathrm{L}_{\underline{x}}\mathrm{G}/\mathrm{L}^{+}_{\underline{x}}\mathrm{G}$ of étale R-stacks. Note that Ran itself is *not* an étale stack, *cf.* [GL14, Warning 2.4.4].

where $\hat{\mathbf{Z}}$ is the pro-(abelian group) $``\lim_{\mathrm{char}\,\mathsf{k}\nmid n}"\mathbf{Z}/n$ and the notation $\Gamma(\mathcal{Y} \text{ mod } \mathcal{Z}, \cdot)$, for any morphism $\mathcal{Z} \to \mathcal{Y}$ of k-prestacks, means the fiber of the pullback map $\Gamma(\mathcal{Y}, \cdot) \to \Gamma(\mathcal{Z}, \cdot)$. We shall refer to (1.7) as the *trace map*.

Indeed, write $\iota : \Gamma_{\underline{x}} \to \mathrm{X}_{\mathrm{R}}$ for the closed immersion. By the formal base change theorem (*cf.* [Fuj95, Corollary 6.6.4], [BM21, Theorem 6.11]), the source of (1.7) is identified as

$$\begin{aligned}\Gamma(\mathrm{D}_{\underline{x}} \text{ mod } \mathring{\mathrm{D}}_{\underline{x}}, \hat{\mathbf{Z}}(1)[2]) &\simeq \Gamma(\mathrm{X}_{\mathrm{R}} \text{ mod } \mathrm{X}_{\mathrm{R}} \smallsetminus \Gamma_{\underline{x}}, \hat{\mathbf{Z}}(1)[2]) \\ &\simeq \Gamma(\Gamma_{\underline{x}}, \iota^!\hat{\mathbf{Z}}(1)[2]) \end{aligned} \tag{1.8}$$

via pullback along $\mathrm{D}_{\underline{x}} \to \mathrm{X}_{\mathrm{R}}$. Since the structural morphism $\mathrm{X}_{\mathrm{R}} \to \operatorname{Spec}\mathrm{R}$ is smooth of relative dimension 1, we may identify $\iota^!\hat{\mathbf{Z}}(1)[2]$ with $\pi^!\hat{\mathbf{Z}}$, for $\pi : \Gamma_{\underline{x}} \to \operatorname{Spec}\mathrm{R}$ the projection. We then have a morphism

$$\begin{aligned}\Gamma(\Gamma_{\underline{x}}, \iota^!\hat{\mathbf{Z}}(1)[2]) &\simeq \Gamma(\Gamma_{\underline{x}}, \pi^!\hat{\mathbf{Z}}) \\ &\simeq \Gamma(\operatorname{Spec}\mathrm{R}, \pi_*\pi^!\hat{\mathbf{Z}}) \to \Gamma(\operatorname{Spec}\mathrm{R}, \hat{\mathbf{Z}}),\end{aligned} \tag{1.9}$$

where the last morphism is the co-unit of the adjunction from properness of π. The desired morphism (1.7) is the composition of (1.8) and (1.9).

Remark 1.1.9. In the construction of (1.7), we worked with coefficients in $\hat{\mathbf{Z}}$. We could instead perform the construction for any complex $\mathcal{A}$ of torsion étale sheaves of invertible order over k (the case of interest being $\mathcal{A} = \mathrm{A}[2]$) and obtain

$$\mathrm{tr}_{\underline{x}} : \Gamma(\mathrm{D}_{\underline{x}} \text{ mod } \mathring{\mathrm{D}}_{\underline{x}}, \mathcal{A}(1)[2]) \to \Gamma(\operatorname{Spec}\mathrm{R}, \mathcal{A}). \tag{1.10}$$

Note that (1.7) and (1.10) are related by a version of the projection formula, asserting that the diagram below commutes

$$\begin{tikzcd}
\Gamma(\operatorname{Spec}\mathrm{R}, \mathcal{A}) \otimes \Gamma(\mathrm{D}_{\underline{x}} \text{ mod } \mathring{\mathrm{D}}_{\underline{x}}, \hat{\mathbf{Z}}(1)[2]) \arrow[r, "\otimes"] \arrow[d, "\mathrm{id}\otimes\mathrm{tr}_{\underline{x}}"] & \Gamma(\mathrm{D}_{\underline{x}} \text{ mod } \mathring{\mathrm{D}}_{\underline{x}}, \mathcal{A}(1)[2]) \arrow[d, "\mathrm{tr}_{\underline{x}}"] \\
\Gamma(\operatorname{Spec}\mathrm{R}, \mathcal{A}) \otimes \Gamma(\operatorname{Spec}\mathrm{R}, \hat{\mathbf{Z}}) \arrow[r, "\otimes"] & \Gamma(\operatorname{Spec}\mathrm{R}, \mathcal{A})
\end{tikzcd}$$

where the horizontal arrows are defined by multiplication of coefficients. This holds because the top horizontal arrow is induced from the canonical morphism

$$\pi^*\mathcal{A} \otimes \pi^!\hat{\mathbf{Z}} \to \pi^!\mathcal{A},$$

which corresponds, under adjunction, to the composition

$$\pi_*(\pi^*\mathcal{A} \otimes \pi^!\hat{\mathbf{Z}}) \simeq \mathcal{A} \otimes \pi_*\pi^!\hat{\mathbf{Z}} \to \mathcal{A} \otimes \hat{\mathbf{Z}} \xrightarrow{\otimes} \mathcal{A}.$$

Remark 1.1.10. Consider the case where $\underline{x}$ is defined by an R-point x of X and write D_x instead of $\mathrm{D}_{\underline{x}}$, *etc.* Denote by

$$\Psi : \mathsf{B}\mathbb{G}_m \to \mathsf{B}^2\hat{\mathbf{Z}}(1)$$

the Kummer morphism, *i.e.* the deloop of the compatible family of boundary maps $\mathbb{G}_m \to \mathsf{B}(\mathbf{Z}/n(1))$ given by the degree-n covers with $\mathrm{char}\,\mathsf{k} \nmid n$.

Consider the line bundle $\mathcal{O}(x)$ over D_x, equipped with its canonical trivialization over $\mathring{\mathrm{D}}_x$. Then $\Psi(\mathcal{O}(x))$ defines an object of (the underlying space of) $\Gamma(\mathrm{D}_{\underline{x}} \text{ mod } \mathring{\mathrm{D}}_{\underline{x}}, \hat{\mathbf{Z}}(1)[2])$. Its image under tr_x is the constant section 1.

Indeed, this can be checked over k-points of R, where it follows from cohomological purity of the closed immersion $\iota : \operatorname{Spec}\mathsf{k} \simeq \Gamma_x \to \mathrm{X}$.

1.1.11 *Construction of* (1.6)**.** Using the cohomological interpretation of $\mathsf{B}^n\mathrm{A}$ (*cf.* §0.4.3), it suffices to construct a map of $\mathrm{H}\mathbf{Z}$-module spectra

$$\Gamma(\mathsf{B}_{\mathrm{X}}\mathrm{G} \bmod \mathrm{X}, \mathrm{A}(1)[4]) \to \Gamma(\mathrm{Hec}_{\mathrm{G}}, \mathrm{A}[2]). \tag{1.11}$$

To construct (1.11), we proceed as follows: Let $\mathrm{P}^0 \overset{\underline{x}}{\sim} \mathrm{P}^1$ be an R-point of $\mathrm{Hec}_{\mathrm{G}}$, which we view as two morphisms $\mathrm{D}_{\underline{x}} \to \mathsf{B}_{\mathrm{X}}\mathrm{G}$ with an identification over $\mathring{\mathrm{D}}_{\underline{x}}$. Pulling back along them and subtracting, we obtain a morphism of $\mathrm{H}\mathbf{Z}$-module spectra

$$(\mathrm{P}^1)^* - (\mathrm{P}^0)^* : \Gamma(\mathsf{B}_{\mathrm{X}}\mathrm{G} \bmod \mathrm{X}, \mathrm{A}(1)[4]) \to \Gamma(\mathrm{D}_{\underline{x}} \bmod \mathring{\mathrm{D}}_{\underline{x}}, \mathrm{A}(1)[4]). \tag{1.12}$$

The desired morphism (1.11) is obtained as the composition of (1.12) with the trace map (1.10) (for $\mathcal{A} \coloneqq \mathrm{A}[2]$), using naturality of the construction in the R-point $\mathrm{P}^0 \overset{\underline{x}}{\sim} \mathrm{P}^1$.

As explained in §1.1.7, this concludes the construction of $\mathcal{G}_{\mathrm{Hec}_{\mathrm{G}}}$.

Remark 1.1.12. The construction of $\mathcal{G}_{\mathrm{Hec}_{\mathrm{G}}}$ also appears in [GL18, §7.3]. As explained in *loc.cit.*, this construction may be used to prove that every factorization A-gerbe on Gr_{G} canonically descends to $\mathrm{Hec}_{\mathrm{G}}$, a result claimed in [Rei12, Theorem III.2.10] but not justified adequately. On the other hand, our approach to the twisted geometric Satake equivalence does *not* need this result.

1.1.13. Next, we shall explain in what sense $\mathcal{G}_{\mathrm{Hec}_{\mathrm{G}}}$ is "multiplicative" with respect to the convolution structure on $\mathrm{Hec}_{\mathrm{G}}$ (*cf.* §1.1.6).

Indeed, the group structure on $\mathsf{B}^2\mathrm{A}$ realizes it as the 1-simplices of a groupoid prestack $\mathsf{B}^2\mathrm{A}^{[\cdot]}$. By pullback, we obtain a groupoid Ran-prestack $\mathsf{B}^2_{\mathrm{Ran}}\mathrm{A}^{[\cdot]}$. We define a *multiplicative* A-*gerbe* over $\mathrm{Hec}_{\mathrm{G}}$ to be a morphism of groupoid Ran-prestacks

$$\mathrm{Hec}_{\mathrm{G}}^{[\cdot]} \to \mathsf{B}^2_{\mathrm{Ran}}\mathrm{A}^{[\cdot]}.$$

Lemma 1.1.14. *The* A-*gerbe* $\mathcal{G}_{\mathrm{Hec}_{\mathrm{G}}}$ *canonically lifts to a multiplicative* A-*gerbe.*

Proof. Given an R-point $\underline{x}$ of Ran, the groupoid $\mathrm{Hec}_{\mathrm{G}}(\mathrm{R}) \times_{\mathrm{Ran}(\mathrm{R})} \{\underline{x}\}$ of lifts of $\underline{x}$ to $\mathrm{Hec}_{\mathrm{G}}$ is the Čech nerve of the morphism in Spc given by pullback:

$$\mathrm{Maps}(\mathrm{D}_{\underline{x}}, \mathsf{B}\mathrm{G}) \to \mathrm{Maps}(\mathring{\mathrm{D}}_{\underline{x}}, \mathsf{B}\mathrm{G}). \tag{1.13}$$

The construction of §1.1.11 supplies a morphism in Spc:

$$\mathrm{Hec}_{\mathrm{G}}(\mathrm{R}) \times_{\mathrm{Ran}(\mathrm{R})} \{\underline{x}\} \to \Gamma(\mathrm{Spec}\,\mathrm{R}, \mathsf{B}^2\mathrm{A}) \tag{1.14}$$

It remains to lift (1.14) to a morphism of groupoids naturally in $\underline{x}$.

The étale level μ induces a morphism from (1.13) to

$$\mathrm{Maps}(\mathrm{D}_{\underline{x}}, \mathsf{B}^4\mathrm{A}(1)) \to \mathrm{Maps}(\mathring{\mathrm{D}}_{\underline{x}}, \mathsf{B}^4\mathrm{A}(1)). \tag{1.15}$$

The morphism (1.15) naturally lifts to a morphism of $\mathrm{H}\mathbf{Z}$-module spectra, thanks to the $\mathbf{Z}$-linear structure on $\mathsf{B}^4\mathrm{A}(1)$ (*cf.* §0.4.3).

Note that given any morphism $f : a \to b$ in a stable ∞-category $\mathcal{C}$, there is a canonical morphism from the Čech nerve of f to $\mathrm{Fib}(f)$ as groupoid objects in $\mathcal{C}$: It is induced from the canonical map from f to $0 \to \mathrm{Cofib}(f)$. Note, furthermore, that on 1-simplices, this morphism specializes to the subtraction map

$$a \times_b a \to \mathrm{Fib}(f).$$

The Čech nerve of (1.15) thus maps to $\Gamma(\mathrm{D}_{\underline{x}} \bmod \mathring{\mathrm{D}}_{\underline{x}}, \mathsf{B}^4\mathrm{A}(1))$, as groupoid objects in $\mathrm{H}\mathbf{Z}$-module spectra. Composing with μ, we see that the morphism

$$\begin{gathered}\mathrm{Hec}_{\mathrm{G}}(\mathrm{R}) \times_{\mathrm{Ran}(\mathrm{R})} \{\underline{x}\} \to \Gamma(\mathrm{D}_{\underline{x}} \bmod \mathring{\mathrm{D}}_{\underline{x}}, \mathsf{B}^4\mathrm{A}(1)) \\ (\mathrm{P}^0 \overset{\underline{x}}{\sim} \mathrm{P}^1) \mapsto \mu(\mathrm{P}^1) - \mu(\mathrm{P}^0)\end{gathered} \tag{1.16}$$

lifts to a morphism of groupoid objects in Spc. The morphism (1.14) is the composition of (1.16) with the trace map (1.10), the latter being a morphism of groupoid objects as it comes from a morphism of $\mathrm{H}\mathbf{Z}$-module spectra. □

1.2. Factorization.

1.2.1. In §1.1, we have constructed a k-prestack $\mathrm{Hec}_{\mathrm{G}}$ together with an A-gerbe $\mathcal{G}_{\mathrm{Hec}_{\mathrm{G}}}$. The pair $(\mathrm{Hec}_{\mathrm{G}}, \mathcal{G}_{\mathrm{Hec}_{\mathrm{G}}})$ furthermore admits a multiplicative structure (*cf.* Lemma 1.1.14).

In this subsection, we shall endow $(\mathrm{Hec}_{\mathrm{G}}, \mathcal{G}_{\mathrm{Hec}_{\mathrm{G}}})$ with an additional structure, which is a precise formulation of its “unital factorization structure” compatible with the given multiplicative structure (*cf.* Proposition 1.2.11).

1.2.2. We begin by lifting Ran to a k-presheaf of colored operads: For a k-algebra R, the set Ran(R) admits the structure of a colored operad, with operations defined by

$$\mathrm{Maps}_{\mathrm{Ran}(\mathrm{R})}(\{\underline{x}_i\}_{i\in\mathrm{I}}, \underline{y}) := \begin{cases} * & \underline{x}_i \text{ and } \underline{x}_j \text{ are disjoint if } i \neq j \text{ and } \bigsqcup_{i\in\mathrm{I}} \underline{x}_i \subset \underline{y} \\ \varnothing & \text{otherwise} \end{cases}$$

Here, two R-points $\underline{x}$, $\underline{y}$ of Ran are called *disjoint* if $\Gamma_{\underline{x}} \cap \Gamma_{\underline{y}} = \varnothing$ as closed subsets of X_{R}.

For any k-presheaf of symmetric monoidal ∞-categories $\mathcal{O}$, we define the ∞-category of Ran-*algebras* in $\mathcal{O}$ as the mapping space

$$\mathsf{Alg}_{\mathrm{Ran}}(\mathcal{O}) := \mathrm{Maps}(\mathrm{Ran}, \mathcal{O})$$

taken in the ∞-category of k-presheaves valued in the ∞-category of ∞-operads. (We view symmetric monoidal ∞-categories as ∞-operads via the tautological forgetful functor.)

Remark 1.2.3. Informally, a Ran-algebra in $\mathcal{O}$ is a compatible family of Ran(R)-algebras $\mathcal{A}_{\mathrm{R}}$ in $\mathcal{O}(\mathrm{R})$, for each k-algebra R.

The datum of a Ran(R)-algebra in $\mathcal{O}(\mathrm{R})$ consists of an object $\mathcal{A}_{\underline{x}} \in \mathcal{O}(\mathrm{R})$ for each $\underline{x} \in \mathrm{Ran}(\mathrm{R})$, along with structural morphisms

$$f_{\{\underline{x}_i\}\to \underline{y}} : \bigotimes_{i\in\mathrm{I}} \mathcal{A}_{\underline{x}_i} \to \mathcal{A}_{\underline{y}} \tag{1.17}$$

for any finite pairwise disjoint set $\underline{x}_i$ ($i \in \mathrm{I}$) of elements of Ran(R) contained in $\underline{y}$. Furthermore, the maps (1.17) admit homotopy coherent data with respect to compositions of such containments.

Remark 1.2.4. For the purpose of this article, the Ran-algebras we shall consider take values in the full subcategory of 1-truncated objects of $\mathcal{O}(\mathrm{R})$ for every k-algebra R.

In this case, homotopy coherence for the structural morphisms (1.17) can be stated explicitly: For a composition of operations $\{\underline{y}_j\}_{j\in\mathrm{J}} \to \underline{z}$ and $\{\underline{x}_i\}_{i\in\mathrm{I}_j} \to \underline{y}_j$ ($j \in \mathrm{J}$), there is

2-isomorphism rendering the diagram

$$\begin{tikzcd}
\bigotimes_{j\in \mathrm{J}}\left(\bigotimes_{i\in \mathrm{I}_j}\mathcal{A}_{\underline{x}_i}\right) \arrow[r, "\simeq"] \arrow[d, "\bigotimes_{j\in\mathrm{J}} f_{\{\underline{x}_i\}\to\underline{y}_j}"] & \bigotimes_{\substack{j\in\mathrm{J}\\ i\in\mathrm{I}_j}}\mathcal{A}_{\underline{x}_i} \arrow[d, "f_{\{\underline{x}_i\}\to\underline{z}}"] \\
\bigotimes_{j\in\mathrm{J}}\mathcal{A}_{\underline{y}_j} \arrow[r, "f_{\{\underline{y}_j\}\to\underline{z}}"] & \mathcal{A}_{\underline{z}}
\end{tikzcd}$$

commute and obeying the coherence condition for triple compositions.

1.2.5. Let $\mathcal{O}$ be a k-presheaf of symmetric monoidal ∞-categories.

A Ran-algebra $\mathcal{A}$ in $\mathcal{O}$ is called a *(unital) factorization algebra* if for any disjoint R-points $\underline{x}$, $\underline{y}$ of Ran, the structural morphism

$$f_{\{\underline{x},\underline{y}\}\to\underline{x}\sqcup\underline{y}} : \mathcal{A}_{\underline{x}}\otimes\mathcal{A}_{\underline{y}}\to\mathcal{A}_{\underline{x}\sqcup\underline{y}}$$

is an isomorphism (*cf.* Remark 1.2.3).

Remark 1.2.6. There is another homotopy-coherent formulation of (unital) factorization structures due to Raskin (*cf.* [Ras14]).

The above formulation has the advantage of fitting into the formalism of *generalized ∞-operads* (*cf.* [Lur17, §2.3.2]). It is independently found by Lin Chen. In forthcoming joint work, we aim to develop the theory of factorization categories from this point of view.

1.2.7. We shall apply the above paradigm to the k-presheaf $\mathrm{Span}(\mathsf{Stk})$ assigning to each k-algebra R the symmetric monoidal ∞-category of spans of étale R-stacks, and endow $\mathrm{Hec}_{\mathrm{G}}$ with the structure of a factorization algebra in $\mathrm{Span}(\mathsf{Stk})$.

Indeed, given a k-algebra R and an operation $\{\underline{x}_i\}_{i\in\mathrm{I}}\to\underline{y}$ in Ran(R), we have natural morphisms of formal disks

$$\bigsqcup_{i\in\mathrm{I}}\mathrm{D}_{\underline{x}_i}\simeq\mathrm{D}_{\underline{x}}\to\mathrm{D}_{\underline{y}}\quad\left(\underline{x}:=\bigsqcup_{i\in\mathrm{I}}\underline{x}_i\right), \tag{1.18}$$

and compatibly, a cospan of formal punctured disks

$$\bigsqcup_{i\in\mathrm{I}}\mathring{\mathrm{D}}_{\underline{x}_i}\simeq\mathring{\mathrm{D}}_{\underline{x}}\to\mathrm{D}_{\underline{y}}\smallsetminus\Gamma_{\underline{x}}\leftarrow\mathring{\mathrm{D}}_{\underline{y}}. \tag{1.19}$$

The assignment $\underline{x}\mapsto\mathrm{D}_{\underline{x}}$ (respectively, $\underline{x}\mapsto\mathring{\mathrm{D}}_{\underline{x}}$) defines a Ran(R)-algebra in the category (respectively, the cospan category) of affine schemes over X equipped with the disjoint union symmetric monoidal structure.

By mapping into BG, we endow $\mathrm{Hec}_{\mathrm{G}}$ with the structure of a Ran-algebra in $\mathrm{Span}(\mathsf{Stk})$. Note that given an operation $\{\underline{x}_i\}\to\underline{y}$ in Ran(R), the structural morphism on $\mathrm{Hec}_{\mathrm{G}}$ is the span of étale R-stacks

$$\begin{tikzcd}
\mathrm{Hec}_{\mathrm{G},\{\underline{x}_i\}\to\underline{y}} \arrow[r, "p_{\underline{y}}"] \arrow[d, "\prod_{i\in\mathrm{I}} p_{\underline{x}_i}"] & \mathrm{Hec}_{\mathrm{G},\underline{y}} \\
\prod_{i\in\mathrm{I}}\mathrm{Hec}_{\mathrm{G},\underline{x}_i} &
\end{tikzcd} \tag{1.20}$$

where $\mathrm{Hec}_{\mathrm{G},\{\underline{x}_i\}\to\underline{y}}$ parametrizes modifications $\mathrm{P}^0\overset{\underline{x}}{\sim}\mathrm{P}^1$ of G-bundles *over* $\mathrm{D}_{\underline{y}}$ *along* $\underline{x}$.

Since both morphisms in (1.19) are isomorphisms for $\underline{x}=\underline{y}$, the Ran-algebra $\mathrm{Hec}_{\mathrm{G}}$ in $\mathrm{Span}(\mathsf{Stk})$ is in fact a factorization algebra.

Remark 1.2.8. We may repeat the construction of §1.2.7 for the affine Grassmannian Gr_{G} (*cf.* §1.1.3), which lifts Gr_{G} to a factorization algebra in Stk, the k-presheaf assigning to R the symmetric monoidal ∞-category of étale R-stacks (as opposed to $\mathrm{Span}(\mathsf{Stk})$).

Indeed, this is because a G-bundle over $\mathrm{D}_{\underline{y}}$ equipped with a trivialization off $\Gamma_{\underline{x}}$ is equivalent to a G-bundle over $\mathrm{D}_{\underline{x}}$ equipped with a trivialization off $\Gamma_{\underline{x}}$, by the Beauville–Laszlo lemma (*cf.* [BL95]).

Under the forgetful functor $\mathsf{Stk} \to \mathrm{Span}(\mathsf{Stk})$, we may realize the structural morphism $\mathrm{Gr}_{\mathrm{G}} \to \mathrm{Hec}_{\mathrm{G}}$ as a morphism of factorization algebras in $\mathrm{Span}(\mathsf{Stk})$.

1.2.9 *Variant: convolution***.** The construction of §1.2.7 has an obvious variant when we take the convolution structure on $\mathrm{Hec}_{\mathrm{G}}$ into account (*cf.* §1.1.6).

Namely, denote by $\mathsf{Gpd}(\mathsf{Stk})$ the k-presheaf assigning to R the symmetric monoidal ∞-category of groupoid étale R-stacks. Then $\mathrm{Hec}_{\mathrm{G}}^{[\cdot]}$ has the natural structure of a factorization algebra in $\mathrm{Span}(\mathsf{Gpd}(\mathsf{Stk}))$.

Note that the structural morphism (1.20) for $\mathrm{Hec}_{\mathrm{G}}^{[\cdot]}$ involves the étale R-stack $\mathrm{Hec}_{\mathrm{G},\{\underline{x}_i\}\to\underline{y}}^{[\cdot]}$ parametrizing chains of modifications of G-bundles over $\mathrm{D}_{\underline{y}}$ along $\underline{x}$.

1.2.10. Finally, we shall involve the multiplicative A-gerbe $\mathcal{G}_{\mathrm{Hec}_{\mathrm{G}}}$.

Denote by $\mathrm{Span}(\mathsf{Gpd}(\mathsf{Stk}))_{/\mathsf{B}^2\mathrm{A}}$ the k-presheaf assigning to a k-algebra R the ∞-category of spans of groupoid étale R-stacks over $\mathsf{B}^2\mathrm{A}^{[\cdot]}$.

Concretely, an object of $\mathrm{Span}(\mathsf{Gpd}(\mathsf{Stk}))_{/\mathsf{B}^2\mathrm{A}}(\mathrm{R})$ is a groupoid étale R-stack $\mathcal{Y}^{[\cdot]}$ equipped with a multiplicative A-gerbe $\mathcal{G} : \mathcal{Y}^{[\cdot]} \to \mathsf{B}^2\mathrm{A}^{[\cdot]}$, and a morphism $(\mathcal{Y}_1^{[\cdot]}, \mathcal{G}_2) \to (\mathcal{Y}_2^{[\cdot]}, \mathcal{G}_2)$ is a correspondence

$$\mathcal{Y}_1^{[\cdot]} \xleftarrow{p_1} \mathcal{Y}_{12}^{[\cdot]} \xrightarrow{p_2} \mathcal{Y}_2^{[\cdot]}$$

of groupoid étale R-stacks along with an isomorphism of multiplicative A-gerbes

$$p_1^*\mathcal{G}_1 \simeq p_2^*\mathcal{G}_2.$$

The symmetric monoidal structure on $\mathrm{Span}(\mathsf{Gpd}(\mathsf{Stk}))_{/\mathsf{B}^2\mathrm{A}}$ is given by Cartesian product of groupoid stacks along with external sum of multiplicative A-gerbes.

Proposition 1.2.11. *The pair* $(\mathrm{Hec}_{\mathrm{G}}, \mathcal{G}_{\mathrm{Hec}_{\mathrm{G}}})$ *canonically lifts to a factorization algebra in* $\mathrm{Span}(\mathsf{Gpd}(\mathsf{Stk}))_{/\mathsf{B}^2\mathrm{A}}$.

Proof. For each k-algebra R, the proof of Lemma 1.1.14 supplies a morphism

$$\mathcal{G}_{\mathrm{Hec}_{\mathrm{G}}} : \mathrm{Hec}_{\mathrm{G}}(\mathrm{R}) \to \Gamma(\operatorname{Spec}\mathrm{R}, \mathsf{B}^2\mathrm{A}), \tag{1.21}$$

where the source $\mathrm{Hec}_{\mathrm{G}}(\mathrm{R})$ lifts to a $\mathrm{Ran}(\mathrm{R})$-algebra in $\mathsf{Gpd}(\mathsf{Spc})$ (*cf.* §1.2.9) while the target $\Gamma(\operatorname{Spec}\mathrm{R}, \mathsf{B}^2\mathrm{A})$ is a commutative algebra in Spc, hence a $\mathrm{Ran}(\mathrm{R})$-algebra. It remains to lift (1.21) to a morphism of $\mathrm{Ran}(\mathrm{R})$-algebras in $\mathsf{Gpd}(\mathsf{Spc})$, naturally in R.

Recall that (1.21) is the composition of two maps: the map induced from (1.16) as $\underline{x}$ varies in $\mathrm{Ran}(\mathrm{R})$

$$\mathrm{Hec}_{\mathrm{G}}(\mathrm{R}) \to \Gamma(\mathrm{D}_{(\cdot)} \bmod \overset{\circ}{\mathrm{D}}_{(\cdot)}, \mathsf{B}^4\mathrm{A}(1)) \tag{1.22}$$

and the trace map (1.10). The morphism (1.22) naturally lifts to a morphism of $\mathrm{Ran}(\mathrm{R})$-algebras in $\mathrm{Span}(\mathsf{Gpd}(\mathsf{Spc}))$, because the $\mathrm{Ran}(\mathrm{R})$-algebra structures are induced from those of $\overset{\circ}{\mathrm{D}}_{(\cdot)}$ and $\mathrm{D}_{(\cdot)}$ (*cf.* §1.2.7). We shall argue that the target of (1.22) lifts to a $\mathrm{Ran}(\mathrm{R})$-algebra in $\mathrm{H}\mathbf{Z}\text{-}\mathsf{Mod}$ (with Cartesian symmetric monoidal structure) and that the trace map

$$\mathrm{tr}_{(\cdot)} : \Gamma(\mathrm{D}_{(\cdot)} \bmod \overset{\circ}{\mathrm{D}}_{(\cdot)}, \mathcal{A}(1)[2]) \to \Gamma(\operatorname{Spec}\mathrm{R}, \mathcal{A}) \tag{1.23}$$

lifts to a map of $\mathrm{Ran}(\mathrm{R})$-algebras in $\mathrm{H}\mathbf{Z}\text{-}\mathsf{Mod}$.

Consider the category $\mathsf{Div}^+(\mathrm{X}_\mathrm{R}/\mathrm{R})$ of effective Cartier divisors of X_R relative to R, with morphisms given by inclusions and symmetric monoidal structure given by sums. The association $\underline{x} \mapsto \Gamma_{\underline{x}}$ defines a $\mathrm{Ran}(\mathrm{R})$-algebra $\Gamma_{(\cdot)}$ in $\mathsf{Div}^+(\mathrm{X}_\mathrm{R}/\mathrm{R})$. The functor

$$\begin{aligned} \mathsf{Div}^+(\mathrm{X}_\mathrm{R}/\mathrm{R}) &\to \mathrm{H}\mathbf{Z}\text{-}\mathsf{Mod} \\ \mathrm{Z} &\mapsto \Gamma(\mathrm{X}_\mathrm{R} \bmod \mathrm{X}_\mathrm{R} \smallsetminus \mathrm{Z}, \mathcal{A}(1)[2]) \end{aligned}$$

is lax symmetric monoidal, so it carries the $\mathrm{Ran}(\mathrm{R})$-algebra $\Gamma_{(\cdot)}$ to the $\mathrm{Ran}(\mathrm{R})$-algebra $\Gamma(\mathrm{X}_\mathrm{R} \bmod \mathrm{X}_\mathrm{R} \smallsetminus \Gamma_{(\cdot)}, \mathcal{A}(1)[2])$ in $\mathrm{H}\mathbf{Z}\text{-}\mathsf{Mod}$. The latter provides the desired lift of the target of (1.22) by formal base change (*cf.* §1.1.8).

It remains to lift the morphism

$$\Gamma(\mathrm{X}_\mathrm{R} \bmod \mathrm{X}_\mathrm{R} \smallsetminus \Gamma_{(\cdot)}, \mathcal{A}(1)[2]) \simeq \Gamma(\Gamma_{(\cdot)}, \pi^! \mathcal{A}) \to \Gamma(\operatorname{Spec} \mathrm{R}, \mathcal{A}),$$

provided by adjunction for $\pi : \Gamma_{\underline{x}} \to \operatorname{Spec} \mathrm{R}$ (over $\underline{x} \in \mathrm{Ran}(\mathrm{R})$), to a morphism of $\mathrm{Ran}(\mathrm{R})$-algebras in $\mathrm{H}\mathbf{Z}\text{-}\mathsf{Mod}$. The compatibility with structural morphisms are supplied by the naturality of the adjunction map

$$\Gamma(\mathrm{Z}, \pi^! \mathcal{A}) \to \Gamma(\operatorname{Spec} \mathrm{R}, \mathcal{A})$$

for any $\mathrm{Z} \in \mathsf{Div}^+(\mathrm{X}_\mathrm{R}/\mathrm{R})$ with projection $\pi : \mathrm{Z} \to \operatorname{Spec} \mathrm{R}$, with respect to Z. □

1.3. The Satake category.

1.3.1 *Coefficients.* Let ℓ be a prime invertible in k and e be a finite extension of $\mathbf{Q}_\ell$. We assume that A is a subgroup of $\mathsf{e}^\times$ and write $\zeta : \mathrm{A} \hookrightarrow \mathsf{e}^\times$ for the inclusion.

We shall use the functor (B.2) assigning to an object $(\mathrm{X}, \mathcal{G}) \in \mathsf{Sch}_{/\mathrm{B}^2\mathrm{A}}$ the ∞-category $\mathsf{D}_{\mathcal{G},\zeta}(\mathrm{X})$ of $(\mathcal{G}, \zeta)$-twisted constructible complexes over X.

1.3.2. For an indscheme Z equipped with an A-gerbe $\mathcal{G}$, we write $\mathsf{D}_{\mathcal{G},\zeta}(\mathrm{Z})$ for the colimit of the ∞-categories $\mathsf{D}_{\mathcal{G},\zeta}(\mathrm{X})$ indexed by the poset of closed subschemes $\mathrm{X} \hookrightarrow \mathrm{Z}$ along pushforward functors.

In particular, a given indscheme presentation $\mathrm{Z} \simeq \operatorname{colim} \mathrm{Z}_\alpha$ realizes $\mathsf{D}_{\mathcal{G},\zeta}(\mathrm{Z})$ as the colimit of ∞-categories

$$\mathsf{D}_{\mathcal{G},\zeta}(\mathrm{Z}) \simeq \operatorname*{colim}_{\alpha} \mathsf{D}_{\mathcal{G},\zeta}(\mathrm{Z}_\alpha).$$

Given any S-point $\underline{x}$ of Ran (for $\mathrm{S} \in \mathsf{Sch}$), we may apply this construction to the indscheme $\mathrm{Gr}_{\mathrm{G},\underline{x}}$ equipped with the pullback $\mathcal{G}_{\mathrm{Gr}_\mathrm{G}}$ of the A-gerbe $\mathcal{G}_{\mathrm{Hec}_\mathrm{G}}$ (*cf.* §1.1.7) along the structural map $\mathrm{Gr}_{\mathrm{G},\underline{x}} \to \mathrm{Hec}_{\mathrm{G},\underline{x}}$. This yields the ∞-category $\mathsf{D}_{\mathcal{G},\zeta}(\mathrm{Gr}_{\mathrm{G},\underline{x}})$ of $(\mathcal{G}_{\mathrm{Gr}_\mathrm{G}}, \zeta)$-twisted constructible complexes over $\mathrm{Gr}_{\mathrm{G},\underline{x}}$.

1.3.3. We give a slightly *ad hoc* definition of $\mathsf{D}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\underline{x}})$, sufficient for our purposes.

Namely, given an S-point $\underline{x}$ of Ran (for $\mathrm{S} \in \mathsf{Sch}$), we write $\mathsf{D}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\underline{x}})$ for the ∞-category of $\mathrm{L}^+_{\underline{x}}\mathrm{G}$-equivariant objects of $\mathsf{D}_{\mathcal{G},\zeta}(\mathrm{Gr}_{\mathrm{G},\underline{x}})$, formed with respect to the canonical $\mathrm{L}^+_{\underline{x}}\mathrm{G}$-equivariance structure of $\mathcal{G}_{\mathrm{Gr}_\mathrm{G}}$.

In terms of the presentation (1.4), we have

$$\mathsf{D}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\underline{x}}) \simeq \operatorname*{colim}_{\alpha} \operatorname*{colim}_{\beta \gg \alpha} \mathsf{D}_{\mathcal{G},\zeta}(\mathrm{Z}_\alpha)^{\mathrm{H}_\beta},$$

where the colimit over β is taken along forgetful functors. (It stabilizes over the cofinal set of β where $\ker(\mathrm{L}^+_{\underline{x}}\mathrm{G} \to \mathrm{H}_\beta)$ is pro-unipotent and acts trivially on Z_α.)

1.3.4. In the remainder of this subsection, we assume that G *is reductive.*

This hypothesis guarantees that $\mathrm{Gr}_{\mathrm{G}} \to \mathrm{Ran}$ is ind-proper (*cf.* [Zhu17, Theorem 3.1.3]). This in turn implies that each boundary map

$$\mathrm{Hec}_{\mathrm{G}}^{[n]} \to \mathrm{Hec}_{\mathrm{G}}^{[m]}$$

of the groupoid Ran-prestack $\mathrm{Hec}_{\mathrm{G}}$ (*cf.* §1.1.6) is ind-proper.

1.3.5 *Convolution.* Fix $\mathrm{S} \in \mathsf{Sch}$ and an S-point $\underline{x}$ of Ran. We shall construct a monoidal operation on $\mathsf{D}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\underline{x}})$, called the *convolution product.*

Indeed, given $\mathcal{A}_1, \cdots, \mathcal{A}_n \in \mathsf{D}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\underline{x}})$, their convolution product is given by

$$\mathcal{A}_1 \circ \cdots \circ \mathcal{A}_n := m_!(p_1^* \mathcal{A}_1 \otimes \cdots \otimes p_n^* \mathcal{A}_n), \tag{1.24}$$

where m, $p_1, \cdots, p_n$ are the face maps

$$\begin{tikzcd} \mathrm{Hec}_{\mathrm{G},\underline{x}}^{[n]} \arrow[r, "\prod p_j"] \arrow[d, "m"] & \prod_{j=1}^n \mathrm{Hec}_{\mathrm{G},\underline{x}} \\ \mathrm{Hec}_{\mathrm{G},\underline{x}} & \end{tikzcd} \tag{1.25}$$

sending $\mathrm{P}^0 \overset{\underline{x}}{\sim} \cdots \overset{\underline{x}}{\sim} \mathrm{P}^n$ to $\mathrm{P}^0 \overset{\underline{x}}{\sim} \mathrm{P}^n$, respectively the segments $\mathrm{P}^0 \overset{\underline{x}}{\sim} \mathrm{P}^1, \cdots, \mathrm{P}^{n-1} \overset{\underline{x}}{\sim} \mathrm{P}^n$.

The formation of (1.24) appeals to the identification of A-gerbes

$$m^* \mathcal{G}_{\mathrm{Hec}_{\mathrm{G},\underline{x}}} \simeq p_1^* \mathcal{G}_{\mathrm{Hec}_{\mathrm{G},\underline{x}}} + \cdots + p_n^* \mathcal{G}_{\mathrm{Hec}_{\mathrm{G},\underline{x}}} \tag{1.26}$$

supplied by Lemma 1.1.14.

The monoidal unit is $e_!(\mathsf{e})$, where $e : \mathsf{B}\mathrm{L}_{\underline{x}}^+\mathrm{G} \simeq \mathrm{Hec}_{\mathrm{G},\underline{x}}^{[0]} \to \mathrm{Hec}_{\mathrm{G},\underline{x}}$ is the unit section, formed with respect to the trivialization of $e^*(\mathcal{G}_{\mathrm{Hec}_{\mathrm{G},\underline{x}}})$ supplied by Lemma 1.1.14.

Remark 1.3.6. Certainly, we expect the above structure to lift to a monoidal structure on the ∞-category $\mathsf{D}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\underline{x}})$. To give a formal construction, however, we need to view $(\mathrm{Hec}_{\mathrm{G},\underline{x}}, \mathcal{G}_{\mathrm{Hec}_{\mathrm{G},\underline{x}}})$ as a monoid in the ∞-category of spans of $\mathsf{Stk}_{/\mathsf{B}^2\mathrm{A}}$ and apply a "twisted" 6-functor formalism, the latter being unavailable at the time of writing.

1.3.7. Given an S-point $\underline{x}$ of Ran, we define

$$\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\underline{x}}) \subset \mathsf{D}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\underline{x}})$$

for the full subcategory consisting of objects $\mathcal{A}$ whose pullback to $\mathrm{Gr}_{\mathrm{G},\underline{x}}$ are universally locally acyclic (ULA) and perverse relative to S (*cf.* Definition B.1.7, §B.1.12).

Remark 1.3.8. Pullback along $\mathrm{Gr}_{\mathrm{G},\underline{x}} \to \mathrm{Hec}_{\mathrm{G},\underline{x}}$ realizes $\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\underline{x}})$ as the full subcategory of $\mathsf{D}_{\mathcal{G},\zeta}(\mathrm{Gr}_{\mathrm{G},\underline{x}})$ consisting of objects which are ULA and perverse relative to S and constant along $\mathrm{L}_{\underline{x}}^+\mathrm{G}$-orbits.

Indeed, this holds because $\mathrm{L}_{\underline{x}}^+\mathrm{G} \to \mathrm{S}$ is pro-smooth with connected geometric fibers: It is an inverse limit of Weil restrictions of G along finite locally free morphisms.

Proposition 1.3.9. *Given* $\mathrm{S} \in \mathsf{Sch}$ *and an* S*-point* $\underline{x}$ *of* Ran,

1. *the monoidal unit* $e_!(\mathsf{e})$ *of* $\mathsf{D}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\underline{x}})$ *belongs to* $\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\underline{x}})$*;*
2. *the convolution product of any* $\mathcal{A}_1, \mathcal{A}_2 \in \mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\underline{x}})$ *belongs to* $\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\underline{x}})$.

1.3.10. The proof of Proposition 1.3.9 will be supplied in §1.4.6.

It implies that $\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\underline{x}})$ inherits a monoidal structure from $\mathsf{D}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\underline{x}})$, which we shall refer to as the *convolution monoidal structure*. We shall refer to the monoidal category $(\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\underline{x}}), \circ)$ as the *Satake category* at $\underline{x}$.

Next, we shall show that when $\underline{x}$ is defined by a finite product of X, the Satake category can be upgraded to a symmetric monoidal category.

Remark 1.3.11. Let I be a finite set. It determines a morphism $\mathrm{X}^{\mathrm{I}} \to \mathrm{Ran}$, sending $(x^i)_{i\in\mathrm{I}}$ to $\underline{x} := \bigcup_{i\in\mathrm{I}} x^i$, hence an étale X^{I}-stack $\mathrm{Hec}_{\mathrm{G},\mathrm{I}}$ by pullback (and likewise $\mathrm{Gr}_{\mathrm{G},\mathrm{I}}$, $\mathrm{L}^+_{\mathrm{I}}\mathrm{G}$, *etc.*).

The factorization structure on $\mathrm{Hec}_{\mathrm{G}}$ includes, in particular, the following span of étale stacks associated to each map of finite sets $\varphi : \mathrm{I} \to \mathrm{J}$ (*cf.* (1.20))

$$\begin{tikzcd}
\mathrm{Hec}_{\mathrm{G},\mathrm{I}\to\mathrm{J}} \arrow[r, "\iota"] \arrow[d, "\pi"] & \mathrm{Hec}_{\mathrm{G},\mathrm{J}} \\
\mathrm{Hec}_{\mathrm{G},\mathrm{I}} &
\end{tikzcd} \tag{1.27}$$

where $\mathrm{Hec}_{\mathrm{G},\mathrm{I}\to\mathrm{J}}$ parametrizes a point y^{J} of X^{J} (with image x^{I} in X^{I}) and a modification of G-bundles over $\mathrm{D}_{\underline{y}}$ along $\underline{x}$. (Here, $\underline{x}$, $\underline{y}$ denote the images of x^{I}, y^{J} in Ran.) Note that $\mathrm{Hec}_{\mathrm{G},\mathrm{I}\to\mathrm{J}}$ may alternatively be presented as the quotient $\mathrm{L}^+_{\mathrm{J}}\mathrm{G}\backslash(\mathrm{Gr}_{\mathrm{G},\mathrm{I}} \times_{\mathrm{X}^{\mathrm{I}}} \mathrm{X}^{\mathrm{J}})$ of étale stacks, showing that ι is a closed immersion.

Moreover, the factorization structure on $(\mathrm{Hec}_{\mathrm{G}}, \mathcal{G}_{\mathrm{Hec}_{\mathrm{G}}})$ (*cf.* Proposition 1.2.11) allows us to lift (1.27) to a span in $\mathsf{Stk}_{/\mathrm{B}^2\mathrm{A}}$. Thus, $\iota_!\pi^*$ restricts to a monoidal functor

$$\varphi_! : \mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\mathrm{I}}) \to \mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\mathrm{J}}) \tag{1.28}$$

This shows that the assignment of $\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\mathrm{I}})$ to I is functorial.

1.3.12 ***Fusion product*.** For any integer $n \geq 1$, denote by $\mathrm{X}^{\sqcup_n \mathrm{I},\mathrm{disj}} \subset \mathrm{X}^{\sqcup_n \mathrm{I}}$ the open subscheme consisting of I-tuples $(x_1^i)_{i\in\mathrm{I}}, \cdots, (x_n^i)_{i\in\mathrm{I}}$ whose images $\underline{x}_1, \cdots, \underline{x}_n$ in Ran are pairwise disjoint (*cf.* §1.2.2) and by $\mathrm{Hec}^{\mathrm{disj}}_{\mathrm{G},\sqcup_n \mathrm{I}}$ the base change of $\mathrm{Hec}_{\mathrm{G},\sqcup \mathrm{I}_n}$ to $\mathrm{X}^{\sqcup_n \mathrm{I},\mathrm{disj}}$.

We have natural open immersions of étale $\mathrm{X}^{\sqcup_n \mathrm{I}}$-stacks

$$\begin{tikzcd}
\mathrm{Hec}^{\mathrm{disj}}_{\mathrm{G},\sqcup_n \mathrm{I}} \arrow[r, "f"] \arrow[d, "j"] & (\mathrm{Hec}_{\mathrm{G},\mathrm{I}})^n \\
\mathrm{Hec}_{\mathrm{G},\sqcup_n \mathrm{I}} &
\end{tikzcd} \tag{1.29}$$

where the map f comes from the factorization structure on $\mathrm{Hec}_{\mathrm{G}}$ (*cf.* §1.2.7). Pulling back the external tensor product of $\mathcal{A}_1, \cdots, \mathcal{A}_n \in \mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\mathrm{I}})$ along f yields

$$f^*(\mathcal{A}_1 \boxtimes \cdots \boxtimes \mathcal{A}_n) \in \mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}^{\mathrm{disj}}_{\mathrm{G},\sqcup_n \mathrm{I}}) \tag{1.30}$$

whose formation uses the isomorphism of A-gerbes

$$f^*(\mathcal{G}_{\mathrm{Hec}_{\mathrm{G},\mathrm{I}}} \boxplus \cdots \boxplus \mathcal{G}_{\mathrm{Hec}_{\mathrm{G},\mathrm{I}}}) \simeq j^*\mathcal{G}_{\mathrm{Hec}_{\mathrm{G},\sqcup_n \mathrm{I}}} \tag{1.31}$$

obtained from the lift of f to a morphism in $\mathsf{Stk}_{/\mathrm{B}^2\mathrm{A}}$ (*cf.* Proposition 1.2.11). Proposition 1.3.13 below shows that (1.30) canonically extends along j to an object

$$\mathcal{A}_1 \star_{\mathrm{X}^{\mathrm{I}}} \cdots \star_{\mathrm{X}^{\mathrm{I}}} \mathcal{A}_n \in \mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\sqcup_n \mathrm{I}}), \tag{1.32}$$

called the *external fusion product* of $\mathcal{A}_1, \cdots, \mathcal{A}_n$.

The pullback of (1.32) to the diagonal $\mathrm{X}^{\mathrm{I}} \to \mathrm{X}^{\sqcup_n \mathrm{I}}$ is called the *fusion product*

$$\mathcal{A}_1 \star \cdots \star \mathcal{A}_n \in \mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\mathrm{I}}). \tag{1.33}$$

Proposition 1.3.13. *For any finite set* I *and integer* $n \geq 1$*,*

1. *the pullback functor*

$$j^* : \mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\sqcup_n \mathrm{I}}) \to \mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}^{\mathrm{disj}}_{\mathrm{G},\sqcup_n \mathrm{I}}) \tag{1.34}$$

is fully faithful;

2. *the object* (1.30) *belongs to the essential image of* (1.34)*.*

1.3.14. The proof of Proposition 1.3.13 will be supplied in §1.4.6. For now, let us explain how the fusion product (1.33) defines a symmetric monoidal category $(\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\mathrm{I}}), \star)$.

Indeed, we use (1.33) as the n-ary operation on $\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\mathrm{I}})$. To see its Σ_n-equivariance, it suffices to construct an isomorphism in $\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}^{\mathrm{disj}}_{\mathrm{G},\sqcup_n \mathrm{I}})$ for each $\sigma \in \Sigma_n$:

$$\sigma^* f^*(\mathcal{A}_1 \boxtimes \cdots \boxtimes \mathcal{A}_n) \simeq f^*(\mathcal{A}_{\sigma(1)} \boxtimes \cdots \mathcal{A}_{\sigma(n)}), \tag{1.35}$$

compatible with compositions. The isomorphism (1.35) comes from the Σ_n-equivariance of f as a morphism in $\mathsf{Stk}_{/\mathrm{B}^2\mathrm{A}}$.

Next, we argue that $e_!(\mathsf{e})$ (*cf.* §1.3.5) is a unit for $\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\mathrm{I}})$ also with respect to the fusion product. This follows from the canonical isomorphism in $\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}^{\mathrm{disj}}_{\mathrm{G},\mathrm{I}\sqcup\mathrm{I}})$

$$f^*(\mathcal{A} \boxtimes e_!(\mathsf{e})) \simeq j^*(\varphi_! \mathcal{A}),$$

where $\varphi : \mathrm{I} \to \mathrm{I} \sqcup \mathrm{I}$ is the inclusion of the first summand (*cf.* Remark 1.3.11), as $\varphi_! \mathcal{A}$ restricts to $\mathcal{A}$ along the diagonal.

Proposition 1.3.15 ("Convolution = fusion")**.** *For any finite set* I*, there is a canonical equivalence of monoidal categories*

$$(\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\mathrm{I}}), \circ) \simeq (\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\mathrm{I}}), \star). \tag{1.36}$$

Proof. We shall argue that the convolution monoidal structure lifts $(\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\mathrm{I}}), \star)$ to a monoid in the (2-)category $\mathsf{CAlg}(\mathsf{Cat})$ of symmetric monoidal categories.

Concretely, this means that given $\mathcal{A}_1, \mathcal{A}_2, \mathcal{B}_1, \mathcal{B}_2 \in \mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\mathrm{I}})$, we have commutation of fusion and convolution products

$$(\mathcal{A}_1 \circ \mathcal{A}_2) \star (\mathcal{B}_1 \circ \mathcal{B}_2) \simeq (\mathcal{A}_1 \star \mathcal{B}_1) \circ (\mathcal{A}_2 \star \mathcal{B}_2).$$

This follows from the construction of the fusion product and the fact that $(\mathrm{Hec}_{\mathrm{G}}, \mathcal{G}_{\mathrm{Hec}_{\mathrm{G}}})$ is a factorization algebra in $\mathrm{Span}(\mathsf{Gpd}(\mathsf{Stk}))_{/\mathrm{B}^2\mathrm{A}}$ (*cf.* Proposition 1.2.11).

The equivalence (1.36) now follows from a categorical analogue of the Eckmann–Hilton argument (*cf.* [Lur17, Proposition 2.4.3.9, Proposition 3.2.4.7]). ☐

1.3.16 *Normalization***.** By Proposition 1.3.15, we may regard $\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\mathrm{I}})$ as endowed with either the convolution or the fusion symmetric monoidal structure.

Next, we shall "normalize" its commutativity constraint to ensure that the constant term functors are symmetric monoidal (*cf.* Lemma 1.5.8). There is a locally constant function

$$d_{\mathrm{G}} : \mathrm{Hec}_{\mathrm{G},\mathrm{I}} \to \mathbf{Z}/2 \tag{1.37}$$

defined as follows: Étale locally over X^{I}, we may assume that G is split, with a Borel subgroup B and maximal quotient torus $\mathrm{B} \twoheadrightarrow \mathrm{T}$. Connected components of $\mathrm{Hec}_{\mathrm{G},\mathrm{I}}$ are in bijection with $\pi_1\mathrm{G}$, which is a quotient of the cocharacter lattice of T. We define (1.37) by

$$d_{\mathrm{G}}(\theta) := \langle 2\check{\rho}, \theta\rangle \bmod 2,$$

where $2\check{\rho}$ is the sum of positive roots. This expression is independent of the choice of B (as it varies in a connected family), so we obtain (1.37) globally over X^{I}.

Given $\mathcal{A}_1, \mathcal{A}_2 \in \mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\mathrm{I}})$ supported on connected components θ_1, θ_2 of $\mathrm{Hec}_{\mathrm{G},\mathrm{I}}$, we modify their commutativity constraint $c_{\mathcal{A}_1,\mathcal{A}_2}$ in $\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\mathrm{I}})$ by

$$\tilde{c}_{\mathcal{A}_1,\mathcal{A}_2} := (-1)^{d_{\mathrm{G}}(\theta_1)d_{\mathrm{G}}(\theta_2)} c_{\mathcal{A}_1,\mathcal{A}_2}. \tag{1.38}$$

Denote by ${}^{+}\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\mathrm{I}})$ the symmetric monoidal category $\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\mathrm{I}})$ equipped with the *modified* commutativity constraint (1.38).

We shall refer to ${}^{+}\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\mathrm{I}})$ as the *normalized Satake category.*

Remark 1.3.17. By definition, we have an equivalence of *monoidal* categories

$${}^{+}\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\mathrm{I}}) \simeq \mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\mathrm{I}})$$

which is *incompatible* with the symmetric monoidal structures.

1.4. Outer convolution diagram.

1.4.1. We remain in the context of §1.1.1 and §1.3.1, and assume that G is reductive.

In the previous subsection, we have defined the Satake category subject to two unproven assertions: Proposition 1.3.9 and Proposition 1.3.13. The goal of this subsection is to supply their proofs.

In fact, we shall deduce both assertions from the "outer convolution diagram". This is an adaptation of an argument due to Gaitsgory [Gai01] to the twisted setting.

1.4.2. Let $n \geq 0$ be an integer. Denote by $\widetilde{\mathrm{Hec}}{}^{[n]}_{\mathrm{G}}$ the prestack parametrizing n points $\underline{x}_1, \cdots, \underline{x}_n$ of Ran, together with a chain of modifications

$$\mathrm{P}^0 \overset{\underline{x}_1}{\sim} \mathrm{P}^1 \overset{\underline{x}_2}{\sim} \cdots \overset{\underline{x}_n}{\sim} \mathrm{P}^n \tag{1.39}$$

of G-bundles over $\mathrm{D}_{\underline{x}_1 \cup \cdots \cup \underline{x}_n}$.

The structural morphism $\widetilde{\mathrm{Hec}}{}^{[n]}_{\mathrm{G}} \to \mathrm{Ran}^n$ realizes $\widetilde{\mathrm{Hec}}{}^{[n]}_{\mathrm{G}}$ as a Ran^n-stack. Given n finite sets $\mathrm{I}_1, \cdots, \mathrm{I}_n$, we may base change $\widetilde{\mathrm{Hec}}{}^{[n]}_{\mathrm{G}}$ along the product $\mathrm{X}^{\mathrm{I}_1 \sqcup \cdots \sqcup \mathrm{I}_n} \to \mathrm{Ran}^n$ to obtain a k-stack $\widetilde{\mathrm{Hec}}{}^{\mathrm{I}_1,\cdots,\mathrm{I}_n}_{\mathrm{G}}$. For our applications, only the case $\mathrm{I}_1 = \cdots = \mathrm{I}_n := \mathrm{I}$ for some finite set I is relevant. Let us display the structural morphisms

$$\begin{tikzcd} \widetilde{\mathrm{Hec}}{}^{\mathrm{I},\cdots,\mathrm{I}}_{\mathrm{G}} \arrow[r, "\prod \widetilde{p}_j"] \arrow[d, "\widetilde{m}"] & \prod_{j=1}^{n} \mathrm{Hec}_{\mathrm{G},\mathrm{I}} \\ \mathrm{Hec}_{\mathrm{G},\sqcup_n \mathrm{I}} & \end{tikzcd} \tag{1.40}$$

where $\widetilde{m}, \widetilde{p}_1, \cdots, \widetilde{p}_n$ send (1.39) to $\mathrm{P}^0 \overset{\underline{x}_1 \cup \cdots \underline{x}_n}{\sim} \mathrm{P}^n$, respectively the segments $\mathrm{P}^0 \overset{\underline{x}_1}{\sim} \mathrm{P}^1, \cdots, \mathrm{P}^{n-1} \overset{\underline{x}_n}{\sim} \mathrm{P}^n$. It is known that $\widetilde{m}$ is ind-proper (*cf.* [Ric14, Corollary 2.10]).

Note that the pullback of (1.40) along the diagonal $\Delta : \mathrm{X}^{\mathrm{I}} \to \mathrm{X}^{\sqcup_n \mathrm{I}}$ recovers (1.25), whereas pulling back $\widetilde{\mathrm{Hec}}{}^{\mathrm{I},\cdots,\mathrm{I}}_{\mathrm{G}}$ to the disjoint locus $\mathrm{X}^{\sqcup_n \mathrm{I},\mathrm{disj}} \subset \mathrm{X}^{\sqcup_n \mathrm{I}}$ in (1.40) yields (1.29).

Lemma 1.4.3. *There is a canonical isomorphism of* A*-gerbes over* $\widetilde{\mathrm{Hec}}_{\mathrm{G}}^{\mathrm{I},\cdots,\mathrm{I}}$*:*

$$\widetilde{m}^*\mathcal{G}_{\mathrm{Hec}_{\mathrm{G},\mathrm{I}}} \simeq \widetilde{p}_1^*\mathcal{G}_{\mathrm{Hec}_{\mathrm{G},\mathrm{I}}} + \cdots + \widetilde{p}_n^*\mathcal{G}_{\mathrm{Hec}_{\mathrm{G},\mathrm{I}}} \tag{1.41}$$

with the following properties:

1. *its pullback to the diagonal* X^{I} *is identified with* (1.26)*;*
2. *its pullback to the disjoint locus* $\mathrm{X}^{\sqcup_n \mathrm{I},\mathrm{disj}}$ *is identified with* (1.31)*.*

Proof. For ease of notation, we will treat the case $n = 2$, the general case being similar.

Given an R-point of $\widetilde{\mathrm{Hec}}_{\mathrm{G}}^{\mathrm{I},\mathrm{I}}$, corresponding to R-points x_1^{I}, x_2^{I} of X^{I} and modifications $\mathrm{P}^0 \overset{x_1}{\sim} \mathrm{P}^1 \overset{x_2}{\sim} \mathrm{P}^2$, we interpret P^0, P^1, P^2 as morphisms $\mathrm{D}_{\underline{x}_1 \cup \underline{x}_2} \to \mathsf{B}\mathrm{G}$ and form

$$\mu(\mathrm{P}^2) - \mu(\mathrm{P}^0) \in \Gamma(\mathrm{D}_{\underline{x}_1 \cup \underline{x}_2} \bmod \mathring{\mathrm{D}}_{\underline{x}_1 \cup \underline{x}_2}, \mathsf{B}^4\mathrm{A}(1)). \tag{1.42}$$

By definition, its image under $\mathrm{tr}_{\underline{x}_1 \cup \underline{x}_2}$ yields the value of $\widetilde{m}^*\mathcal{G}_{\mathrm{Hec}_{\mathrm{G},\mathrm{I}}}$ (*cf.* §1.1.11).

Let us express (1.42) as the sum

$$\begin{aligned} \mu(\mathrm{P}^2) - \mu(\mathrm{P}^0) \simeq\ & \mu(\mathrm{P}^2) - \mu(\mathrm{P}^1) \\ & + \mu(\mathrm{P}^1) - \mu(\mathrm{P}^0), \end{aligned}$$

where the two summands come from $\Gamma(\mathrm{D}_{\underline{x}_i} \bmod \mathring{\mathrm{D}}_{\underline{x}_i}, \mathsf{B}^4\mathrm{A}(1))$ for $i = 2, 1$, respectively. The desired isomorphism (1.41) now follows from the additivity of the trace map.

Propertes (1) and (2) are immediate consequences of the construction. ☐

1.4.4. Given $\mathcal{A}_1, \cdots, \mathcal{A}_n \in \mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\mathrm{I}})$, we may use the isomorphism (1.41) to form the *outer convolution product*

$$\mathcal{A}_1 \circ_{\mathrm{X}^{\mathrm{I}}} \cdots \circ_{\mathrm{X}^{\mathrm{I}}} \mathcal{A}_n := \widetilde{m}_!(\widetilde{p}_1^*\mathcal{A}_1 \otimes \cdots \otimes \widetilde{p}_n^*\mathcal{A}_2) \tag{1.43}$$

as an object of the ∞-category $\mathsf{D}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\sqcup_n \mathrm{I}})$.

Using Lemma 1.4.3, we obtain isomorphisms

$$\Delta^*(\mathcal{A}_1 \circ_{\mathrm{X}^{\mathrm{I}}} \cdots \circ_{\mathrm{X}^{\mathrm{I}}} \mathcal{A}_n) \simeq \mathcal{A}_1 \circ \cdots \circ \mathcal{A}_n, \tag{1.44}$$

$$j^*(\mathcal{A}_1 \circ_{\mathrm{X}^{\mathrm{I}}} \cdots \circ_{\mathrm{X}^{\mathrm{I}}} \mathcal{A}_n) \simeq f^*(\mathcal{A}_1 \boxtimes \cdots \boxtimes \mathcal{A}_n), \tag{1.45}$$

where the morphisms j and f are as in §1.3.12.

Lemma 1.4.5. *The complex* $\mathcal{A}_1 \circ_{\mathrm{X}^{\mathrm{I}}} \cdots \circ_{\mathrm{X}^{\mathrm{I}}} \mathcal{A}_n$ *belongs to* $\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\sqcup_n \mathrm{I}})$.

Proof. By definition, we need to show that $\mathcal{A}_1 \circ_{\mathrm{X}^{\mathrm{I}}} \cdots \circ_{\mathrm{X}^{\mathrm{I}}} \mathcal{A}_n$ is ULA and perverse relative to $\mathrm{X}^{\sqcup_n \mathrm{I}}$ (*cf.* §1.3.7). The ULA property holds because $\widetilde{m}$ is ind-proper.

To show that $\mathcal{A}_1 \circ_{\mathrm{X}^{\mathrm{I}}} \cdots \circ_{\mathrm{X}^{\mathrm{I}}} \mathcal{A}_n$ is perverse relative to $\mathrm{X}^{\sqcup_n \mathrm{I}}$, we first use the isomorphism (1.45) to see that its pullback along j is perverse relative to $\mathrm{X}^{\sqcup_n \mathrm{I},\mathrm{disj}}$. Then we conclude using its ULA property relative to $\mathrm{X}^{\sqcup_n \mathrm{I}}$ and the fact that the nearby cycles functor is perverse t-exact (*cf.* [Ill94, Corollaire 4.5]). ☐

1.4.6. We now prove Propositions 1.3.9 and 1.3.13 simultaneously.

Proof of Proposition 1.3.13. Assertion (1) follows from [HS23, Theorem 6.8]. Assertion (2) follows from Lemma 1.4.5 and the isomorphism (1.45). ☐

Proof of Proposition 1.3.9. Assertion (1) is immediate.

To prove assertion (2), we note that $\mathcal{A}_1 \circ \cdots \circ \mathcal{A}_n$ is ULA relative to S because m is ind-proper. The statement that it is perverse relative to S can be proved over geometric points, so we reduce to the case where S is the spectrum of an algebraically closed field.

In this case, the S-point $\underline{x}$ of Ran is of the form $\underline{x} = \bigcup_{i \in \mathrm{I}} x_i$, where $\{x_i\}_{i \in \mathrm{I}}$ are pairwise disjoint S-points of X. The factorization structure of $(\mathrm{Hec}_{\mathrm{G}}, \mathcal{G}_{\mathrm{Hec}_{\mathrm{G}}})$ (*cf.* Proposition 1.2.11) allows us to further reduce to the case of a single S-point x of X.

Now, any object $\mathcal{A} \in \mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},x})$ extends to some étale neighborhood of x. Thus the result follows from Lemma 1.4.5 and the isomorphism (1.44) (for $\mathrm{I} = \{1\}$). ☐

1.5. Constant term functors.

1.5.1. We remain in the context of §1.1.1 and §1.3.1 and assume that G is reductive. Furthermore, we assume the existence of, and choose a square root $\mathsf{e}(\frac{1}{2})$ of the Tate twist $\mathsf{e}(1)$ over $\operatorname{Spec} \mathsf{k}$. (This is done for convenience, *cf.* Remark 2.3.3.)

Let P be a parabolic subgroup of G with Levi quotient $\mathrm{P} \twoheadrightarrow \mathrm{M}$. For any S-point $\underline{x}$ of Ran ($\mathrm{S} \in \mathsf{Sch}$), we have a diagram of indschemes:

$$\begin{tikzcd} & \mathrm{Gr}_{\mathrm{P},\underline{x}} \arrow[dl, "q"'] \arrow[dr, "p"] & \\ \mathrm{Gr}_{\mathrm{G},\underline{x}} & & \mathrm{Gr}_{\mathrm{M},\underline{x}} \end{tikzcd}$$

1.5.2 *Definition of* CT_{P}**.** Since the pullback of μ along $\mathrm{P} \subset \mathrm{G}$ canonically descends to M, we have an isomorphism of A-gerbes over $\mathrm{Gr}_{\mathrm{P},\underline{x}}$ (already over $\mathrm{Hec}_{\mathrm{P},\underline{x}}$, in fact):

$$q^* \mathcal{G}_{\mathrm{Gr}_{\mathrm{G}}} \simeq p^* \mathcal{G}_{\mathrm{Gr}_{\mathrm{M}}}, \tag{1.46}$$

which we may use to form the functor

$$q_! p^* : \mathsf{D}_{\mathcal{G},\zeta}(\mathrm{Gr}_{\mathrm{G},\underline{x}}) \to \mathsf{D}_{\mathcal{G},\zeta}(\mathrm{Gr}_{\mathrm{M},\underline{x}}). \tag{1.47}$$

We shall "normalize" (1.47) as follows. Over an étale cover of S, we may split M and choose a Borel subgroup $\mathrm{B}_{\mathrm{M}} \subset \mathrm{M}$, with induced Borel $\mathrm{B} := \mathrm{P} \times_{\mathrm{M}} \mathrm{B}_{\mathrm{M}}$ of G. Denote by T the maximal quotient torus of B_{M}, so T is the abstract Cartan of both G and M. Denote by $2\check{\rho}_{\mathrm{G}}$ (respectively $2\check{\rho}_{\mathrm{M}}$) the sum of positive roots of G (respectively M). We obtain a locally constant function

$$d_{\mathrm{G},\mathrm{M}} : \mathrm{Gr}_{\mathrm{M},\underline{x}} \to \mathbf{Z} \tag{1.48}$$

taking value $\langle 2\check{\rho}_{\mathrm{G}} - 2\check{\rho}_{\mathrm{M}}, \theta \rangle$ over the connected component corresponding to $\theta \in \pi_1 \mathrm{M}$. This expression is independent of the choice of B_{M}, so (1.48) descends to S.

We define the *constant term functor* to be

$$\mathrm{CT}_{\mathrm{P}} := q_! p^* (\frac{d_{\mathrm{G},\mathrm{M}}}{2})[d_{\mathrm{G},\mathrm{M}}], \tag{1.49}$$

using the fixed half-integral Tate twist (*cf.* §1.5.1).

Remark 1.5.3. The formation of (1.49) respects composition in the following sense. Given a parabolic subgroup $\mathrm{P}_1 \subset \mathrm{M}$ with Levi quotient $\mathrm{P}_1 \twoheadrightarrow \mathrm{M}_1$, we have an isomorphism

$$\mathrm{CT}_{\mathrm{P} \times_{\mathrm{M}} \mathrm{P}_1} \simeq \mathrm{CT}_{\mathrm{P}_1} \circ \mathrm{CT}_{\mathrm{P}},$$

where $\mathrm{P} \times_{\mathrm{M}} \mathrm{P}_1$ is viewed as a parabolic subgroup of G. This follows from base change.

Remark 1.5.4. The functor (1.49) carries $\mathrm{L}^+_{\underline{x}}\mathrm{G}$-equivariant objects to $\mathrm{L}^+_{\underline{x}}\mathrm{M}$-equivariant objects. Indeed, it suffices to show that $p_!$ carries $\mathrm{L}^+_{\underline{x}}\mathrm{P}$-equivariant objects to $\mathrm{L}^+_{\underline{x}}\mathrm{M}$-equivariant objects, and this holds because the kernel of $\mathrm{L}^+_{\underline{x}}\mathrm{P} \to \mathrm{L}^+_{\underline{x}}\mathrm{M}$ is pro-unipotent.

Therefore, (1.49) induces a functor

$$\mathrm{CT}_{\mathrm{P}} : \mathsf{D}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\underline{x}}) \to \mathsf{D}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{M},\underline{x}}).$$

1.5.5. Next, we prove some basic properties of the constant term functor (1.49). Our main ingredient is a twisted version of Braden's hyperbolic localization theorem, which we establish over a general base scheme in §B.3, following Richarz's strategy in [Ric19].

Proposition 1.5.6. *Let* $\mathcal{A}$ *be an* $\mathrm{L}^+_{\underline{x}}\mathrm{G}$*-equivariant object of* $\mathsf{D}_{\mathcal{G},\zeta}(\mathrm{Gr}_{\mathrm{G},\underline{x}})$.

(1) $\mathcal{A}$ *vanishes if and only if* $\mathrm{CT}_{\mathrm{P}}(\mathcal{A})$ *does;*

(2) $\mathcal{A}$ *is ULA relative to* S *if and only if* $\mathrm{CT}_{\mathrm{P}}(\mathcal{A})$ *is;*

(3) $\mathcal{A}$ *is connective (respectively, coconnective) in the perverse t-structure relative to* S *if and only if* $\mathrm{CT}_{\mathrm{P}}(\mathcal{A})$ *is.*

Proof. All statements are of étale local nature over S, so we may fix a split maximal torus and a Borel subgroup $\mathrm{T} \subset \mathrm{B} \subset \mathrm{G}$, realizing P as a standard parabolic of G. Since CT_{P} respects composition (*cf.* Remark 1.5.3), we further reduce to the case $\mathrm{P} = \mathrm{B}$.

Consider the $\mathbb{G}_m$-action on G by conjugation by a regular dominant cocharacter. It has attractor B, repeller B^- (the opposite Borel), and fixed point locus T (*cf.* §B.3.1). The induced $\mathbb{G}_m$-action on $\mathrm{Gr}_{\mathrm{G},\underline{x}}$ has attractor $\mathrm{Gr}_{\mathrm{B},\underline{x}}$, repeller $\mathrm{Gr}_{\mathrm{B}^-,\underline{x}}$, and fixed point locus $\mathrm{Gr}_{\mathrm{T},\underline{x}}$. In particular, (B.10) leads to the following diagram of S-indschemes

$$\begin{tikzcd}
 & \mathrm{Gr}_{\mathrm{B},\underline{x}} \arrow[dl, "q^+"'] \arrow[dr, shift left, "p^+"] & \\
\mathrm{Gr}_{\mathrm{G},\underline{x}} & & \mathrm{Gr}_{\mathrm{T},\underline{x}} \arrow[ul, shift left, "i^+"] \arrow[dl, shift right, "i^-"'] \\
 & \mathrm{Gr}_{\mathrm{B}^-,\underline{x}} \arrow[ul, "q^-"] \arrow[ur, shift right, "p^-"'] &
\end{tikzcd} \tag{1.50}$$

and we have a canonical isomorphism in $\mathsf{D}_{\mathcal{G},\zeta}(\mathrm{Gr}_{\mathrm{T},\underline{x}})$ (*cf.* Theorem B.3.4)

$$\mathrm{CT}_{\mathrm{B}}(\mathcal{A}) \simeq (p^-)_*(q^-)^!\mathcal{A}(\frac{d_{\mathrm{G},\mathrm{T}}}{2})[d_{\mathrm{G},\mathrm{T}}]. \tag{1.51}$$

To prove statement (1), we may use Lemma B.4.2 to reduce to the case $\mathrm{S} = \operatorname{Spec} \bar{\mathsf{k}}$. Using the factorization structure of $(\mathrm{Gr}_{\mathrm{G}}, \mathcal{G}_{\mathrm{Gr}_{\mathrm{G}}})$ (*cf.* Proposition 1.2.11), we may further assume that $\underline{x}$ factors through a $\bar{\mathsf{k}}$-point x of X. The implication $\mathrm{CT}_{\mathrm{B}}(\mathcal{A}) \simeq 0 \Rightarrow \mathcal{A} \simeq 0$ holds, because every $\mathrm{L}^+_x\mathrm{G}$-orbit in $\mathrm{Gr}_{\mathrm{G},x}$ meets the image of *some* connected component of $\mathrm{Gr}_{\mathrm{B},x}$ at a single $\bar{\mathsf{k}}$-point (*cf.* [MV07, Proof of Theorem 3.2]).

To prove statement (2), we may use Proposition B.4.3 to handle the "⇒" direction. For the "⇐" direction, we assume that $\mathrm{CT}_{\mathrm{B}}(\mathcal{A})$ is ULA and prove that the canonical morphism below is an isomorphism:

$$\mathbf{D}(\mathcal{A}) \boxtimes \mathcal{A} \to \mathcal{H}om(\overleftarrow{\pi}^*\mathcal{A}, \overrightarrow{\pi}^!\mathcal{A}). \tag{1.52}$$

Here, $\mathbf{D}$ stands for Verdier duality over $\mathrm{Gr}_{\mathrm{G},\underline{x}}$ relative to S and $\overleftarrow{\pi}$, $\overrightarrow{\pi}$ are the two projections from $\mathrm{Gr}_{\mathrm{G},\underline{x}} \times_{\mathrm{S}} \mathrm{Gr}_{\mathrm{G},\underline{x}}$ to $\mathrm{Gr}_{\mathrm{G},\underline{x}}$ (*cf.* §B.1.10).

By statement (1), it suffices to prove that (1.52) becomes an isomorphism after applying $\mathrm{CT}_{\mathrm{B}^-\times\mathrm{B}}$, the constant term functor associated to $\mathrm{G} \times \mathrm{G}$, the étale level $(-\mu, \mu)$, and the parabolic subgroup $\mathrm{B}^- \times \mathrm{B}$. By Lemma B.4.2 and Proposition B.4.5, the image of (1.52) under $\mathrm{CT}_{\mathrm{B}^-\times\mathrm{B}}$ may be identified with

$$\mathbf{D}(\mathrm{CT}_{\mathrm{B}}\,\mathcal{A}) \boxtimes \mathrm{CT}_{\mathrm{B}}(\mathcal{A}) \to \mathcal{H}om(\overleftarrow{\pi}^*\,\mathrm{CT}_{\mathrm{B}}\,\mathcal{A}, \overrightarrow{\pi}^!\,\mathrm{CT}_{\mathrm{B}}\,\mathcal{A}),$$

which is an isomorphism because $\mathrm{CT}_{\mathrm{B}}(\mathcal{A})$ is ULA.

To prove statement (3), we observe that the "⇒" direction follows from (1.51), as the left-hand-side is right t-exact while the right-hand-side is left t-exact, and the "⇐" follows from the "⇒" direction and statement (1). □

1.5.7. By Proposition 1.5.6, (1.49) restricts to a functor on the Satake categories:

$$\mathrm{CT_P} : \mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\underline{x}}) \to \mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{M},\underline{x}}). \tag{1.53}$$

When $\underline{x}$ is given by $\mathrm{X^I} \to \mathrm{Ran}$ for a finite set I, let us incorporate the symmetric monoidal structure (*cf.* §1.3.16) and write (1.53) as a functor

$$\mathrm{CT_P} : {}^{+}\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\mathrm{I}}) \to {}^{+}\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{M},\mathrm{I}}). \tag{1.54}$$

Lemma 1.5.8. *The functor* (1.54) *is naturally symmetric monoidal.*

Proof. This follows from the classical argument (*cf.* [MV07, Proposition 6.4]), as the cohomology shift $[d_{\mathrm{G,M}}]$ in the definition of (1.49) is precisely accounted for by the normalization of the commutativity constraints (*cf.* §1.3.16). □

1.6. Verdier duality.

1.6.1. We remain in the context of §1.1.1 and §1.3.1 and assume that G is reductive.

Fix an S-point $\underline{x}$ of Ran ($\mathrm{S} \in \mathsf{Sch}$). The goal of this subsection is to use the Verdier duality functor over $\mathrm{Gr}_{\mathrm{G},\underline{x}}$ relative to S to construct monoidal duals in $\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\underline{x}})$ with respect to the convolution monoidal structure (*cf.* §1.3.10).

This construction is known in the untwisted setting (*cf.* [MV07, §11], [FS24, Proposition VI.8.2]) and its adaptation to the twisted setting is straightforward.

1.6.2. First, we note that Verdier duality (*cf.* §B.1.10) yields an equivalence

$$\mathbf{D} : \mathsf{D}_{\mathcal{G},\zeta}(\mathrm{Gr}_{\mathrm{G},\underline{x}}) \simeq \mathsf{D}_{-\mathcal{G},\zeta}(\mathrm{Gr}_{\mathrm{G},\underline{x}}). \tag{1.55}$$

preserving perverse ULA objects. After restricting to the full subcategories of perverse ULA objects, it also preserves the property of being constant along $\mathrm{L}^{+}_{\underline{x}}\mathrm{G}$-orbits. These properties also hold for the inverse of (1.55), being itself a Verdier duality functor.

Therefore, (1.55) restricts to an equivalence (*cf.* Remark 1.3.8)

$$\mathbf{D} : \mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\underline{x}}) \simeq \mathsf{Sat}_{-\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\underline{x}}). \tag{1.56}$$

1.6.3. On the other hand, consider the *swap* automorphism

$$\mathrm{sw} : \mathrm{Hec}_{\mathrm{G},\underline{x}} \to \mathrm{Hec}_{\mathrm{G},\underline{x}},$$
$$(\mathrm{P}^0 \overset{x}{\simeq} \mathrm{P}^1) \mapsto (\mathrm{P}^1 \overset{x}{\simeq} \mathrm{P}^0).$$

By construction of $\mathcal{G}_{\mathrm{Hec_G}}$, the pullback $\mathrm{sw}^*(-\mathcal{G}_{\mathrm{Hec_G}})$ is canonically identified with $\mathcal{G}_{\mathrm{Hec_G}}$, so sw^* induces an equivalence

$$\mathrm{sw}^* : \mathsf{Sat}_{-\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\underline{x}}) \simeq \mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\underline{x}}). \tag{1.57}$$

Proposition 1.6.4. *For any* $\mathcal{A} \in \mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\underline{x}})$*, the object* $\mathrm{sw}^*\mathbf{D}(\mathcal{A})$ *is a left dual of* $\mathcal{A}$ *with respect to the convolution monoidal structure.*

Proof. We need to exhibit the unit and co-unit morphisms

$$\mathbf{1} \to \mathrm{sw}^*\mathbf{D}(\mathcal{A}) \circ \mathcal{A} \tag{1.58}$$

$$\mathrm{sw}^*\mathbf{D}(\mathcal{A}) \circ \mathcal{A} \to \mathbf{1}, \tag{1.59}$$

where $\mathbf{1} := e_!(\mathsf{e})$ is the monoidal unit (*cf.* Proposition 1.3.9), and show that they obey the adjunction property.

In order to do so, we shall use the Cartesian square:

$$\begin{tikzcd}
\mathrm{Hec}_{\mathrm{G},\underline{x}} \arrow[r, "\delta"] \arrow[d, "t"] & \mathrm{Hec}^{[2]}_{\mathrm{G},\underline{x}} \arrow[d, "m"] \\
\mathrm{L}^+_{\underline{x}}\mathsf{B}\mathrm{G} \arrow[r, "e"] & \mathrm{Hec}_{\mathrm{G},\underline{x}}
\end{tikzcd} \tag{1.60}$$

where m is the multiplication map (*cf.* §1.3.5) and t, δ sends $\mathrm{P}^0 \overset{\underline{x}}{\sim} \mathrm{P}^1$ to P^1, respectively the concatenation with its own inverse $\mathrm{P}^1 \overset{\underline{x}}{\sim} \mathrm{P}^0 \overset{\underline{x}}{\sim} \mathrm{P}^1$. Note that if we express $\mathrm{Hec}^{[2]}_{\mathrm{G},\underline{x}}$ as the fiber product $\mathrm{Hec}_{\mathrm{G},\underline{x}} \times_{\mathrm{L}^+_{\underline{x}}\mathsf{B}\mathrm{G}} \mathrm{Hec}_{\mathrm{G},\underline{x}}$ along the projections p_1, p_2, then δ corresponds to the morphism $(\mathrm{sw}, \mathrm{id})$.

To construct (1.58), we express the right-hand-side as $m_*(p_1^*\mathrm{sw}^*\mathbf{D}(\mathcal{A}) \otimes p_2^*\mathcal{A})$. By adjunction and base change along (1.60), it suffices to construct a global section of the !-restriction of $\mathbf{D}(\mathcal{A}) \boxtimes \mathcal{A}$ along the diagonal of the morphism $i : \mathrm{Hec}_{\mathrm{G},\underline{x}} \to \mathrm{L}^+_{\underline{x}}\mathsf{B}\mathrm{G}$ sending $\mathrm{P}^0 \overset{\underline{x}}{\sim} \mathrm{P}^1$ to P^0. However, our assumption on $\mathcal{A}$ implies that it is ULA with respect to i. Thus this !-restriction is isomorphic to $\mathcal{H}om(\mathcal{A}, \mathcal{A})$ (*cf.* Remark B.1.11). The required global section is the identity.

To construct (1.59), we perform a similar analysis and reduce to constructing a morphism $\mathbf{D}(\mathcal{A}) \otimes \mathcal{A} \to i^!(\mathsf{e})$, which is given by the co-unit of Verdier duality. We omit verifying the adjunction property satisfied by (1.58) and (1.59). ☐

2. The Satake equivalence

The goal of this section is to formulate the twisted geometric Satake equivalence (*cf.* Theorem 2.3.2). We define metaplectic dual data in §2.1 and use it to construct the dual category of "twisted representations" in §2.2. We state the equivalence in §2.3 and list a few of its properties, which will be established in the course of its proof.

Prior works on the twisted geometric Satake equivalence include [Lys06, FL10, Lys14, Rei12, GL18]. We compare our statement to the literature in Appendix C.

2.1. The dual data.

2.1.1. We remain in the context of §1.1.1 and assume in addition that G is reductive. We shall construct a pair (H, ν), which we call *metaplectic dual data*, where

(1) H is a locally constant étale sheaf over X of pinned split reductive group $\mathbf{Z}$-schemes;
(2) $\nu : \hat{\mathrm{Z}}_{\mathrm{H}} \to \mathsf{B}^2_{\mathrm{X}}\mathrm{A}$ is morphism of étale sheaves over X of $\mathbb{E}_\infty$-monoids.

Here, Z_{H} denotes the center of H and $\hat{\mathrm{Z}}_{\mathrm{H}}$ its character group, viewed as a (locally constant) étale sheaf of abelian groups over X.

The construction of (H, ν) is insensitive to the geometry of X, so up to §2.1.20, we shall replace X by an arbitrary scheme S over which A has invertible order.

2.1.2. We first fix some notation asssociated to the reductive group G.

(1) G_{sc} (respectively G_{ad}) is the simply connected (respectively adjoint) form G. The kernel of $\mathrm{G} \to \mathrm{G}_{\mathrm{ad}}$ is the center Z, and the stack quotient $\mathrm{G}/\mathrm{G}_{\mathrm{sc}}$ is the cocenter G_{ab} of G, *cf.* [SZ25, §1.1.5].

(2) T (respectively T_{sc}, T_{ad}) is the universal Cartan of G (respectively G_{sc}, G_{ad}). The locally constant étale sheaf of *cocharacters* of T (respectively T_{sc}, T_{ad}) is denoted by Λ (respectively Λ_{sc}, Λ_{ad}), with dual $\check{\Lambda}$ (respectively $\check{\Lambda}_{\mathrm{sc}}$, $\check{\Lambda}_{\mathrm{ad}}$).

In particular, we have

$$\mathrm{Z} \simeq \mathrm{Fib}(\Lambda \to \Lambda_{\mathrm{ad}}) \otimes_{\mathbf{Z}} \mathbb{G}_m$$
$$\mathrm{G}_{\mathrm{ab}} \simeq \pi_1\mathrm{G} \otimes_{\mathbf{Z}} \mathbb{G}_m \text{ for } \pi_1\mathrm{G} := \Lambda/\Lambda_{\mathrm{sc}}.$$

(3) $\Delta \subset \Lambda$ (respectively $\check{\Delta} \subset \check{\Lambda}$) is the subsheaf of simple coroots (respectively simple roots). Then Δ spans Λ_{sc} and $\check{\Delta}$ spans $\check{\Lambda}_{\mathrm{ad}}$.

2.1.3. Recall that we have a canonical fiber sequence of étale sheaves of **Z**-linear spaces[11] over S (*cf.* [Zha22, Proposition 5.1.11])

$$\mathcal{M}aps_{\mathbf{Z}}(\pi_1\mathrm{G}, \mathsf{B}^2\mathrm{A}) \to \mathcal{M}aps_*(\mathsf{B}\mathrm{G}, \mathsf{B}^4\mathrm{A}(1)) \to \mathcal{Q}uad(\Lambda, \mathrm{A}(-1))_{\mathrm{st}}, \tag{2.1}$$

where $\mathcal{M}aps_{\mathbf{Z}}(\cdot,\cdot)$ (respectively $\mathcal{M}aps_*(\cdot,\cdot)$) is the étale sheaf of mapping spaces of sheaves of **Z**-linear spaces (respectively pointed spaces), and $\mathcal{Q}uad(\Lambda, \mathrm{A}(-1))_{\mathrm{st}}$ denotes the étale sheaf of *strictly Weyl-invariant* quadratic forms $\Lambda \to \mathrm{A}(-1)$ (to be recalled in §2.1.5).

The first map in (2.1) is given by tensoring with $\mathbb{G}_m$ and pulling back along $\mathrm{G} \to \mathrm{G}_{\mathrm{ab}}$. The second map in (2.1) is induced from the canonical isomorphism

$$\pi_0\mathcal{M}aps_*(\mathsf{B}\mathrm{G}, \mathsf{B}^4\mathrm{A}(1)) \simeq \mathcal{Q}uad(\Lambda, \mathrm{A}(-1))_{\mathrm{st}}.$$

Remark 2.1.4. For $\mathrm{G} = \mathbb{G}_m$, the fiber sequence (2.1) reads

$$\mathsf{B}^2\mathrm{A} \to \mathcal{M}aps_*(\mathsf{B}\mathbb{G}_m, \mathsf{B}^4\mathrm{A}(1)) \to \mathrm{A}(-1). \tag{2.2}$$

This fiber sequence admits a canonical splitting, sending $a \in \mathrm{A}(-1)$ to $a \otimes (\Psi \otimes \Psi)$, where $\Psi : \mathsf{B}\mathbb{G}_m \to \mathsf{B}^2\hat{\mathbf{Z}}(1)$ is the Kummer morphism (*cf.* [Zha22, Remark 4.2.8]).

2.1.5. By definition, the étale level μ is a global section of $\mathcal{M}aps_*(\mathsf{B}\mathrm{G}, \mathsf{B}^4\mathrm{A}(1))$. Let us use (2.1) to fix notation associated to μ.

(1) Q is the strictly Weyl-invariant quadratic form $\Lambda \to \mathrm{A}(-1)$ associated to μ, *i.e.* its image under the second map of (2.1);

(2) b is the symmetric form on Λ associated to Q, defined by the identity

$$b(\lambda_1, \lambda_2) := \mathrm{Q}(\lambda_1 + \lambda_2) - \mathrm{Q}(\lambda_1) - \mathrm{Q}(\lambda_2).$$

The condition of strict Weyl-invariance on Q means that the equality below holds for any $\alpha \in \Delta$ and $\lambda \in \Lambda$:

$$b(\alpha, \lambda) = \mathrm{Q}(\alpha)\langle\check{\alpha}, \lambda\rangle. \tag{2.3}$$

2.1.6 *Construction of* H**.** Denote by $\Lambda^\sharp \subset \Lambda$ the kernel of b and by $\check{\Lambda}^\sharp$ its dual, which we may view as a lattice inside $\check{\Lambda} \otimes_{\mathbf{Z}} \mathbf{Q}$. For each $\alpha \in \Delta$, we write

$$\alpha^\sharp := \mathrm{ord}(\mathrm{Q}(\alpha)) \cdot \alpha,$$
$$\check{\alpha}^\sharp := \mathrm{ord}(\mathrm{Q}(\alpha))^{-1} \cdot \check{\alpha},$$

where $\mathrm{ord}(\mathrm{Q}(\alpha))$ is the order of $\mathrm{Q}(\alpha) \in \mathrm{A}(-1)$.

It follows from strict Weyl-invariance (2.3) that $\alpha^\sharp$ belongs to $\Lambda^\sharp$ and $\check{\alpha}^\sharp$ belongs to $\check{\Lambda}^\sharp$. Furthermore, the quadruple $(\{\alpha^\sharp\} \subset \Lambda^\sharp, \{\check{\alpha}^\sharp\} \subset \check{\Lambda}^\sharp)$, together with the correspondence $\alpha^\sharp \leftrightarrow \check{\alpha}^\sharp$ for each $\alpha \in \Delta$, defines a locally constant étale sheaf of root data.

---

[11] Recall that a "**Z**-linear space" means a connective H**Z**-module spectrum (*cf.* §0.4.2).

We write H for the locally constant étale sheaf of pinned split reductive group $\mathbf{Z}$-schemes with root data $(\{\alpha^\sharp\} \subset \Lambda^\sharp, \{\check{\alpha}^\sharp\} \subset \check{\Lambda}^\sharp)$. (Here, $\{\alpha^\sharp\}$ are the simple *roots* of H, *etc.*)

Remark 2.1.7. By definition of H, the étale sheaf $\hat{\mathrm{Z}}_{\mathrm{H}}$ may be realized as the quotient

$$\hat{\mathrm{Z}}_{\mathrm{H}} \simeq \Lambda^\sharp / \Lambda^\sharp_{\mathrm{sc}}, \tag{2.4}$$

where $\Lambda^\sharp_{\mathrm{sc}}$ is the $\mathbf{Z}$-linear span of $\alpha^\sharp$ $(\alpha \in \Delta)$.

2.1.8 *The form* b_1**.** For later purposes, we shall also need the bilinear form

$$b_1 : \Lambda_{\mathrm{sc}} \otimes \Lambda_{\mathrm{ad}} \to \mathrm{A}(-1) \tag{2.5}$$

defined as the $\mathbf{Z}$-linear extension of the expression (2.3) for $\alpha \in \Delta$ and $\lambda \in \Lambda_{\mathrm{ad}}$, where we interpret $\langle \cdot, \cdot \rangle$ as the canonical pairing between Λ_{ad} and the root lattice.

Remark 2.1.9. The form b_1 provides us another interpretation of the sublattice $\Lambda^\sharp_{\mathrm{sc}} \subset \Lambda_{\mathrm{sc}}$ (*cf.* Remark 2.1.7). Namely, it is the (one-sided) kernel of b_1.

To see this, we write an arbitrary element of Λ_{sc} as a sum $\sum_{\alpha \in \Delta} d_\alpha \alpha$ for $d_\alpha \in \mathbf{Z}$. This element belongs to the kernel of b_1 if and only if its pairing with every fundamental coweight $\omega_\alpha \in \Lambda_{\mathrm{ad}}$ $(\alpha \in \Delta)$ vanishes. However, this pairing is computed by $\mathrm{Q}(\alpha) d_\alpha$, so it vanishes if and only if d_α is a multiple of $\operatorname{ord} \mathrm{Q}(\alpha)$.

2.1.10. For the moment, let us assume that G is split and choose a Borel subgroup $\mathrm{B} \subset \mathrm{G}$ as well as a splitting $\mathrm{T} \subset \mathrm{B}$ of the projection $\mathrm{B} \twoheadrightarrow \mathrm{T}$.

Pulling back μ along $\mathrm{T}^\sharp \to \mathrm{T} \to \mathrm{G}$, for $\mathrm{T}^\sharp := \Lambda^\sharp \otimes_{\mathbf{Z}} \mathbb{G}_m$, yields an $\mathbb{E}_\infty$-monoidal morphism $\mu_{\mathrm{T}^\sharp} : \mathsf{B}\mathrm{T}^\sharp \to \mathsf{B}^4 \mathrm{A}(1)$ (*cf.* [Zha22, Proposition 4.6.2]). Taking sections over $\mathsf{B}\mathbb{G}_m$ then yields an $\mathbb{E}_\infty$-monoidal morphism

$$\begin{aligned} \Lambda^\sharp &\simeq \mathcal{M}aps_*(\mathsf{B}\mathbb{G}_m, \mathsf{B}\mathrm{T}^\sharp) \\ &\to \mathcal{M}aps_*(\mathsf{B}\mathbb{G}_m, \mathsf{B}^4\mathrm{A}(1)) \simeq \mathsf{B}^2\mathrm{A} \oplus \mathrm{A}(-1), \end{aligned} \tag{2.6}$$

where the last isomorphism is given by the canonical splitting of (2.2). Projecting (2.6) onto the first factor yields an $\mathbb{E}_\infty$-monoidal morphism

$$\Lambda^\sharp \to \mathsf{B}^2\mathrm{A}, \tag{2.7}$$

while projecting it onto the second factor yields the restriction of Q to $\Lambda^\sharp$.

In Lemma 2.1.14 below, we shall endow (2.7) with a canonical equivariance structure with respect to the Weyl-action on $\Lambda^\sharp$.

2.1.11 G-*equivariance of* μ**.** Consider the G-action on itself by inner automorphisms. It induces a G-action on the pointed stack $\mathsf{B}\mathrm{G}$. This construction is functorial in sheaves of $\mathbb{E}_1$-monoids. Applying to the loop space $\Omega\mu : \mathrm{G} \to \mathsf{B}^3\mathrm{A}(1)$ of μ, we see that

$$\mu : \mathsf{B}\mathrm{G} \to \mathsf{B}^4\mathrm{A}(1) \tag{2.8}$$

is equivariant with respect to the G-, and $\mathsf{B}^3\mathrm{A}(1)$-actions via $\Omega\mu$.

Moreover, using the $\mathbb{E}_\infty$-monoid structure on $\mathsf{B}^3\mathrm{A}(1)$, we may trivialize the $\mathsf{B}^3\mathrm{A}(1)$-action on itself by inner automorphisms, obtaining a trivialization of the induced $\mathsf{B}^3\mathrm{A}(1)$-action on $\mathsf{B}^4\mathrm{A}(1)$. This shows that (2.8) factors through a morphism

$$\widetilde{\mu} : (\mathsf{B}\mathrm{G})/\mathrm{G} \to \mathsf{B}^4\mathrm{A}(1). \tag{2.9}$$

In other words, μ admits a canonical G-equivariance structure.[12]

---

[12] Informally, if one thinks of μ as a "central extension" E of G by $\mathsf{B}^2\mathrm{A}(1)$, then this G-equivariance structure corresponds to an extension of the conjugation action of G along $\mathrm{E} \to \mathrm{G}$.

Remark 2.1.12. Let us point out a confusing aspect of (2.9). Indeed, the G-action on $\mathsf{B}\mathrm{G}$ as an étale stack is canonically trivial: Given a G-bundle P and its twist $g(\mathrm{P}) := \mathrm{P}\times^{\mathrm{G}}\mathrm{G}$ by a point g of G (where we use the conjugation map $\mathrm{G}\to\mathrm{G}$, $h\mapsto ghg^{-1}$ in the formation of $\mathrm{P}\times^{\mathrm{G}}\mathrm{G}$), we have the canonical isomorphism of G-bundles

$$\begin{aligned} \mathrm{P} &\simeq g(\mathrm{P}), \\ p &\mapsto (p,g). \end{aligned} \tag{2.10}$$

This is *not* a trivialization of the G-action on $\mathsf{B}\mathrm{G}$ as a *pointed* stack.

Nevertheless, let us use this trivialization to express the isomorphism $\mu(g(\mathrm{P}))\simeq\mu(\mathrm{P})$ provided by (2.9). Namely, by identifying its source with $\mu(\mathrm{P})$ using (2.10), then this becomes the automorphism of $\mu(\mathrm{P})$ defined by the section $(\Omega\mu)(g)$ of $\mathsf{B}^3\mathrm{A}(1)$.

2.1.13 *Weyl-equivariance***.** Denote by $\mathrm{N}_{\mathrm{G}}(\mathrm{T})$ the normalizer of T in G and by $\mathrm{W} := \mathrm{N}_{\mathrm{G}}(\mathrm{T})/\mathrm{T}$ the Weyl group. The natural W-action on Λ induces a W-action on $\Lambda^\sharp$. The restriction of (2.9) along $\mathrm{T}^\sharp\to\mathrm{G}$ and $\mathrm{N}_{\mathrm{G}}(\mathrm{T})\subset\mathrm{G}$ yields an extension of $\mu_{\mathrm{T}^\sharp}$:

$$\widetilde{\mu}_{\mathrm{T}^\sharp} : (\mathsf{B}\mathrm{T}^\sharp)/\mathrm{N}_{\mathrm{G}}(\mathrm{T}) \to \mathsf{B}^4\mathrm{A}(1). \tag{2.11}$$

The following result shows that (2.11) equips $\mu_{\mathrm{T}^\sharp}$ with a W-equivariance structure.

Lemma 2.1.14. *The morphism* (2.11) *canonically factors through* $(\mathsf{B}\mathrm{T}^\sharp)/\mathrm{W}$.

Proof. Consider the Cartesian diagram of étale stacks

$$\begin{tikzcd} \mathsf{B}\mathrm{T} \arrow[r] \arrow[d] & \mathsf{B}\mathrm{N}_{\mathrm{G}}(\mathrm{T}) \arrow[d, "p"] \\ \mathrm{S} \arrow[r] & \mathsf{B}\mathrm{W} \end{tikzcd} \tag{2.12}$$

and the étale sheaf $\mathcal{M}aps_*(\mathsf{B}\mathrm{T}^\sharp,\mathsf{B}^4\mathrm{A}(1))$ over $\mathsf{B}\mathrm{W}$. The unit map

$$\mathcal{M}aps_*(\mathsf{B}\mathrm{T}^\sharp,\mathsf{B}^4\mathrm{A}(1)) \to p_*p^*\mathcal{M}aps_*(\mathsf{B}\mathrm{T}^\sharp,\mathsf{B}^4\mathrm{A}(1))$$

induces a fiber sequence by taking global sections:

$$\mathrm{Maps}_*((\mathsf{B}\mathrm{T}^\sharp)/\mathrm{W},\mathsf{B}^4\mathrm{A}(1)) \to \mathrm{Maps}_*((\mathsf{B}\mathrm{T}^\sharp)/\mathrm{N}_{\mathrm{G}}(\mathrm{T}),\mathsf{B}^4\mathrm{A}(1)) \to \mathrm{Hom}(\Lambda\otimes\Lambda^\sharp,\mathrm{A}(-1))^{\mathrm{W}}. \tag{2.13}$$

Here, the third term arises by base change along (2.12): The complex corresponding to $\mathcal{M}aps_*(\mathsf{B}\mathrm{T}^\sharp,\mathsf{B}^4\mathrm{A}(1))$ is concentrated in degrees ≥ -2, with cohomology $\mathcal{H}om(\Lambda^\sharp,\mathrm{A})$ in the lowest degree (*cf.* §2.1.3), while reduced cohomology of $\mathsf{B}\mathrm{T}$ is concentrated in degrees ≥ 2, with $\mathrm{Hom}(\Lambda,\cdot)$ in the lowest degree.

From (2.13), we see that W-equivariant pointed morphisms $\mathsf{B}\mathrm{T}^\sharp\to\mathsf{B}^4\mathrm{A}(1)$ form a full subcategory of $\mathrm{N}_{\mathrm{G}}(\mathrm{T})$-equivariant ones. It remains to show that the image of (2.11) in $\mathrm{Hom}(\Lambda\otimes\Lambda^\sharp,\mathrm{A}(-1))^{\mathrm{W}}$ vanishes. For this, we may restrict (2.11) along $\mathsf{B}\mathrm{T}\to\mathsf{B}\mathrm{N}_{\mathrm{G}}(\mathrm{T})$ and identify the resulting morphism

$$\widetilde{\mu}_{\mathrm{T}^\sharp} : (\mathsf{B}\mathrm{T}^\sharp)/\mathrm{T} \to \mathsf{B}^4\mathrm{A}(1) \tag{2.14}$$

as a family of pointed morphisms $\mathsf{B}\mathrm{T}^\sharp\to\mathsf{B}^4\mathrm{A}(1)$ parametrized by $\mathsf{B}\mathrm{T}$. By the commutator computation of [Zha22, §5.2], (2.14) is the sum of the constant family $\mu_{\mathrm{T}^\sharp}$ with the morphism

$$b\otimes(\Psi\boxtimes\Psi) : \mathsf{B}\mathrm{T}\times\mathsf{B}\mathrm{T}^\sharp \to \mathsf{B}^4\mathrm{A}(1),$$

which vanishes because $\Lambda^\sharp$ is the kernel of b. This implies that (2.14) is the constant family $\mu_{\mathrm{T}^\sharp}$ over $\mathsf{B}\mathrm{T}$, so its class in $\mathrm{Hom}(\Lambda\otimes\Lambda^\sharp,\mathrm{A}(-1))$ vanishes. ☐

Remark 2.1.15. The W-equivariant $\mathbb{E}_\infty$-monoidal morphism (2.7) is canonically assigned to G, *i.e.* independent of the choice of B and the splitting of $\mathrm{B} \twoheadrightarrow \mathrm{T}$. While this can be verified directly following [Zha22, §5.2.6], let us offer a cleaner treatment.

Denote by Torel the S-scheme parametrizing subgroup schemes $\mathrm{T} \subset \mathrm{B} \subset \mathrm{G}$, where B is a Borel subgroup and T is a maximal torus of G. Any choice of a base point $e : \mathrm{S} \to \mathrm{Torel}$ realizes Torel as isomorphic to G/T. Thus, the geometric fibers of the structural map

$$\pi : \mathrm{Torel} \to \mathrm{S} \tag{2.15}$$

have the cohomology of the flag variety of G. From this, we deduce that for any complex of torsion étale sheaves of invertible order over S concentrated in degrees ≥ -2, the cofiber of the pullback map

$$\pi^* : \Gamma(\mathrm{S}, \mathcal{A}) \to \Gamma(\mathrm{Torel}, \mathcal{A}) \tag{2.16}$$

is concentrated in degrees ≥ 0. Furthermore, the choice of a base point e realizes H^0 of the cofiber of (2.16) as $\mathrm{H}^2(\mathrm{Torel} \bmod e, \mathrm{H}^{-2}\mathcal{A})$.

Let us perform the construction of the W-equivariant $\mathbb{E}_\infty$-monoidal morphism (2.7) with respect to the *universal* Borel and maximal torus over Torel. This yields a W-equivariant $\mathbb{E}_\infty$-monoidal morphism

$$\mu_{\mathrm{T}^\sharp} : \mathsf{B}_{\mathrm{Torel}}\mathrm{T}^\sharp \simeq \mathrm{Torel} \times \mathsf{B}\mathrm{T}^\sharp \to \mathsf{B}^4\mathrm{A}(1). \tag{2.17}$$

It remains to show that (2.17) descends along (2.15). Note that this is a *condition* instead of additional data, by the computation of the cofiber of (2.16) above, applied to $\mathcal{A}$ the complex corresponding to $\mathcal{M}aps_{\mathbb{E}_\infty}(\mathsf{B}\mathrm{T}^\sharp, \mathsf{B}^4\mathrm{A}(1))^{\mathrm{W}}$. To check that this condition is met, we may work étale locally on S and choose a base point e of Torel, realizing the obstruction as an element of

$$\begin{aligned}\mathrm{H}^2(\mathrm{Torel} \bmod e, \mathrm{H}^{-2}\mathcal{A}) &\simeq \mathrm{H}^2(\mathrm{Torel} \bmod e, \mathrm{Hom}(\Lambda^\sharp, \mathrm{A})^{\mathrm{W}}) \\ &\simeq \mathrm{Hom}(\Lambda_{\mathrm{sc}}, \mathrm{Hom}(\Lambda^\sharp, \mathrm{A})^{\mathrm{W}}(-1)) \subset \mathrm{Hom}(\Lambda_{\mathrm{sc}} \otimes \Lambda^\sharp, \mathrm{A}(-1)).\end{aligned}$$

However, by the commutator computation of [Zha22, §5.2], this element is the restriction of the symmetric form b along $\Lambda_{\mathrm{sc}} \otimes \Lambda^\sharp \subset \Lambda \otimes \Lambda$, which vanishes by definition of $\Lambda^\sharp$.

2.1.16 G_{ad}*-equivariance of* $\mu_{\mathrm{G}_{\mathrm{sc}}}$**.** We note a variant of the construction of §2.1.11.

Consider the G_{ad}-action on G_{sc} by conjugation. It induces a G_{ad}-action on $\mathsf{B}\mathrm{G}_{\mathrm{sc}}$. Since the pullback $\mu_{\mathrm{G}_{\mathrm{sc}}}$ of μ to G_{sc} is uniquely determined by its quadratic form (*cf.* §2.1.3), it is naturally G_{ad}-equivariant, *i.e.* $\mu_{\mathrm{G}_{\mathrm{sc}}}$ factors through

$$\widetilde{\mu}_{\mathrm{G}_{\mathrm{sc}}} : (\mathsf{B}\mathrm{G}_{\mathrm{sc}})/\mathrm{G}_{\mathrm{ad}} \to \mathsf{B}^4\mathrm{A}(1). \tag{2.18}$$

Note that $(\mathsf{B}\mathrm{G}_{\mathrm{sc}})/\mathrm{G}_{\mathrm{ad}}$ is canonically identified with the classifying stack of $\mathrm{G}_{\mathrm{sc}} \rtimes \mathrm{G}_{\mathrm{ad}}$, so (2.18) may be viewed as an étale level for the reductive group scheme $\mathrm{G}_{\mathrm{sc}} \rtimes \mathrm{G}_{\mathrm{ad}}$ with universal Cartan $\mathrm{T}_{\mathrm{sc}} \times \mathrm{T}_{\mathrm{ad}}$.

Let us compute the symmetric form

$$\widetilde{b} : (\Lambda_{\mathrm{sc}} \oplus \Lambda_{\mathrm{ad}}) \otimes (\Lambda_{\mathrm{sc}} \oplus \Lambda_{\mathrm{ad}}) \to \mathrm{A}(-1) \tag{2.19}$$

associated to (2.18).

Lemma 2.1.17. *The symmetric form* (2.19) *equals the matrix*

$$\widetilde{b} = \begin{pmatrix} b_{\mathrm{sc}} & b_1 \\ (b_1)^\dagger & 0 \end{pmatrix} \tag{2.20}$$

where b_{sc} *is the restriction of* b *to* Λ_{sc} *and* $(b_1)^\dagger$ *is the transpose of* b_1.

Proof. We may choose a Borel subgroup $\mathrm{B} \subset \mathrm{G}$ and a splitting of $\mathrm{B} \twoheadrightarrow \mathrm{T}$, realizing T_{sc} (respectively T_{ad}) as a maximal torus of G_{sc} (respectively G_{ad}). The restriction of (2.18) along $\mathrm{T}_{\mathrm{sc}} \subset \mathrm{G}_{\mathrm{sc}}$, $\mathrm{T}_{\mathrm{ad}} \subset \mathrm{G}_{\mathrm{ad}}$ yields

$$\widetilde{\mu}_{\mathrm{T}_{\mathrm{sc}}} : (\mathsf{B}\mathrm{T}_{\mathrm{sc}})/\mathrm{T}_{\mathrm{ad}} \to \mathsf{B}^4\mathrm{A}(1), \tag{2.21}$$

which we may view as a family of pointed morphism $\mathsf{B}\mathrm{T}_{\mathrm{sc}} \to \mathsf{B}^4\mathrm{A}(1)$ parametrized by $\mathsf{B}\mathrm{T}_{\mathrm{ad}}$.

By the commutator computation of [Zha22, Proposition 5.5.4], (2.21) is the sum of the constant family $\mu_{\mathrm{T}_{\mathrm{sc}}}$ with the morphism

$$b_1 \otimes (\Psi \boxtimes \Psi) : \mathsf{B}\mathrm{T}_{\mathrm{sc}} \times \mathsf{B}\mathrm{T}_{\mathrm{ad}} \to \mathsf{B}^4\mathrm{A}(1)$$

where b_1 is the form (2.5). The equality (2.20) follows. □

2.1.18 *Trivialization along* $\Lambda^{\sharp}_{\mathrm{sc}}$**.** We are now ready to construct the canonical trivialization of (2.7) over the sublattice $\Lambda^{\sharp}_{\mathrm{sc}} \subset \Lambda^{\sharp}$. Since the pullback of $\mu_{\mathrm{T}^{\sharp}}$ to $\mathrm{T}^{\sharp}_{\mathrm{sc}} \coloneqq \Lambda^{\sharp}_{\mathrm{sc}} \otimes_{\mathbf{Z}} \mathbb{G}_m$ is canonically $\mathbf{Z}$-linear (*cf.* §2.1.3), it suffices to trivialize the value of ν on the basis $\{\alpha^{\sharp}\}_{\alpha\in\Delta}$ of $\Lambda^{\sharp}_{\mathrm{sc}}$. We shall do so using the extension $\widetilde{\mu}_{\mathrm{G}_{\mathrm{sc}}}$ (*cf.* §2.1.16).

Indeed, Lemma 2.1.17 implies that $\Lambda^{\sharp}_{\mathrm{sc}} \oplus \Lambda_{\mathrm{ad}}$ belongs to the kernel of the symmetric form $\widetilde{b}$ attached to $\widetilde{\mu}_{\mathrm{G}_{\mathrm{sc}}}$ (*cf.* Remark 2.1.9). The constructions of §2.1.10–§2.1.13, applied to $\widetilde{\mu}_{\mathrm{G}_{\mathrm{sc}}}$, yields a Weyl-equivariant $\mathbb{E}_\infty$-monoidal (in fact, $\mathbf{Z}$-linear) morphism

$$\widetilde{\nu} : \Lambda^{\sharp}_{\mathrm{sc}} \oplus \Lambda_{\mathrm{ad}} \to \mathsf{B}^2\mathrm{A}.$$

On the other hand, the Weyl group W of G_{sc} embeds naturally in the Weyl group of $\mathrm{G}_{\mathrm{sc}} \rtimes \mathrm{G}_{\mathrm{ad}}$, where its action on $\Lambda_{\mathrm{sc}} \oplus \Lambda_{\mathrm{ad}}$ is determined by the formula

$$w(0,\lambda) - (0,\lambda) = (w(\lambda) - \lambda, 0) \tag{2.22}$$

for any $w \in \mathrm{W}$ and $\lambda \in \Lambda_{\mathrm{ad}}$. This follows, for example, by realizing T as a maximal torus of G and performing the following computation in $\mathrm{G}_{\mathrm{sc}} \rtimes \mathrm{T}_{\mathrm{ad}}$ for any $\dot{w} \in \mathrm{G}_{\mathrm{sc}}$ lifting w:

$$\begin{aligned}(\dot{w},1)(1,t)(\dot{w}^{-1},1) &= (\dot{w},1)(t\dot{w}^{-1}t^{-1},1)(1,t)\\ &= (\dot{w}t\dot{w}^{-1}t^{-1},1)(1,t) = (w(t)t^{-1},1)(1,t).\end{aligned}$$

Altogether, we arrive at a diagram of W-equivariant $\mathbb{E}_\infty$-monoidal morphisms

$$\begin{tikzcd}
\Lambda^{\sharp}_{\mathrm{sc}} \arrow[r, phantom, "\subset"] \arrow[d, phantom, "\cap"] & \Lambda^{\sharp}_{\mathrm{sc}} \oplus \Lambda_{\mathrm{ad}} \arrow[d, "\widetilde{\nu}"] \\
\Lambda^{\sharp} \arrow[r, "\nu"] & \mathsf{B}^2\mathrm{A}
\end{tikzcd} \tag{2.23}$$

Writing $\omega \in \Lambda_{\mathrm{ad}}$ for the fundamental coweight associated to α and $d \coloneqq \operatorname{ord}\mathrm{Q}(\alpha)$, we obtain canonical isomorphisms using (2.22) and the equivariance of $\widetilde{\nu}$ with respect to the simple reflection $s_{\check{\alpha}} \in \mathrm{W}$:

$$\begin{aligned}\nu(\alpha^{\sharp}) \simeq \widetilde{\nu}(\alpha^{\sharp},0) &\simeq \widetilde{\nu}(s_{\check{\alpha}}(0,-d\omega)) - \widetilde{\nu}(0,-d\omega)\\ &\simeq \widetilde{\nu}(0,-d\omega) - \widetilde{\nu}(0,-d\omega) \simeq 0.\end{aligned}$$

This provides the desired trivialization of $\nu(\alpha^{\sharp})$.

Remark 2.1.19. An *a priori* different trivialization of (2.7) over $\Lambda^{\sharp}_{\mathrm{sc}}$ is constructed in [Zha22, §6.1.5]. We have not compared these trivializations, although it seems likely that they coincide. The advantage of the trivialization constructed in §2.1.18 is that it can be directly related to the affine Grassmannian (*cf.* Lemma 6.3.6). This relation will ultimately be responsible for the *canonicity* of our twisted geometric Satake equivalence.

2.1.20 *Construction of ν.* Assuming that G is split, we have constructed the $\mathbb{E}_\infty$-monoidal morphism (2.7) together with a trivialization over $\Lambda^\sharp_{\mathrm{sc}}$ (*cf.* §2.1.18). In view of (2.4), the morphism (2.7) factors through an $\mathbb{E}_\infty$-monoidal morphism

$$\nu : \hat{\mathrm{Z}}_{\mathrm{H}} \to \mathsf{B}^2_{\mathrm{X}}\mathrm{A}. \tag{2.24}$$

The $\mathbb{E}_\infty$-monoidal morphism (2.24) is functorially assigned to (G, μ), so for any reductive group S-scheme G, we obtain ν by étale descent.

2.1.21 *The ϑ-shift.* Finally, we involve the geometry of the base curve X.

Let us assume that A admits nontrivial 2-torsion. By our assumption that $|\mathrm{A}|$ is invertible in k, this forces $\operatorname{char}\mathsf{k} \neq 2$.

Since the restriction of b to $\Lambda^\sharp$ vanishes, the restriction of Q to $\Lambda^\sharp$ is a linear form valued in $\mathrm{A}(-1)_{2\text{-tors}}$, the subsheaf of $\mathrm{A}(-1)$ of 2-torsion elements. It factors through $\hat{\mathrm{Z}}_{\mathrm{H}}$ via (2.4), defining a character

$$\epsilon : \hat{\mathrm{Z}}_{\mathrm{H}} \to \mathrm{A}(-1)_{2\text{-tors}}. \tag{2.25}$$

Denote by Ω_{X} the canonical line bundle of X relative to k and consider the image $\Psi(\Omega_{\mathrm{X}}^{-1})$ of Ω_{X}^{-1} under Ψ (*cf.* Remark 1.1.10), viewed as a global section of $\mathsf{B}^2_{\mathrm{X}}\hat{\mathbf{Z}}(1)$. Taking tensor product with (2.25), we obtain a $\mathbf{Z}$-linear morphism

$$\vartheta := \epsilon \otimes \Psi(\Omega_{\mathrm{X}}^{-1}) : \hat{\mathrm{Z}}_{\mathrm{H}} \to \mathsf{B}^2_{\mathrm{X}}(\mathrm{A}_{2\text{-tors}}). \tag{2.26}$$

We may then add (2.26) to (2.24) to obtain the $\mathbb{E}_\infty$-monoidal morphism

$$\nu + \vartheta : \hat{\mathrm{Z}}_{\mathrm{H}} \to \mathsf{B}^2_{\mathrm{X}}\mathrm{A}, \tag{2.27}$$

which we call the *ϑ-shifted dual datum.*

By convention, the notation $\nu + \vartheta$ stands for ν itself when A has trivial 2-torsion.

Remark 2.1.22. Since (2.25) is 2-torsion-valued, the formation of (2.26) only depends on the reduction of $\Psi(\Omega_{\mathrm{X}}^{-1})$ mod 2. In particular, we may replace Ω_{X}^{-1} by Ω_{X} without changing (2.26). The fact that we prefer Ω_{X}^{-1} is because it arises naturally from the geometry of the affine Grassmannian (*cf.* the proof of Proposition 4.2.6).

Note that the reduction of $\Psi(\Omega_{\mathrm{X}}^{-1})$ mod 2 is the μ_2-gerbe of ϑ-characteristics over X. In particular, a choice of a ϑ-characteristic trivializes (2.26) and renders the ϑ-shift irrelevant. When the ground field k is algebraically closed or a finite field, then ϑ-characteristics over X always exist (*cf.* [Ati71]).

2.2. Twisted H-representations.

2.2.1. We will now invoke the coefficient data of §1.3.1.

For each $\mathrm{S} \in \mathsf{Sch}$, we write $\mathsf{Lis}(\mathrm{S})$ for the full subcategory of $\mathsf{D}(\mathrm{S})$ consisting of locally constant sheaf of finite-dimensional e-vector spaces, viewed as a symmetric monoidal category under tensor product.

Given a coalgebra $\mathcal{A}$ in $\operatorname{Ind}\mathsf{Lis}(\mathrm{S})$, we write $\mathcal{A}\text{-}\mathsf{Comod}(\mathsf{Lis}(\mathrm{S}))$ for the symmetric monoidal e-linear category of $\mathcal{A}$-comodules in $\mathsf{Lis}(\mathrm{S})$.

The structure sheaf $\mathcal{O}_{\mathrm{H}}$ admits the structure of a Hopf algebra in $\operatorname{Ind}\mathsf{Lis}(\mathrm{X})$. Thus, we obtain an étale sheaf of symmetric monoidal e-linear categories $\mathcal{R}ep_{\mathrm{H}}$ over X, sending S to $\mathcal{O}_{\mathrm{H}}\text{-}\mathsf{Comod}(\mathsf{Lis}(\mathrm{S}))$. We write $\mathsf{Rep}_{\mathrm{H}}$ for its global section.

2.2.2. For a moment, let us assume that $\hat{\mathrm{Z}}_{\mathrm{H}}$ is constant. (This happens for example when G is split, *cf.* §2.1.6.) Then for any $\xi \in \hat{\mathrm{Z}}_{\mathrm{H}}$, we have the full subsheaf $\mathcal{R}ep^{\xi}_{\mathrm{H}} \subset \mathcal{R}ep_{\mathrm{H}}$ consisting of objects on which Z_{H} acts via the character ξ. The assignment $\xi \mapsto \mathcal{R}ep^{\xi}_{\mathrm{H}}$ realizes $\mathcal{R}ep_{\mathrm{H}}$ as a $\hat{\mathrm{Z}}_{\mathrm{H}}$-graded sheaf of e-linear symmetric monoidal categories (*cf.* §A.2.4, §A.2.6).

Let us apply the twisting construction of §A.2.7, with the $\mathbb{E}_\infty$-monoidal morphism $\hat{\mathrm{Z}}_{\mathrm{H}} \to \mathsf{B}^2\mathsf{e}^\times$ given by the composition of (2.27) with (the deloop of) ζ. This yields a $\hat{\mathrm{Z}}_{\mathrm{H}}$-graded étale sheaf of e-linear symmetric monoidal categories $\mathcal{R}ep_{\mathrm{H},\nu+\vartheta}$.

Since étale sheaves of symmetric monoidal e-linear categories satisfy étale descent, the construction of §2.2.2 generalizes to the case where $\hat{\mathrm{Z}}_{\mathrm{H}}$ is locally constant. Thus, we obtain an étale sheaf of symmetric monoidal e-linear categories $\mathcal{R}ep_{\mathrm{H},\nu+\vartheta}$, whose global section is denoted by $\mathsf{Rep}_{\mathrm{H},\nu+\vartheta}$.

Objects of $\mathsf{Rep}_{\mathrm{H},\nu+\vartheta}$ are called $(\nu+\vartheta)$*-twisted* H*-representations on* e*-local systems.*

2.2.3. Next, we extend the construction of $\mathcal{R}ep_{\mathrm{H},\nu+\vartheta}$ to finite products of copies of X. By étale descent, it again suffices to treat the case where $\hat{\mathrm{Z}}_{\mathrm{H}}$ is constant.

For any finite set I, the external tensor product $\mathcal{O}^{\boxtimes \mathrm{I}}_{\mathrm{H}}$ is a Hopf algebra in $\operatorname{Ind}\mathsf{Lis}(\mathrm{X}^{\mathrm{I}})$. We write $\mathcal{R}ep_{\mathrm{H}^{\boxtimes \mathrm{I}}}$ for the étale sheaf of symmetric monoidal e-linear categories over X^{I}, sending S to $\mathcal{O}^{\boxtimes \mathrm{I}}_{\mathrm{H}}\text{-}\mathsf{Comod}(\mathsf{Lis}(\mathrm{X}^{\mathrm{I}}))$. Then $\mathcal{R}ep_{\mathrm{H}^{\boxtimes \mathrm{I}}}$ lifts to a $(\hat{\mathrm{Z}}_{\mathrm{H}})^{\boxplus \mathrm{I}}$-graded sheaf of symmetric monoidal e-linear categories.

Applying the twisting construction of §A.2.7 with the $\mathbb{E}_\infty$-monoidal morphism

$$\begin{aligned}(\nu+\vartheta)^{\boxplus \mathrm{I}} : (\hat{\mathrm{Z}}_{\mathrm{H}})^{\boxplus \mathrm{I}} &\to (\mathsf{B}^2_{\mathrm{X}^{\mathrm{I}}}\mathrm{A})^{\oplus \mathrm{I}} \\ &\xrightarrow{\Sigma} \mathsf{B}^2_{\mathrm{X}^{\mathrm{I}}}\mathrm{A} \xrightarrow{\zeta} \mathsf{B}^2_{\mathrm{X}^{\mathrm{I}}}\mathsf{e}^\times \end{aligned} \tag{2.28}$$

we obtain a $(\hat{\mathrm{Z}}_{\mathrm{H}})^{\boxplus \mathrm{I}}$-graded étale sheaf of e-linear symmetric monoidal categories $\mathcal{R}ep_{\mathrm{H}^{\boxtimes \mathrm{I}},(\nu+\vartheta)^{\boxplus \mathrm{I}}}$ over X^{I}, with global section $\mathsf{Rep}_{\mathrm{H}^{\boxtimes \mathrm{I}},(\nu+\vartheta)^{\boxplus \mathrm{I}}}$.

Remark 2.2.4. Let us explain the functoriality of $\mathsf{Rep}_{\mathrm{H}^{\boxtimes \mathrm{I}},(\nu+\vartheta)^{\boxplus \mathrm{I}}}$ with respect to the finite set I, in parallel with Remark 1.3.11.

Indeed, given a morphism of finite sets $\varphi : \mathrm{I} \to \mathrm{J}$, we have an induced morphism $\Delta_\varphi : \mathrm{X}^{\mathrm{J}} \to \mathrm{X}^{\mathrm{I}}$, hence a pullback functor $(\Delta_\varphi)^* : \mathsf{Lis}(\mathrm{X}^{\mathrm{I}}) \to \mathsf{Lis}(\mathrm{X}^{\mathrm{J}})$. There is a natural morphism of Hopf algebras $(\Delta_\varphi)^*(\mathcal{O}^{\boxtimes \mathrm{I}}_{\mathrm{H}}) \to \mathcal{O}^{\boxtimes \mathrm{J}}_{\mathrm{H}}$ in $\operatorname{Ind}\mathsf{Lis}(\mathrm{X}^{\mathrm{J}})$, compatible with restrictions of (2.28), so we obtain a functor of symmetric monoidal e-linear categories

$$(\Delta_\varphi)^* : \mathsf{Rep}_{\mathrm{H}^{\boxtimes \mathrm{I}},(\nu+\vartheta)^{\boxplus \mathrm{I}}} \to \mathsf{Rep}_{\mathrm{H}^{\boxtimes \mathrm{J}},(\nu+\vartheta)^{\boxplus \mathrm{J}}},$$

which respects compositions of morphisms of finite sets.

2.2.5. Next, we shall rewrite $\mathsf{Rep}_{\mathrm{H}^{\boxtimes \mathrm{I}},(\nu+\vartheta)^{\boxplus \mathrm{I}}}$ in terms of an "L-group". To do so, we need to extract the **Z**-linear part of ν (*cf.* §2.1.20).

Let us perform this construction in a more abstract setting: Let Ξ be an étale sheaf of abelian groups over a base scheme S over which A has invertible order. There is a fiber sequence of étale sheaves of **Z**-linear spaces

$$\mathcal{M}aps_{\mathbf{Z}}(\Xi, \mathsf{B}^2\mathrm{A}) \to \mathcal{M}aps_{\mathbb{E}_\infty}(\Xi, \mathsf{B}^2\mathrm{A}) \to \mathcal{H}om(\Xi, \mathrm{A}_{2\text{-tors}}), \tag{2.29}$$

where the first map is induced from the forgetful functor and the second map is defined as follows: Given an $\mathbb{E}_\infty$-monoidal morphism $\Xi \to \mathsf{B}^2\mathrm{A}$, its fiber is a symmetric monoidal extension $\widetilde{\widetilde{\Xi}}$ of Ξ by $\mathsf{B}\mathrm{A}$. Associating to each $\xi \in \widetilde{\widetilde{\Xi}}$ the commutativity constraint $c_{\xi,\xi} : \xi \otimes \xi \simeq \xi \otimes \xi$ defines a character $\Xi \to \mathrm{A}_{2\text{-tors}}$.

A key observation, due to Gaitsgory and Lysenko (*cf.* [GL18, §4.8]), is that (2.29) canonically splits. To construct this splitting, we may assume $\mathrm{A}_{\text{2-tors}} \simeq \mathbf{Z}/2$ and treat the universal case $\Xi := \mathbf{Z}/2$, where we lift the identity character of $\mathbf{Z}/2$ to the $\mathbb{E}_\infty$-monoidal morphism $\mathbf{Z}/2 \to \mathsf{B}^2\mathbf{Z}/2$ which is trivial as an $\mathbb{E}_1$-monoidal morphism, but with commutativity constraint specified by the pairing

$$\mathbf{Z}/2 \otimes \mathbf{Z}/2 \to \mathbf{Z}/2, \quad a \otimes b \mapsto ab.$$

2.2.6. Let us apply the above construction to $\mathrm{S} := \mathrm{X}$ and $\Xi := \hat{\mathrm{Z}}_{\mathrm{H}}$. Using the splitting of (2.29), we may project ν onto a $\mathbf{Z}$-linear morphism

$$^0\nu : \hat{\mathrm{Z}}_{\mathrm{H}} \to \mathsf{B}^2_{\mathrm{X}}\mathrm{A}. \tag{2.30}$$

We shall also involve the ϑ-shift (*cf.* §2.1.21): Using $^0\nu$ instead of ν in the construction of (2.27), we obtain a $\mathbf{Z}$-linear morphism

$$^0\nu + \vartheta : \hat{\mathrm{Z}}_{\mathrm{H}} \to \mathsf{B}^2_{\mathrm{X}}\mathrm{A}. \tag{2.31}$$

Remark 2.2.7. The image of ν along the second map of (2.29) is a homomorphism $\hat{\mathrm{Z}}_{\mathrm{H}} \to \mathrm{A}_{\text{2-tors}}$. By [Zha22, Proposition 4.6.6], this coincides with the homomorphism (2.25) under the canonical isomorphism

$$\mathrm{A}(-1)_{\text{2-tors}} \simeq \mathrm{A}_{\text{2-tors}}.$$

2.2.8 *The* L-*group***.** We now assume that X is geometrically connected and affine. This hypothesis implies that X is an algebraic $\mathrm{K}(\pi,1)$-space (*cf.* [MO15, §1.4]).

We fix a geometric point $\bar{x}$ of X and write $\mathrm{Z}_{\mathrm{H},\bar{x}}$ (respectively $\mathrm{H}_{\bar{x}}$) for the fiber of Z_{H} (respectively H) at $\bar{x}$, which is equipped with a natural $\pi_1^{\text{ét}}(\mathrm{X},\bar{x})$-action.

We also fix an algebraic closure $\overline{\mathbf{Q}}_\ell$ of $\mathbf{Q}_\ell$ containing e. Inducing (2.31) along ζ, we obtain a $\mathbf{Z}$-linear morphism $\hat{\mathrm{Z}}_{\mathrm{H}} \to \mathsf{B}^2_{\mathrm{X}}\overline{\mathbf{Q}}_\ell^\times$, or equivalently an étale $\mathrm{Z}_{\mathrm{H}}(\overline{\mathbf{Q}}_\ell)$-gerbe. Choose a trivialization of this $\mathrm{Z}_{\mathrm{H}}(\overline{\mathbf{Q}}_\ell)$-gerbe over $\bar{x}$ and denote by ${}^{\mathrm{L}}\mathrm{Z}_{\mathrm{H},\mathrm{X}}$ its fundamental group, which is well-defined by the $\mathrm{K}(\pi,1)$-property of X (*cf.* [Wei18, Theorem 19.6]). By construction, ${}^{\mathrm{L}}\mathrm{Z}_{\mathrm{H},\mathrm{X}}$ fits into a short exact sequence

$$1 \to \mathrm{Z}_{\mathrm{H},\bar{x}}(\overline{\mathbf{Q}}_\ell) \to {}^{\mathrm{L}}\mathrm{Z}_{\mathrm{H},\mathrm{X}} \to \pi_1^{\text{ét}}(\mathrm{X},\bar{x}) \to 1 \tag{2.32}$$

satisfying the following properties:

(1) the conjugation action of ${}^{\mathrm{L}}\mathrm{Z}_{\mathrm{H},\mathrm{X}}$ on $\mathrm{Z}_{\mathrm{H},\bar{x}}(\overline{\mathbf{Q}}_\ell)$ factors through the natural action of $\pi_1^{\text{ét}}(\mathrm{X},\bar{x})$ on $\mathrm{Z}_{\mathrm{H},\bar{x}}(\overline{\mathbf{Q}}_\ell)$;

(2) there is a finite extension $\mathsf{e} \subset \mathsf{e}_1$ contained in $\overline{\mathbf{Q}}_\ell$ and a finite quotient $\pi_1^{\text{ét}}(\mathrm{X},\bar{x}) \twoheadrightarrow \Gamma$ such that (2.32) is induced from an extension of Γ by $\mathrm{Z}_{\mathrm{H},\bar{x}}(\mathsf{e}_1)$.

Finally, we define the L-*group* to be the short exact sequence induced from (2.32) along the $\pi_1^{\text{ét}}(\mathrm{X},\bar{x})$-equivariant inclusion $\mathrm{Z}_{\mathrm{H},\bar{x}}(\overline{\mathbf{Q}}_\ell) \hookrightarrow \mathrm{H}_{\bar{x}}(\overline{\mathbf{Q}}_\ell)$:

$$1 \to \mathrm{H}_{\bar{x}}(\overline{\mathbf{Q}}_\ell) \to {}^{\mathrm{L}}\mathrm{H}_{\mathrm{X}} \to \pi_1^{\text{ét}}(\mathrm{X},\bar{x}) \to 1. \tag{2.33}$$

Remark 2.2.9. The extension (2.32) is constructed using (2.31) as input, the latter being the sum of $\mathbf{Z}$-linear morphisms $^0\nu$ and ϑ. We may apply the same construction to ϑ and $^0\nu$ individually, obtaining extensions ${}^{\mathrm{L}}\mathrm{Z}^{(1)}_{\mathrm{H},\mathrm{X}}$, respectively ${}^{\mathrm{L}}\mathrm{Z}^{(2)}_{\mathrm{H},\mathrm{X}}$, of $\pi_1^{\text{ét}}(\mathrm{X},\bar{x})$ by $\mathrm{Z}_{\mathrm{H},\bar{x}}(\overline{\mathbf{Q}}_\ell)$. Then (2.32) is the Baer sum

$${}^{\mathrm{L}}\mathrm{Z}_{\mathrm{H},\mathrm{X}} \simeq {}^{\mathrm{L}}\mathrm{Z}^{(1)}_{\mathrm{H},\mathrm{X}} + {}^{\mathrm{L}}\mathrm{Z}^{(2)}_{\mathrm{H},\mathrm{X}}.$$

We shall explain in §7.4 that ${}^{\mathrm{L}}\mathrm{Z}^{(1)}_{\mathrm{H},\mathrm{X}}$ (defined by ϑ) is Weissman's "meta-Galois twist" (*cf.* [Wei18, §4], also called the "first twist" in [Wei14]) when k is a finite field.

2.2.10. We remain in the context of §2.2.8 and fix a finite set I. The product X^{I} is also an algebraic $\mathrm{K}(\pi,1)$-space. Applying the construction of *loc.cit.*, with (2.31) replaced by the $\mathbf{Z}$-linear morphism

$$({}^{0}\nu + \vartheta)^{\boxplus \mathrm{I}} : (\hat{\mathrm{Z}}_{\mathrm{H}})^{\boxplus \mathrm{I}} \to \mathsf{B}^2_{\mathrm{X}^{\mathrm{I}}}\mathrm{A},$$

we obtain a short exact sequence

$$1 \to \mathrm{H}^{\mathrm{I}}_{\bar{x}}(\overline{\mathbf{Q}}_\ell) \to {}^{\mathrm{L}}\mathrm{H}_{\mathrm{X}^{\mathrm{I}}} \to \pi_1^{\text{ét}}(\mathrm{X}^{\mathrm{I}}, \bar{x}^{\mathrm{I}}) \to 1.$$

Denote by $\mathsf{Rep}({}^{\mathrm{L}}\mathrm{H}_{\mathrm{X}^{\mathrm{I}}})$ the symmetric monoidal category of continuous ${}^{\mathrm{L}}\mathrm{H}_{\mathrm{X}^{\mathrm{I}}}$-representations on finite-dimensional $\overline{\mathbf{Q}}_\ell$-vector spaces which are algebraic over $\mathrm{H}^{\mathrm{I}}_{\bar{x}}(\overline{\mathbf{Q}}_\ell)$. On the other hand, recall the symmetric monoidal category $\mathsf{Rep}_{\mathrm{H}^{\boxtimes \mathrm{I}},(\nu+\vartheta)^{\boxplus \mathrm{I}}}$ (*cf.* §2.2.3), with e replaced by $\overline{\mathbf{Q}}_\ell$.

Proposition 2.2.11. *For any finite set* I*, there is a canonical equivalence of* $\overline{\mathbf{Q}}_\ell$*-linear* monoidal *categories*

$$\mathsf{Rep}_{\mathrm{H}^{\boxtimes \mathrm{I}},(\nu+\vartheta)^{\boxplus \mathrm{I}}} \simeq \mathsf{Rep}({}^{\mathrm{L}}\mathrm{H}_{\mathrm{X}^{\mathrm{I}}}). \tag{2.34}$$

Proof. It follows from [Zha22, Proposition 6.4.8] that $\mathsf{Rep}({}^{\mathrm{L}}\mathrm{H}_{\mathrm{X}^{\mathrm{I}}})$ is canonically equivalent to $\mathsf{Rep}_{\mathrm{H}^{\boxtimes \mathrm{I}},({}^{0}\nu+\vartheta)^{\boxplus \mathrm{I}}}$ as $\overline{\mathbf{Q}}_\ell$-linear *symmetric* monoidal categories.

It remains to observe that $({}^{0}\nu + \vartheta)^{\boxplus \mathrm{I}}$ and $(\nu + \vartheta)^{\boxplus \mathrm{I}}$ are isomorphic as $\mathbb{E}_1$-monoidal morphisms, but this follows from the fact that the splitting of §2.2.5 is defined by an $\mathbb{E}_1$-monoidally trivial morphism. ☐

Remark 2.2.12. The equivalence (2.34) is natural in I in the following sense: Given a map of finite sets $\varphi : \mathrm{I} \to \mathrm{J}$, we have a commutative square

$$\begin{tikzcd} \mathsf{Rep}_{\mathrm{H}^{\boxtimes \mathrm{I}},(\nu+\vartheta)^{\boxplus \mathrm{I}}} \arrow[r, phantom, "\simeq"] \arrow[d, "(\Delta_\varphi)^*"] & \mathsf{Rep}({}^{\mathrm{L}}\mathrm{H}_{\mathrm{X}^{\mathrm{I}}}) \arrow[d, "(\Delta_\varphi)^*"] \\ \mathsf{Rep}_{\mathrm{H}^{\boxtimes \mathrm{J}},(\nu+\vartheta)^{\boxplus \mathrm{J}}} \arrow[r, phantom, "\simeq"] & \mathsf{Rep}({}^{\mathrm{L}}\mathrm{H}_{\mathrm{X}^{\mathrm{J}}}). \end{tikzcd} \tag{2.35}$$

where the left vertical arrow is defined as in Remark 2.2.4 and the right vertical arrow is restriction along the natural map $\Delta_\varphi : {}^{\mathrm{L}}\mathrm{H}_{\mathrm{X}^{\mathrm{J}}} \to {}^{\mathrm{L}}\mathrm{H}_{\mathrm{X}^{\mathrm{I}}}$ corresponding to φ. Furthermore, (2.35) is compatible with compositions.

2.3. Statement of the equivalence.

2.3.1. Let X be a smooth curve over a field k and G be a reductive group X scheme. Let e be a finite extension of $\mathbf{Q}_\ell$, for a prime ℓ invertible in k. We assume the existence of a square root $\mathsf{e}(\frac{1}{2})$ of $\mathsf{e}(1)$ over $\operatorname{Spec}\mathsf{k}$ and fix it. Let $\zeta : \mathrm{A} \hookrightarrow \mathsf{e}^\times$ be a finite subgroup and μ be an A-valued étale level of G.

For any finite set I, we have defined the symmetric monoidal e-linear abelian categories ${}^{+}\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\mathrm{I}})$ (*cf.* §1.3.16) and $\mathsf{Rep}_{\mathrm{H}^{\boxtimes \mathrm{I}},(\nu+\vartheta)^{\boxplus \mathrm{I}}}$ (*cf.* §2.2.3).

The statement below is our version of the geometric Satake equivalence.

Theorem 2.3.2. *For any finite set* I*, there is a canonical equivalence of symmetric monoidal* e*-linear categories*

$${}^{+}\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\mathrm{I}}) \simeq \mathsf{Rep}_{\mathrm{H}^{\boxtimes \mathrm{I}},(\nu+\vartheta)^{\boxplus \mathrm{I}}}. \tag{2.36}$$

Remark 2.3.3. In our formulation of Theorem 2.3.2, we sacrificed some generality by invoking the square root $\mathsf{e}(\frac{1}{2})$. This is done so that the dual group H appears naturally in the equivalence (2.36).

As in the classical setting, one can remove half-integral Tate twists by *not* incorporating any Tate twist in the definition of the constant term functor (*cf.* §1.5.2). This would lead to a different fiber functor, hence a different Tannaka dual $\mathrm{H}^{\mathrm{geom}}$. The analysis is parallel to [Zhu15, Appendix A], so we will omit it.

Remark 2.3.4. We formulated Theorem 2.3.2 for ℓ-adic constructible sheaves. This is motivated by our main application for global function fields.

When the ground field k is $\mathbf{C}$, we may also define ${}^{+}\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\mathrm{I}})$ using constructible sheaves of e-vector spaces (for any field e of characteristic zero) in the classical topology, or using algebraic D-modules. Our proof of Theorem 2.3.2 applies in these contexts.

2.3.5 *Functoriality in* I**.** The equivalence (2.36) will be natural in I with respect to the functorialities specified in Remark 1.3.11 and Remark 2.2.4.

More precisely, for any map of finite sets $\varphi : \mathrm{I} \to \mathrm{J}$, there is a canonical 2-isomorphism rendering the following diagram of symmetric monoidal e-linear categories commute:

$$\begin{tikzcd}
{}^{+}\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\mathrm{I}}) \arrow[r, phantom, "\simeq"] \arrow[d, "\varphi_!"] & \mathsf{Rep}_{\mathrm{H}^{\boxtimes \mathrm{I}},(\nu+\vartheta)^{\boxplus \mathrm{I}}} \arrow[d, "(\Delta_{\varphi})^*"] \\
{}^{+}\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\mathrm{J}}) \arrow[r, phantom, "\simeq"] & \mathsf{Rep}_{\mathrm{H}^{\boxtimes \mathrm{J}},(\nu+\vartheta)^{\boxplus \mathrm{J}}}
\end{tikzcd} \tag{2.37}$$

Furthermore, the 2-isomorphism in (2.37) respects compositions of maps of finite set.

2.3.6 *Compatibility with constant terms***.** Let P be a parabolic subgroup of G with Levi quotient $\mathrm{P} \twoheadrightarrow \mathrm{M}$. The pullback of the étale level μ along $\mathrm{P} \subset \mathrm{G}$ canonically descends to an étale level μ_{M} of M. Applying the construction of the metaplectic dual data to μ_{M}, we obtain $(\mathrm{H}_{\mathrm{M}}, \nu_{\mathrm{M}})$ (*cf.* §2.1).

By construction, there is a morphism

$$h_{\mathrm{P}} : \mathrm{H}_{\mathrm{M}} \to \mathrm{H} \tag{2.38}$$

of locally constant étale sheaves over X of pinned split reductive group $\mathbf{Z}$-schemes. It induces a morphism $\hat{h}_{\mathrm{P}} : \hat{\mathrm{Z}}_{\mathrm{H}} \to \hat{\mathrm{Z}}_{\mathrm{H}_{\mathrm{M}}}$ of characters of the centers with $\nu_{\mathrm{M}} \circ \hat{h}_{\mathrm{P}} \simeq \nu$.

The equivalence (2.36) renders the following diagram commute

$$\begin{tikzcd}
{}^{+}\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\mathrm{I}}) \arrow[r, phantom, "\simeq"] \arrow[d, "\mathrm{CT}_{\mathrm{P}}"] & \mathsf{Rep}_{\mathrm{H}^{\boxtimes \mathrm{I}},(\nu+\vartheta)^{\boxplus \mathrm{I}}} \arrow[d, "h_{\mathrm{P}}^*"] \\
{}^{+}\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{M},\mathrm{I}}) \arrow[r, phantom, "\simeq"] & \mathsf{Rep}_{\mathrm{H}_{\mathrm{M}}^{\boxtimes \mathrm{I}},(\nu_{\mathrm{M}}+\vartheta)^{\boxplus \mathrm{I}}}
\end{tikzcd}$$

where CT_{P} is the constant term functor (1.54) and h_{P}^* is restriction along (2.38).

Remark 2.3.7. In the context of §2.2.8, we have defined the category $\mathsf{Rep}({}^{\mathrm{L}}\mathrm{H}_{\mathrm{X}})$ (and more generally $\mathsf{Rep}({}^{\mathrm{L}}\mathrm{H}_{\mathrm{X}^{\mathrm{I}}})$ for a finite set I). By combining (2.34) and (2.36) and replacing e by $\overline{\mathbf{Q}}_{\ell}$, we obtain a canonical equivalence of *monoidal* $\overline{\mathbf{Q}}_{\ell}$-linear categories

$$\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\mathrm{I}}) \simeq \mathsf{Rep}({}^{\mathrm{L}}\mathrm{H}_{\mathrm{X}^{\mathrm{I}}}) \tag{2.39}$$

natural in I. Here, we omitted the normalization of the Satake category as it does not affect the underlying monoidal category (*cf.* Remark 1.3.17).

3. Pointwise studies of the Satake category

Throughout this section, we fix a smooth curve X over an *algebraically closed* field k and a *split* reductive group X-scheme G. Let e be a finite extension of $\mathbf{Q}_\ell$, for a prime ℓ invertible in k. We shall also fix a square root $\mathsf{e}(\frac{1}{2})$ of $\mathsf{e}(1)$. Let $\zeta : A \hookrightarrow \mathsf{e}^\times$ be a finite subgroup and μ be an A-valued étale level of G.

The goal of this section is to study e-linear abelian category $\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{G,x})$ for a k-point x of X (*cf.* §1.3.7). Namely, we will prove that it is semisimple and identify its simple objects. In fact, we will prove a stronger assertion where we allow x to be any S-point of X (*cf.* Theorem 3.3.5). The proof of this theorem is an adaptation of Lusztig and Yun's method in handling monodromic Hecke categories (*cf.* [LY20]). This idea of using [LY20] is suggested by Gurbir Dhillon.[13]

3.1. Schubert cells.

3.1.1. We use the notation §2.1.2 for notions associated to G. Furthermore, we fix a maximal torus and a Borel subgroup $T \subset B \subset G$. (In particular, T is identified with the universal Cartan of G.) Denote by $\Lambda^+ \subset \Lambda$ the submonoid of dominant coweights.

We shall work over one copy of X and write Hec_G for $\mathrm{Hec}_{G,\{1\}}$ (*cf.* Remark 1.3.11). We use similar notation Gr_G, L^+G, LG, *etc.*

3.1.2. For any $\lambda \in \Lambda$, we write Hec_G^λ (respectively Gr_G^λ) for the corresponding Schubert cell of Hec_G (respectively Gr_G).

Denote by $\varpi^\lambda : X \to \mathrm{Gr}_T$ the map sending x to the modification $\mathcal{O} \overset{x}{\sim} \mathcal{O}(\lambda\Gamma_x)$ of T-bundles. We use the same notation for its composition with the inclusion into Gr_G (and also for its further composition with the projection onto Hec_G). By definition, the L^+G-action on $\varpi^\lambda : X \to \mathrm{Gr}_G$ has orbit Gr_G^λ and stabilizer $L^+G \cap \varpi^\lambda L^+G \varpi^{-\lambda}$.

Denote by $P^\lambda \subset G$ the parabolic subgroup generated by root subgroups N_α corresponding to those roots $\check{\alpha}$ with $\langle \check{\alpha}, \lambda \rangle \le 0$.[14] Denote by M^λ the Levi quotient of P^λ. With the above notation, we have morphisms

$$\begin{tikzcd}
X \arrow[r, "\varpi^\lambda"] & \mathrm{Hec}_G^\lambda \arrow[r, phantom, "\subset"] \arrow[d, "\pi^\lambda"] & \mathrm{Hec}_G \\
& \mathsf{B}_X(M^\lambda) &
\end{tikzcd} \tag{3.1}$$

where π^λ is the composition (*cf.* [Zhu17, §2.1])

$$\mathrm{Hec}_G^\lambda \simeq \mathsf{B}_X(L^+G \cap \varpi^\lambda L^+G \varpi^{-\lambda}) \to \mathsf{B}_X(P^\lambda) \to \mathsf{B}_X(M^\lambda). \tag{3.2}$$

3.1.3. Denote by $\mathcal{G}_{\mathrm{Hec}_G^\lambda}$ the restriction of the A-gerbe $\mathcal{G}_{\mathrm{Hec}_G}$ (*cf.* §1.1.7) to Hec_G^λ. We shall determine $\mathcal{G}_{\mathrm{Hec}_G^\lambda}$ in relation to the morphisms in (3.1).

Write $\mathcal{G}_{\varpi^\lambda}$ for the pullback of $\mathcal{G}_{\mathrm{Hec}_G}$ along ϖ^λ. It is an A-gerbe over X, so it defines an A-gerbe $\mathcal{G}_{\varpi^\lambda}|_{\mathcal{Z}}$ on any X-prestack $\mathcal{Z}$ by pullback.

---

[13] At least when the étale level μ admits an integral lift (*cf.* §C.1), the recent work [DLYZ25] goes much further than what we prove in this section.

[14] If we assume $\lambda \in \Lambda^+$, then P^λ is *opposite* to the standard parabolic subgroup corresponding to simple roots orthogonal to λ.

On the other hand, the fiber sequence (2.1) yields an isomorphism

$$\mathcal{H}om(\pi_1(\mathrm{M}^\lambda), \mathrm{A}(-1)) \simeq \mathcal{M}aps_*(\mathsf{B}_\mathrm{X}(\mathrm{M}^\lambda), \mathsf{B}^2_\mathrm{X}\mathrm{A}),$$

so any character $\chi : \pi_1(\mathrm{M}^\lambda) \to \mathrm{A}(-1)$ induces a pointed morphism $\chi \otimes \Psi : \mathsf{B}_\mathrm{X}(\mathrm{M}^\lambda) \to \mathsf{B}^2_\mathrm{X}\mathrm{A}$, where Ψ is the Kummer morphism (*cf.* Remark 1.1.10).

Let b the symmetric form associated to μ (*cf.* §2.1.5). By construction, all roots of M^λ are orthogonal to λ. The equality (2.3) thus implies that the character $b(\lambda, \cdot) : \Lambda \to \mathrm{A}(-1)$ factors through $\pi_1(\mathrm{M}^\lambda)$, so we obtain a pointed morphism

$$b(\lambda, \cdot) \otimes \Psi : \mathsf{B}_\mathrm{X}(\mathrm{M}^\lambda) \to \mathsf{B}^2_\mathrm{X}\mathrm{A}.$$

The following result is an analogue of [FL10, Lemma 2.4] for étale levels.

Proposition 3.1.4. *There is a canonical isomorphism of* A*-gerbes*

$$\mathcal{G}_{\mathrm{Hec}^\lambda_\mathrm{G}} - (\mathcal{G}_{\varpi^\lambda}|_{\mathrm{Hec}^\lambda_\mathrm{G}}) \simeq (\pi^\lambda)^*(b(\lambda, \cdot) \otimes \Psi). \tag{3.3}$$

3.1.5 *The forms* b_2**.** For the proof of Proposition 3.1.4, as well as for many later purposes, we shall recall the canonical quadratic structure on μ (*cf.* [SZ25, Proposition 3.1.3]).

To state it, we need the bilinear pairing

$$b_2 : \pi_1\mathrm{G} \otimes \mathrm{Fib}(\Lambda \to \Lambda_{\mathrm{ad}}) \to \mathrm{A}(-1),$$

characterized by the property that the adjoints of b_2, b, b_1 (*cf.* §2.1.5, §2.1.8) fit into a map of fiber sequences

$$\begin{tikzcd}
\mathrm{Fib}(\Lambda \to \Lambda_{\mathrm{ad}}) \arrow[r, "b_2"] \arrow[d] & \mathcal{H}om(\pi_1\mathrm{G}, \mathrm{A}(-1)) \arrow[d] \\
\Lambda \arrow[r, "b"] \arrow[d] & \mathcal{H}om(\Lambda, \mathrm{A}(-1)) \arrow[d] \\
\Lambda_{\mathrm{ad}} \arrow[r, "b_1"] & \mathcal{H}om(\Lambda_{\mathrm{sc}}, \mathrm{A}(-1))
\end{tikzcd} \tag{3.4}$$

In particular, by tensoring b_2 with the self-tensor product of the Kummer morphism $\Psi^{\otimes 2} : \mathsf{B}\mathbb{G}_m \otimes \mathsf{B}\mathbb{G}_m \to \mathsf{B}^4\hat{\mathbf{Z}}(2)$, we obtain a pairing

$$b_2 \otimes \Psi^{\otimes 2} : \mathsf{B}\mathrm{G}_{\mathrm{ab}} \otimes \mathsf{B}\mathrm{Z} \to \mathsf{B}^4\mathrm{A}(1). \tag{3.5}$$

3.1.6 *Canonical quadratic structure***.** Consider the $\mathsf{B}\mathrm{Z}$-action on $\mathsf{B}\mathrm{G}$ induced from the Z-action on G. Denote by

$$a : \mathsf{B}\mathrm{G} \times \mathsf{B}\mathrm{Z} \to \mathsf{B}\mathrm{G}$$

the action morphism and by p_1, p_2 the projections of $\mathsf{B}\mathrm{G} \times \mathsf{B}\mathrm{Z}$ onto $\mathsf{B}\mathrm{G}$, respectively $\mathsf{B}\mathrm{Z}$.

By [SZ25, Proposition 3.1.3], there is a canonical isomorphism

$$a^*\mu \simeq (p_1)^*\mu + (p_2)^*\mu_\mathrm{Z} + b_2 \otimes \Psi^{\otimes 2}, \tag{3.6}$$

where μ_Z denotes the pullback of μ along $\mathrm{Z} \subset \mathrm{G}$, and we (slightly abusively) use $b_2 \otimes \Psi^{\otimes 2}$ to denote the pullback of (3.5) to $\mathsf{B}\mathrm{G} \times \mathsf{B}\mathrm{Z}$.

Furthermore, (3.6) is compatible with the canonical trivializations of the two sides along $e \times \mathsf{B}\mathrm{Z}$, respectively $\mathsf{B}\mathrm{G} \times e$, and admits natural cocycle data over $\mathsf{B}\mathrm{G} \times \mathsf{B}\mathrm{Z} \times \mathsf{B}\mathrm{Z}$.

Remark 3.1.7. For $\mathrm{G} = \mathrm{T}$ a split torus, we have $b_2 = b$ and (3.6) reduces to the canonical quadratic structure constructed in [Zha22, Proposition 4.7.3]. For the proof of Proposition 3.1.4 (but not for later purposes), this special case is sufficient.

3.1.8. Let us now construct the isomorphism (3.3).

Proof of Proposition 3.1.4. Recall that π^λ is the quotient of the affine-space fibration $\mathrm{Gr}_{\mathrm{G}}^{\lambda} \to \mathrm{G}/\mathrm{P}^{\lambda}$ by the action of $\mathrm{L}^+\mathrm{G} \to \mathrm{G}$, which has a pro-unipotent kernel. Thus, pullback by π^λ induces an equivalence of (discrete) groupoids

$$(\pi^\lambda)^* : \mathrm{Maps}_*(\mathsf{B}_{\mathrm{X}}(\mathrm{M}^\lambda), \mathsf{B}_{\mathrm{X}}^2\mathrm{A}) \simeq \mathrm{Maps}_*(\mathrm{Hec}_{\mathrm{G}}^{\lambda}, \mathsf{B}_{\mathrm{X}}^2\mathrm{A}), \tag{3.7}$$

where $\mathrm{Hec}_{\mathrm{G}}^{\lambda}$ is viewed as a pointed X-stack via ϖ^λ.

The left-hand-side of (3.3) admits the natural structure of a *pointed* morphism $\mathrm{Hec}_{\mathrm{G}}^{\lambda} \to \mathsf{B}_{\mathrm{X}}^2\mathrm{A}$, so it remains to identify it with the image of $b(\lambda, \cdot) \otimes \Psi$ under (3.7).

The inclusion $\mathrm{T} \subset \mathrm{G}$ gives rise to a commutative square

$$\begin{tikzcd}
\mathrm{Maps}_*(\mathsf{B}_{\mathrm{X}}(\mathrm{M}^\lambda), \mathsf{B}_{\mathrm{X}}^2\mathrm{A}) \arrow[r, phantom, "\simeq"] \arrow[d] & \mathrm{Maps}_*(\mathrm{Hec}_{\mathrm{G}}^{\lambda}, \mathsf{B}_{\mathrm{X}}^2\mathrm{A}) \arrow[d] \\
\mathrm{Maps}_*(\mathsf{B}_{\mathrm{X}}\mathrm{T}, \mathsf{B}_{\mathrm{X}}^2\mathrm{A}) \arrow[r, phantom, "\simeq"] & \mathrm{Maps}_*(\mathrm{Hec}_{\mathrm{T}}^{\lambda}, \mathsf{B}_{\mathrm{X}}^2\mathrm{A})
\end{tikzcd}$$

where the vertical arrows are fully faithful (*cf.* §3.1.3). Thus, it suffices to construct the isomorphism (3.3) after restriction to $\mathrm{Hec}_{\mathrm{T}}^{\lambda}$, or equivalently to its reduced locus $\mathsf{B}_{\mathrm{X}}\mathrm{L}^+\mathrm{T}$.

Let P be an R-point of $\mathsf{B}_{\mathrm{X}}\mathrm{L}^+\mathrm{T}$, viewed as a T-bundle over D_x with $x \in \mathrm{X}(\mathrm{R})$. It defines the R-ponit $\mathrm{P} \overset{x}{\sim} \mathrm{P}(\lambda x)$ of $\mathrm{Hec}_{\mathrm{T}}^{\lambda}$. The value of $\mathcal{G}_{\mathrm{Hec}_{\mathrm{T}}}$ at this R-point is the image of

$$\mu(\mathrm{P}(\lambda x)) - \mu(\mathrm{P}) \in \Gamma(\mathrm{D}_x \bmod \mathring{\mathrm{D}}_x, \mathsf{B}^4\mathrm{A}(1))$$

under the trace map (*cf.* §1.1.11). We shall use the canonical quadratic structure (3.6) (for $\mathrm{G} = \mathrm{T}$) to identify this section:

$$\mu(\mathrm{P}(\lambda x)) - \mu(\mathrm{P}) \simeq \mu(\mathcal{O}(\lambda x)) + (b \otimes \Psi^{\otimes 2})(\mathrm{P}, \mathcal{O}(\lambda x)).$$

Note that under the trace map, $\mu(\mathcal{O}(\lambda x))$ yields the value of $\mathcal{G}_{\varpi^\lambda}|_{\mathrm{Hec}_{\mathrm{T}}}$.

It thus remains to identify the image of $(b \otimes \Psi^{\otimes 2})(\mathrm{P}, \mathcal{O}(\lambda x))$ under the trace map with the image of $\mathrm{P}|_{\Gamma_x}$ under $b(\lambda, \cdot) \otimes \Psi$. To do so, we shall write

$$(b \otimes \Psi^{\otimes 2})(\mathrm{P}, \mathcal{O}(\lambda x)) \simeq (b(\lambda, \cdot) \otimes \Psi)(\mathrm{P}) \otimes \Psi(\mathcal{O}(x)),$$

where $(b(\lambda, \cdot) \otimes \Psi)(\mathrm{P})$ is viewed as an object of $\Gamma(\mathrm{D}_x, \mathsf{B}^2\mathrm{A})$ and $\Psi(\mathcal{O}(x))$ is viewed as an object of $\Gamma(\mathrm{D}_x \bmod \mathring{\mathrm{D}}_x, \mathsf{B}^2\hat{\mathbf{Z}}(1))$.

Note that $(b(\lambda, \cdot) \otimes \Psi)(\mathrm{P})$ canonically descends to $(b(\lambda, \cdot) \otimes \Psi)(\mathrm{P}|_{\Gamma_x})$ along $\mathrm{D}_x \to \operatorname{Spec} \mathrm{R}$. We thus obtain the desired identification

$$\begin{aligned}
\mathrm{tr}_x((b \otimes \Psi^{\otimes 2})(\mathrm{P}, \mathcal{O}(\lambda x))) &\simeq \mathrm{tr}_x((b(\lambda, \cdot) \otimes \Psi)(\mathrm{P}) \otimes \Psi(\mathcal{O}(x))) \\
&\simeq (b(\lambda, \cdot) \otimes \Psi)(\mathrm{P}|_{\Gamma_x}) \otimes \mathrm{tr}_x(\Psi(\mathcal{O}(x))) \simeq (b(\lambda, \cdot) \otimes \Psi)(\mathrm{P}|_{\Gamma_x})
\end{aligned}$$

from Remark 1.1.9 and Remark 1.1.10. □

3.1.9. Given an S-point x of X ($\mathrm{S} \in \mathsf{Sch}$), we have ∞-category $\mathsf{D}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},x})$ (*cf.* §1.3.3).

We shall use Proposition 3.1.4 to show that its objects are supported on (the base change of) Schubert cells $\mathrm{Hec}_{\mathrm{G},x}^{\lambda}$ corresponding to $\lambda \in \Lambda^\sharp$. (Recall that $\Lambda^\sharp$ denotes the kernel of b, *cf.* §2.1.6.) More precisely, we have the following vanishing statement.

Corollary 3.1.10. *Let x be an* S*-point of* X *(*$\mathrm{S} \in \mathsf{Sch}$*) and $\lambda \in \Lambda \smallsetminus \Lambda^\sharp$. Then*

$$\mathsf{D}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},x}^{\lambda}) \simeq 0.$$

Proof. The vanishing of any $\mathcal{A} \in \mathsf{D}_{\mathcal{G},\zeta}(\mathrm{Hec}^{\lambda}_{\mathrm{G},x})$ may be checked on fibers, so we reduce to the case $\mathrm{S} = \operatorname{Spec} \mathsf{k}$. The condition $\lambda \notin \Lambda^{\sharp}$ is equivalent to the nontriviality of the A-gerbe $b(\lambda, \cdot) \otimes \Psi$ over the classifying stack $\mathsf{B}(\mathrm{M}^{\lambda})$ (*cf.* §3.1.3).

By Proposition 3.1.4, the restriction of $\mathcal{G}_{\mathrm{Hec}_{\mathrm{G}}}$ to $\mathrm{Hec}^{\lambda}_{\mathrm{G},x}$ is *non-canonically* isomorphic to the pullback of $b(\lambda, \cdot) \otimes \Psi$ along π^{λ}. Thus the result follows from Lemma B.2.2, applied to $\operatorname{Spec} \mathsf{k}$ and a finite type quotient of $\mathrm{L}^+_x\mathrm{G} \cap \varpi^{\lambda}\mathrm{L}^+_x\mathrm{G}\varpi^{-\lambda}$. ☐

3.1.11. Next, let us study the case $\lambda \in \Lambda^{\sharp}$ more closely.

Corollary 3.1.12. *For each $\lambda \in \Lambda^{\sharp}$, there is a canonical isomorphism of* A*-gerbes*

$$\mathcal{G}_{\mathrm{Hec}^{\lambda}_{\mathrm{G}}} \simeq \mathcal{G}_{\varpi^{\lambda}}|_{\mathrm{Hec}^{\lambda}_{\mathrm{G}}}. \tag{3.8}$$

Proof. The isomorphism (3.8) is supplied by (3.3) and the trivialization of its right-hand-side coming from the vanishing of $b(\lambda, \cdot)$. ☐

3.1.13. By Corollary 3.1.12, the A-gerbe $\mathcal{G}_{\mathrm{Hec}^{\lambda}_{\mathrm{G}}}$ descends to X for any $\lambda \in \Lambda^{\sharp}$. In particular, given $\lambda_1 \in \mathrm{W}\lambda$, where W is the Weyl group, we obtain an identification

$$\mathcal{G}_{\varpi^{\lambda_1}} \simeq \mathcal{G}_{\varpi^{\lambda}} \tag{3.9}$$

from the fact that ω^{λ}, ω^{λ_1} belong to the same $\mathrm{L}^+\mathrm{G}$-orbit.

On the other hand, the restriction of μ to $\mathrm{T}^{\sharp}$ is W-equivariant by Lemma 2.1.14. It follows that the A-gerbe $\mathcal{G}_{\mathrm{Hec}_{\mathrm{T}}}$ is also W-equivariant. This yields another identification

$$\mathcal{G}_{\varpi^{\lambda_1}} \simeq \mathcal{G}_{\varpi^{\lambda}}, \tag{3.10}$$

which may *a priori* differ from (3.9). Let us show that this is not the case.

Proposition 3.1.14. *Given $\lambda, \lambda_1 \in \Lambda^{\sharp}$ in the same Weyl-orbit, the identifications* (3.9) *and* (3.10) *are canonically isomorphic.*

Proof. Recall the canonical G-equivariance structure on μ (*cf.* §2.1.11). It induces an $\mathrm{L}^+\mathrm{G}$-equivariance structure on $\mathcal{G}_{\mathrm{Gr}_{\mathrm{G}}}$ as follows: Given R-points g of $\mathrm{L}^+\mathrm{G}$ and $\mathrm{P}^0 \overset{x}{\sim} \mathrm{P}^1$ of Gr_{G} lying over the same R-point x of X, the G-equivariance structure on μ yields an isomorphism in $\Gamma(\mathrm{D}_x \bmod \mathring{\mathrm{D}}_x, \mathsf{B}^4\mathrm{A}(1))$:

$$\mu(g(\mathrm{P}^1)) - \mu(g(\mathrm{P}^0)) \simeq \mu(\mathrm{P}^1) - \mu(\mathrm{P}^0), \tag{3.11}$$

where $g(\mathrm{P}^1)$, $g(\mathrm{P}^0)$ are the twists of P^1, P^0 by g (*cf.* Remark 2.1.12). The desired $\mathrm{L}^+\mathrm{G}$-equivariance structure on $\mathcal{G}_{\mathrm{Gr}_{\mathrm{G}}}$ is obtained from (3.11) via the trace map (*cf.* §1.1.11).

It suffices to prove that this $\mathrm{L}^+\mathrm{G}$-equivariance structure on $\mathcal{G}_{\mathrm{Gr}_{\mathrm{G}}}$ agrees with the one coming from the descent data of $\mathcal{G}_{\mathrm{Hec}_{\mathrm{G}}}$. Indeed, the former $\mathrm{L}^+\mathrm{G}$-equivariance structure induces (3.10) while the latter induces (3.9).

Note that the $\mathrm{L}^+\mathrm{G}$-equivariance structure coming from the descent data of $\mathcal{G}_{\mathrm{Hec}_{\mathrm{G}}}$ corresponds to the isomorphism

$$\mu(g(\mathrm{P}^1)) - \mu(g(\mathrm{P}^0)) \simeq \mu(\mathrm{P}^1) - \mu(\mathrm{P}^0) \tag{3.12}$$

arising from the identification of $g(\mathrm{P}^0) \overset{x}{\sim} g(\mathrm{P}^1)$ and $\mathrm{P}^0 \overset{x}{\sim} \mathrm{P}^1$ as R-points of $\mathrm{Hec}_{\mathrm{G}}$. This identification is in turn defined by the canonical isomorphisms of G-bundles $g(\mathrm{P}^0) \simeq \mathrm{P}^0$, $g(\mathrm{P}^1) \simeq \mathrm{P}^1$, *cf.* Remark 2.1.12. It follows from the *loc.cit.* that the difference between (3.11) and (3.12) is given by the trivial section

$$(\Omega\mu)(g) - (\Omega\mu)(g) \simeq 0$$

of $\Gamma(\mathrm{D}_x \bmod \mathring{\mathrm{D}}_x, \mathsf{B}^3\mathrm{A}(1))$, so (3.11) and (3.12) agree. ☐

3.2. Properties of ULA sheaves.

3.2.1. In this subsection, we establish a few useful properties of ULA sheaves over the local Hecke stack, by characterizing them in terms of Schubert cells. These are analogues of results in [FS24, §VI.6].

We remain in the same context as §3.1.1. Moreover, let us write $\Lambda^{\sharp,+} := \Lambda^\sharp \cap \Lambda^+$. This is the monoid parametrizing Schubert cells which support nonzero $(\mathcal{G}_{\mathrm{Hec}_{\mathrm{G}}}, \zeta)$-twisted constructible complexes (*cf.* Corollary 3.1.10).

Proposition 3.2.2. *Let x be an S-point of X ($\mathrm{S} \in \mathsf{Sch}$) and $\mathcal{A}$ be an object of $\mathsf{D}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},x})$. The following are equivalent:*

(1) $\mathcal{A}$ is ULA relative to S;

(2) for each $\lambda \in \Lambda^{\sharp,+}$, the pullback of $\mathcal{A}$ along $\varpi^\lambda : \mathrm{S} \to \mathrm{Hec}_{\mathrm{G},x}$ is locally constant.

Proof. For each $\lambda \in \Lambda^+$, we write $j^\lambda : \mathrm{Hec}^\lambda_{\mathrm{G},x} \hookrightarrow \mathrm{Hec}_{\mathrm{G},x}$ for the immersion of the Schubert cell. Recall that $\mathrm{Hec}^\lambda_{\mathrm{G},x}$ is the quotient of S by an affine group scheme which is an extension of M^λ by a pro-unipotent affine group scheme (*cf.* §3.1.2).

(2) ⇒ (1). By Corollary 3.1.10, the pullback of $\mathcal{A}$ along ϖ^λ vanishes for $\lambda \in \Lambda^+ \smallsetminus \Lambda^{\sharp,+}$. Thus, if the pullback of $\mathcal{A}$ along ϖ^λ is locally constant for $\lambda \in \Lambda^{\sharp,+}$, then $\mathcal{A}$ is an iterated extension of objects of the form $(j^\lambda)_!\mathcal{B}$, where $\mathcal{B} \in \mathsf{D}_{\mathcal{G},\zeta}(\mathrm{Hec}^\lambda_{\mathrm{G},x})$ is locally constant.

We shall argue that $(j^\lambda)_!\mathcal{B}$ is ULA relative to S. Indeed, by [HS23, Corollary 3.9], we may assume that $x : \mathrm{S} \to \mathrm{X}$ is the spectrum of the henselian local ring of a k-point $\bar{x}$ of X. By (2.1), the pullback of μ along $\mathrm{S} \to \mathrm{X}$ then descends to $\mathrm{Spec}\,\mathsf{k}$. By choosing a uniformizer ϖ at x, we realize $\mathrm{Hec}_{\mathrm{G},x}$ as the base change of $\mathrm{Hec}_{\mathrm{G},\bar{x}}$ along $\mathrm{S} \to \mathrm{Spec}\,\mathsf{k}$, compatibly with the A-gerbe $\mathcal{G}_{\mathrm{Hec}_{\mathrm{G}}}$. The assertion now follows from base change along

$$\begin{tikzcd}
\mathrm{Hec}^\lambda_{\mathrm{G},x} \arrow[r, "j^\lambda"] \arrow[d] & \mathrm{Hec}_{\mathrm{G},x} \arrow[d] \\
\mathrm{Hec}^\lambda_{\mathrm{G},\bar{x}} \arrow[r, "j^\lambda_{\bar{x}}"] & \mathrm{Hec}_{\mathrm{G},\bar{x}}
\end{tikzcd}$$

and the fact that any object of $\mathsf{D}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\bar{x}})$ is ULA relative to $\mathrm{Spec}\,\mathsf{k}$ (*cf.* Remark B.1.9).

(1) ⇒ (2). Suppose that $\mathcal{A}$ is ULA relative to S. Let λ be an element of Λ^+ whose corresponding Schubert cell is open in the support of $\mathcal{A}$. In particular, $\lambda \in \Lambda^{\sharp,+}$ (*cf.* Corollary 3.1.10). Then the pullback of $\mathcal{A}$ along ϖ^λ is locally constant. This implies that $\mathcal{B} := (j^\lambda)^*\mathcal{A}$ is locally constant as well, so $(j^\lambda)_!\mathcal{B}$ is ULA relative to S by the argument above.

We may now replace $\mathcal{A}$ by the cofiber of $(j^\lambda)_!\mathcal{B} \to \mathcal{A}$ and proceed by induction. □

3.2.3. Let x be an S-point of X ($\mathrm{S} \in \mathsf{Sch}$). Given $\mathcal{A}, \mathcal{B} \in \mathsf{D}_{\mathcal{G},\zeta}(\mathrm{Gr}_{\mathrm{G},x})$, assumed ULA relative to S, the formation of their internal Hom

$$\mathcal{H}om(\mathcal{A}, \mathcal{B}) \in \mathsf{D}(\mathrm{Gr}_{\mathrm{G},x}) \tag{3.13}$$

commutes with arbitrary base change in $\mathrm{S} \in \mathsf{Sch}$.

Indeed, writing $\mathbf{D}$ for the Verdier duality functor over $\mathrm{Gr}_{\mathrm{G},x}$ relative to S, we may express $\mathcal{H}om(\mathcal{A}, \mathcal{B})$ as $\mathbf{D}(\mathcal{A} \otimes \mathbf{D}\mathcal{B})$ (*cf.* Remark B.1.11). Furthermore, $\mathcal{A} \otimes \mathbf{D}\mathcal{B}$ is ULA relative to S by Proposition 3.2.2, so the formation of its Verdier dual commutes with arbitrary base change in $\mathrm{S} \in \mathsf{Sch}$ (*cf.* §B.1.10).

3.2.4. We shall use the base change property of (3.13) to relate the "stalk" and "fiber" of the category of ULA sheaves over the local Hecke stack.

To be more precise, let $\bar{x}$ be a k-point of X and $x : \mathrm{S} \to \mathrm{X}$ be the inverse limit of étale neighborhoods of $\bar{x}$. Pullback yields a functor

$$\mathsf{D}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},x}) \to \mathsf{D}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\bar{x}}). \tag{3.14}$$

Proposition 3.2.5. *The functor* (3.14) *restricts to an equivalence on the full subcategories consisting of ULA objects (relative to* S*, respectively* k*).*

Proof. Given ULA objects $\mathcal{A}, \mathcal{B} \in \mathsf{D}_{\mathcal{G},\zeta}(\mathrm{Gr}_{\mathrm{G},x})$, we have an identification

$$(\pi_* \mathcal{H}om(\mathcal{A}, \mathcal{B}))_{\bar{x}} \simeq \mathrm{Hom}(\mathcal{A}_{\bar{x}}, \mathcal{B}_{\bar{x}}), \tag{3.15}$$

where $\pi : \mathrm{Gr}_{\mathrm{G},x} \to \mathrm{S}$ is the structural map and $\mathcal{A}_{\bar{x}}, \mathcal{B}_{\bar{x}}$ are the pullbacks of $\mathcal{A}, \mathcal{B}$ to $\mathrm{Gr}_{\mathrm{G},\bar{x}}$. Indeed, (3.15) follows from the ind-properness of π and the fact that (3.13) commutes with arbitrary base change.

From (3.15), we deduce an isomorphism

$$\mathrm{Hom}(\mathcal{A}, \mathcal{B}) \simeq \mathrm{Hom}(\mathcal{A}_{\bar{x}}, \mathcal{B}_{\bar{x}}), \tag{3.16}$$

where the Homs are taken in $\mathsf{D}_{\mathcal{G},\zeta}(\mathrm{Gr}_{\mathrm{G},x})$, respectively $\mathsf{D}_{\mathcal{G},\zeta}(\mathrm{Gr}_{\mathrm{G},\bar{x}})$. Now, assuming that $\mathcal{A}$, $\mathcal{B}$ are endowed with $\mathrm{L}^+_x\mathrm{G}$-equivariance, we may compute their Homs in $\mathsf{D}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},x})$, respectively $\mathsf{D}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\bar{x}})$ by taking $\mathrm{L}^+_x\mathrm{G}$-, respectively $\mathrm{L}^+_{\bar{x}}\mathrm{G}$-invariants on (3.16). They coincide by base change properties of the cohomology of $\mathsf{B}\mathrm{G}$. This implies that (3.14) is fully faithful on ULA objects.

To show that it is essentially surjective, we observe that $\mathsf{D}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\bar{x}})$ is generated by !-extensions of constant sheaves from Schubert cells. These objects lift along (3.14) to objects of the same kind, which are ULA by Proposition 3.2.2. ☐

3.3. Generators.

3.3.1. We fix $\mathrm{T} \subset \mathrm{B} \subset \mathrm{G}$ as in §3.1.1. Let $\mathrm{S} \in \mathsf{Sch}$ and x be an S-point of X.

For each $\lambda \in \Lambda^{\sharp,+}$ (*cf.* the notation of §3.2.1), we denote by $\mathsf{Lis}_{\mathcal{G}_{\varpi^\lambda},\zeta}(\mathrm{S})$ the e-linear abelian category of $(\mathcal{G}_{\varpi^\lambda}, \zeta)$-twisted e-local systems over S.

Pullback along the structural morphism $\pi : \mathrm{Hec}^\lambda_{\mathrm{G},x} \to \mathrm{S}$, in view of the identification (3.8), yields a functor

$$\pi^* : \mathsf{Lis}_{\mathcal{G}_{\varpi^\lambda},\zeta}(\mathrm{S}) \to \mathsf{D}_{\mathcal{G},\zeta}(\mathrm{Hec}^\lambda_{\mathrm{G},x}). \tag{3.17}$$

3.3.2. To make the target of (3.17) perverse[15] relative to S and to incorporate the "correct" amount of Tate twists, we shall instead consider the functor

$$\pi^*(\langle \check{\rho}, \lambda \rangle)[\langle 2\check{\rho}, \lambda \rangle] : \mathsf{Lis}_{\mathcal{G}_{\varpi^\lambda},\zeta}(\mathrm{S}) \to \mathsf{D}_{\mathcal{G},\zeta}(\mathrm{Hec}^\lambda_{\mathrm{G},x}). \tag{3.18}$$

Since $\langle \check{\rho}, \lambda \rangle$ is a half-integer in general, the formation of (3.18) invokes $\mathsf{e}(\frac{1}{2})$.

For each $\mathcal{E} \in \mathsf{Lis}_{\mathcal{G}_{\varpi^\lambda},\zeta}(\mathrm{S})$, we write $\widetilde{\mathcal{E}}$ for its image under (3.18). Denote by $j^\lambda : \mathrm{Hec}^\lambda_{\mathrm{G},x} \to \mathrm{Hec}_{\mathrm{G},x}$ the locally closed immersion. We may form the following functors, called the *standard*, respectively *costandard functors*:

$$\Delta^\lambda : \mathsf{Lis}_{\mathcal{G}_{\varpi^\lambda},\zeta}(\mathrm{S}) \to \mathsf{D}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},x}), \quad \mathcal{E} \mapsto {}^p\mathrm{H}^0(j^\lambda)_!\widetilde{\mathcal{E}}$$

$$\nabla^\lambda : \mathsf{Lis}_{\mathcal{G}_{\varpi^\lambda},\zeta}(\mathrm{S}) \to \mathsf{D}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},x}), \quad \mathcal{E} \mapsto {}^p\mathrm{H}^0(j^\lambda)_*\widetilde{\mathcal{E}}$$

---

[15] As always, this means perverse relative to S after pulling back to $\mathrm{Gr}^\lambda_{\mathrm{G},x}$, *cf.* §1.3.7.

Denote by $\mathbf{D}$ the Verdier duality functor over $\mathrm{Gr}_{\mathrm{G},x}$ relative to S (*cf.* §B.1.10). We have the following basic result concerning Δ^λ and ∇^λ.

Proposition 3.3.3. *Given* $\lambda \in \Lambda^{\sharp,+}$ *and* $\mathcal{E} \in \mathsf{Lis}_{\mathcal{G}_{\varpi^\lambda},\zeta}(\mathrm{S})$, *there holds*

(1) the objects $\Delta^\lambda(\mathcal{E})$, $\nabla^\lambda(\mathcal{E})$ *belong to* $\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},x})$*;*

(2) the formations of $\Delta^\lambda(\mathcal{E})$, $\nabla^\lambda(\mathcal{E})$ *commute with base change in* $\mathrm{S} \in \mathsf{Sch}$*;*

(3) there is a canonical isomorphism

$$\mathbf{D}\Delta^\lambda(\mathcal{E}) \simeq \nabla^\lambda(\mathcal{E}^\vee),\ \textit{for}\ \mathcal{E}^\vee := \mathcal{H}om(\mathcal{E},\mathsf{e}) \in \mathsf{Lis}_{-\mathcal{G}_{\varpi^\lambda},\zeta}(\mathrm{S}). \tag{3.19}$$

Proof. By Proposition 3.2.2, the complex $(j^\lambda)_!\widetilde{\mathcal{E}} \in \mathrm{D}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},x})$ is ULA relative to S, for any $\mathcal{E} \in \mathsf{Lis}_{\mathcal{G}_{\varpi^\lambda},\zeta}(\mathrm{S})$. We have an isomorphism

$$\mathbf{D}(j^\lambda)_! \simeq (j^\lambda)_*\mathbf{D}, \tag{3.20}$$

where by a slight abuse of notation, we also use $\mathbf{D}$ for the Verdier duality functor over $\mathrm{Gr}^\lambda_{\mathrm{G},x}$ relative to S. It follows from (3.20) that $(j^\lambda)_*\widetilde{\mathcal{E}}$ is also ULA relative to S. Moreover, the formation of both $(j^\lambda)_!\widetilde{\mathcal{E}}$ and $(j^\lambda)_*\widetilde{\mathcal{E}}$ commutes with base change in $\mathrm{S} \in \mathsf{Sch}$.

Statement (1) now follows from the fact that truncation functor ${}^p\mathrm{H}^0$ preserves universal local acyclicity (*cf.* Remark B.1.13).

Statement (2) follows because ${}^p\mathrm{H}^0$ commutes with base change in $\mathrm{S} \in \mathsf{Sch}$.

To prove statement (3), we shall argue that the morphism

$$\mathbf{D}\Delta^\lambda(\mathcal{E}) \to \mathbf{D}(j^\lambda)_!\widetilde{\mathcal{E}}, \tag{3.21}$$

obtained by dualizing the degree-0 truncation $(j^\lambda)_!\widetilde{\mathcal{E}} \to \Delta^\lambda(\mathcal{E})$, is itself the degree-0 truncation of the connective complex $\mathbf{D}(j^\lambda)_!\widetilde{\mathcal{E}}$.

This claim will yield the isomorphism (3.19), in view of the isomorphisms

$$\begin{aligned}\mathbf{D}\Delta^\lambda(\mathcal{E}) &\simeq {}^p\mathrm{H}^0\mathbf{D}(j^\lambda)_!\widetilde{\mathcal{E}}\\ &\simeq {}^p\mathrm{H}^0(j^\lambda)_*\mathbf{D}(\widetilde{\mathcal{E}}) \simeq \nabla^\lambda(\mathcal{E}^\vee),\end{aligned}$$

where we used (3.20) and the fact that $\mathrm{Gr}^\lambda_{\mathrm{G},x} \to \mathrm{S}$ is smooth of relative dimension $\langle 2\check{\rho}, \lambda\rangle$.

To prove the claim, we shall apply CT_B to (3.21). Realizing CT_B as a hyperbolic localization (*cf.* the proof of Proposition 1.5.6), the image may be identified with

$$\mathbf{D}\,\mathrm{CT}_{\mathrm{B}^-}\,\Delta^\lambda(\mathcal{E}) \to \mathbf{D}\,\mathrm{CT}_{\mathrm{B}^-}(j^\lambda)_!\widetilde{\mathcal{E}},$$

where $\mathbf{D}$ now stands for Verdier duality over $\mathrm{Gr}_{\mathrm{T},x}$ relative to S, *i.e.* the functor $\mathcal{H}om(\cdot,\mathsf{e})$. Since $\mathrm{CT}_{\mathrm{B}^-}$ is t-exact (*cf.* Proposition 1.5.6), we may identify $\mathrm{CT}_{\mathrm{B}^-}\,\Delta^\lambda(\mathcal{E})$ with the degree-0 trunction of the coconnective complex $\mathrm{CT}_{\mathrm{B}^-}(j^\lambda)_!\widetilde{\mathcal{E}}$. It follows that $\mathbf{D}\,\mathrm{CT}_{\mathrm{B}^-}\,\Delta^\lambda(\mathcal{E})$ is the degree-0 truncation of the connective complex $\mathbf{D}\,\mathrm{CT}_{\mathrm{B}^-}(j^\lambda)_!\widetilde{\mathcal{E}}$. The claim now follows from the t-exactness and conservativity of CT_B (*cf.* Proposition 1.5.6). ☐

3.3.4. Given $\lambda \in \Lambda^{\sharp,+}$, we define the functor, called the *intersection cohomology*

$$\mathrm{IC}^\lambda : \mathsf{Lis}_{\mathcal{G}_{\varpi^\lambda},\zeta}(\mathrm{S}) \to \mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},x}), \tag{3.22}$$

which assigns to each $\mathcal{E} \in \mathsf{Lis}_{\mathcal{G}_{\varpi^\lambda},\zeta}(\mathrm{S})$ the image of the natural morphism $\Delta^\lambda(\mathcal{E}) \to \nabla^\lambda(\mathcal{E})$ in $\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},x})$ (*cf.* Proposition 3.3.3). Then the formation of $\mathrm{IC}^\lambda(\mathcal{E})$ also commutes with base change in $\mathrm{S} \in \mathsf{Sch}$. Summing (3.22) yields a functor of e-linear abelian categories

$$\bigoplus_{\lambda \in \Lambda^{\sharp,+}} \mathsf{Lis}_{\mathcal{G}_{\varpi^\lambda},\zeta}(\mathrm{S}) \to \mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},x}); \tag{3.23}$$

By construction, we also have the following morphisms of functors:

$$\Delta^{\lambda} \twoheadrightarrow \mathrm{IC}^{\lambda} \hookrightarrow \nabla^{\lambda}. \tag{3.24}$$

The remainder of this section is devoted to the proof of the following result.

Theorem 3.3.5. *The functor* (3.23) *is an equivalence.*

3.3.6. Theorem 3.3.5 has a number of pleasant consequences.

Corollary 3.3.7. *If x is a geometric point of* X*, then* $\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},x})$ *is semisimple.*

Proof. When S is the spectrum of an algebraically closed field, the category $\mathsf{Lis}_{\mathcal{G}_{\varpi^{\lambda}},\zeta}(\mathrm{S})$ is *non-canonically* equivalent to the category of finite-dimensional e-vector spaces. Hence the assertion follows from Theorem 3.3.5. ☐

Corollary 3.3.8. *The morphisms in* (3.24) *are isomorphisms.*

Proof. Since the formation of Δ^{λ} and ∇^{λ} commutes with base change in S (*cf.* Proposition 3.3.3), we reduce to the case $\mathrm{S} = \operatorname{Spec}\mathsf{k}$. The same result implies that $\mathrm{IC}^{\lambda} \hookrightarrow \nabla^{\lambda}$ is the Verdier dual of $\Delta^{\lambda} \twoheadrightarrow \mathrm{IC}^{\lambda}$, so it suffices to prove that the latter is an isomorphism.

Let $\mathcal{E}$ be a $(\mathcal{G}_{\varpi^{\lambda}}, \zeta)$-twisted local system over S. The kernel $\mathcal{A}$ of the surjection $\Delta^{\lambda}(\mathcal{E}) \twoheadrightarrow \mathrm{IC}^{\lambda}(\mathcal{E})$ is supported on the complement of $\mathrm{Gr}^{\lambda}_{\mathrm{G},x}$ in its closure $\overline{\mathrm{Gr}}^{\lambda}_{\mathrm{G},x}$. By Corollary 3.3.7, the injection $\mathcal{A} \hookrightarrow \Delta^{\lambda}(\mathcal{E})$ admits a retraction $\Delta^{\lambda}(\mathcal{E}) \twoheadrightarrow \mathcal{A}$, but any such morphism vanishes. ☐

3.4. Monodromic affine Hecke categories.

3.4.1. The main input in Theorem 3.3.5 is the parity vanishing statement for intersection cohomology sheaves on the affine flag variety. Contrary to the untwisted situation, the proof of this statement requires considering sheaves twisted by a *family* of A-gerbes.

We fix $\mathrm{T} \subset \mathrm{B} \subset \mathrm{G}$ as in §3.1.1 and a k-point x of X. Denote by W the Weyl group of (G, T). It naturally acts on the cocharacter lattice Λ, and we may form the extended affine Weyl group $\mathrm{W}^{\mathrm{aff}} := \Lambda \rtimes \mathrm{W}$, whose elements may be represented by $\mathsf{t}(\lambda)w$ ($\lambda \in \Lambda$, $w \in \mathrm{W}$) where $\mathsf{t} : \Lambda \subset \mathrm{W}^{\mathrm{aff}}$ is the natural inclusion.

The group $\mathrm{W}^{\mathrm{aff}}$ admits a length function $\ell : \mathrm{W}^{\mathrm{aff}} \to \mathbf{Z}_{\geq 0}$, which may be expressed explicitly by Matsumoto's formula (*cf.* [BR18, §4.2]).

3.4.2 *The ∞-category* ${}_{\chi_1}(\mathsf{D}^{\mathrm{aff}}_{\zeta})_{\chi_2}$**.** Denote by $\mathrm{I} \subset \mathrm{L}^{+}_{x}\mathrm{G}$ the preimage of B^{-} under the projection map $\mathrm{L}^{+}_{x}\mathrm{G} \to \mathrm{G}$, where $\mathrm{B}^{-} \subset \mathrm{G}$ is the Borel subgroup *opposite* to B.

The *Iwahori–Hecke stack* $\mathrm{Hec}^{\mathrm{aff}}_{\mathrm{G},x} := \mathrm{I}\backslash\mathrm{L}_{x}\mathrm{G}/\mathrm{I}$ admits the Bruhat stratification indexed by $\mathrm{W}^{\mathrm{aff}}$. Furthermore, it is equipped with three projection maps

$$\begin{tikzcd} & \mathrm{Hec}^{\mathrm{aff}}_{\mathrm{G},x} \arrow[r, "p"] \arrow[dl, "\overleftarrow{\pi}"'] \arrow[dr, "\overrightarrow{\pi}"] & \mathrm{Hec}_{\mathrm{G},x} \\ \mathsf{BI} & & \mathsf{BI} \end{tikzcd} \tag{3.25}$$

where p is induced from the inclusion $\mathrm{I} \subset \mathrm{L}^{+}_{x}\mathrm{G}$, and $\overleftarrow{\pi}$, $\overrightarrow{\pi}$ are defined by the left, respectively right I-action on $\mathrm{L}_{x}\mathrm{G}$. Denote by $\mathcal{G}^{\mathrm{aff}}$ the pullback of the A-gerbe $\mathcal{G}_{\mathrm{Hec}_{\mathrm{G}}}$ along p.

Since the projection $\mathrm{I} \twoheadrightarrow \mathrm{B}^{-} \twoheadrightarrow \mathrm{T}$ has a pro-unipotent kernel, we have a canonical isomorphism of abelian groups

$$\mathrm{Hom}(\Lambda, \mathrm{A}(-1)) \simeq \mathrm{Maps}_{*}(\mathsf{BI}, \mathsf{B}^{2}\mathrm{A}),$$

so any character $\chi : \Lambda \to \mathrm{A}(-1)$ defines an A-gerbe $\chi \otimes \Psi$ over BI (*cf.* §3.1.3).

Given a pair of characters $\chi_1, \chi_2 \in \mathrm{Hom}(\Lambda, \mathrm{A}(-1))$, we write ${}_{\chi_1}(\mathsf{D}^{\mathrm{aff}}_{\zeta})_{\chi_2}$ for the ∞-category of $(\overleftarrow{\pi}^*(\chi_1 \otimes \Psi) + \mathcal{G}^{\mathrm{aff}} - \overrightarrow{\pi}^*(\chi_2 \otimes \Psi), \zeta)$-twisted constructible complexes over $\mathrm{Hec}^{\mathrm{aff}}_{\mathrm{G},x}$ (*cf.* §1.3.3), which we refer to as the *monodromic affine Hecke category.*

3.4.3 *Convolution***.** The monodromic affine Hecke categories have the following "convolution product": Given a triple of characters $\chi_1, \chi_2, \chi_3 \in \mathrm{Hom}(\Lambda, \mathrm{A}(-1))$, there is a functor

$$\begin{aligned} {}_{\chi_1}(\mathsf{D}^{\mathrm{aff}}_{\zeta})_{\chi_2} \times {}_{\chi_2}(\mathsf{D}^{\mathrm{aff}}_{\zeta})_{\chi_3} &\to {}_{\chi_1}(\mathsf{D}^{\mathrm{aff}}_{\zeta})_{\chi_3}, \\ \mathcal{A}, \mathcal{B} &\mapsto \mathcal{A} \circ \mathcal{B} := m_!(p_1^*\mathcal{A} \otimes p_2^*\mathcal{B}), \end{aligned} \tag{3.26}$$

where m, p_1, p_2 refer to the three projections from $\mathrm{Hec}^{\mathrm{aff},[2]}_{\mathrm{G},x}$ to $\mathrm{Hec}^{\mathrm{aff}}_{\mathrm{G},x}$ defined by the natural groupoid structure on $\mathrm{Hec}^{\mathrm{aff}}_{\mathrm{G},x}$ (*cf.* §1.3.5).

The formation of (3.26) relies on the following identification of A-gerbes over $\mathrm{Hec}^{\mathrm{aff},[2]}_{\mathrm{G},x}$:

$$\begin{aligned} &(p_1)^*(\overleftarrow{\pi}^*(\chi_1 \otimes \Psi) + \mathcal{G}^{\mathrm{aff}} - \overrightarrow{\pi}^*(\chi_2 \otimes \Psi)) + (p_2)^*(\overleftarrow{\pi}^*(\chi_2 \otimes \Psi) + \mathcal{G}^{\mathrm{aff}} - \overrightarrow{\pi}^*(\chi_3 \otimes \Psi)) \\ &\quad\simeq (p_1)^*\overleftarrow{\pi}^*(\chi_1 \otimes \Psi) + (p_1)^*\mathcal{G}^{\mathrm{aff}} + (p_2)^*\mathcal{G}^{\mathrm{aff}} - (p_2)^*\overrightarrow{\pi}^*(\chi_3 \otimes \Psi) \\ &\quad\simeq m^*(\overleftarrow{\pi}^*(\chi_1 \otimes \Psi) + \mathcal{G}^{\mathrm{aff}} - \overrightarrow{\pi}^*(\chi_3 \otimes \Psi)), \end{aligned}$$

where we used the identification $\overrightarrow{\pi} \circ p_1 = \overleftarrow{\pi} \circ p_2$ in the first isomorphism to cancel out the $(\chi_2 \otimes \Psi)$-term, and the multiplicative structure on $\mathcal{G}^{\mathrm{aff}}$ (*cf.* Lemma 1.1.14) with $\overleftarrow{\pi} \circ p_1 = \overleftarrow{\pi} \circ m$, $\overrightarrow{\pi} \circ p_2 = \overrightarrow{\pi} \circ m$ for the second isomorphism.

The convolution product (3.26) admits an associativity constraint in the evident sense, although we will not attempt to lift it to full homotopy coherence data.

3.4.4 $\mathrm{W}^{\mathrm{aff}}$*-action on characters***.** Observe that $\mathrm{W}^{\mathrm{aff}}$ acts on the set $\mathrm{Hom}(\Lambda, \mathrm{A}(-1))$: Given an element $v = \mathsf{t}(\lambda)w \in \mathrm{W}^{\mathrm{aff}}$ and $\chi \in \mathrm{Hom}(\Lambda, \mathrm{A}(-1))$, we define

$$v(\chi) := w(\chi) - b(\lambda, \cdot), \tag{3.27}$$

where $w(\chi)$ is the contragredient action $w(\chi) := \chi(w^{-1}(\cdot))$.

The fact that the formula (3.27) defines a $\mathrm{W}^{\mathrm{aff}}$-action follows from the W-invariance of b, which is a consequence of (2.3).

3.4.5. For each $v = \mathsf{t}(\lambda)w \in \mathrm{W}^{\mathrm{aff}}$, we have the Bruhat cell

$$j^v : \mathrm{Hec}^{\mathrm{aff},v}_{\mathrm{G},x} \to \mathrm{Hec}^{\mathrm{aff}}_{\mathrm{G},x}.$$

The corresponding Bruhat cell $\mathrm{Fl}^v_{\mathrm{G},x}$ in the affine flag variety $\mathrm{Fl}_{\mathrm{G},x} := \mathrm{L}_x\mathrm{G}/\mathrm{I}$ has dimension $\ell(v)$. We shall consider the restrictions of the morphisms in (3.25) along j^v without changing the notation. For a pair of characters $\chi_1, \chi_2 \in \mathrm{Hom}(\Lambda, \mathrm{A}(-1))$, we write ${}_{\chi_1}(\mathsf{D}^{\mathrm{aff},v}_{\zeta})_{\chi_2}$ for the ∞-category of $(\overleftarrow{\pi}^*(\chi_1 \otimes \Psi) + \mathcal{G}^{\mathrm{aff}} - \overrightarrow{\pi}^*(\chi_2 \otimes \Psi), \zeta)$-twisted constructible complexes over $\mathrm{Hec}^{\mathrm{aff},v}_{\mathrm{G},x}$.

By Proposition 3.1.4, the restriction of the A-gerbe $\mathcal{G}^{\mathrm{aff}}$ along j^v may be identified as

$$(j^v)^*\mathcal{G}^{\mathrm{aff}} \simeq \overleftarrow{\pi}^*(b(\lambda, \cdot) \otimes \Psi) + (\mathcal{G}_{\varpi^\lambda}|_{\mathrm{Hec}^{\mathrm{aff},v}_{\mathrm{G},x}}). \tag{3.28}$$

This allows us to define pullback along $\pi : \mathrm{Hec}^{\mathrm{aff},v}_{\mathrm{G},x} \to \mathrm{Spec}\,\mathsf{k} \simeq x$ as a functor

$$\pi^* : \mathsf{Lis}_{\mathcal{G}_{\varpi^\lambda},\zeta}(x) \to {}_{-b(\lambda,\cdot)}(\mathsf{D}^{\mathrm{aff},v}_{\zeta})_0. \tag{3.29}$$

3.4.6. To proceed further, we will make additional choices. Let ϖ be a uniformizer at x. It allows us to identify $\mathrm{L}_x^+\mathrm{G}$ (respectively $\mathrm{L}_x\mathrm{G}$) with $\mathrm{G}[[\varpi]]$ (respectively $\mathrm{G}((\varpi))$).

We furthermore choose a lift $\dot{w} \in \mathrm{G}$ for each $w \in \mathrm{W}$. With these choices, every $v = \mathsf{t}(\lambda)w \in \mathrm{W}^{\mathrm{aff}}$ admits a lift to $\mathrm{G}((\varpi))$, namely $\dot{v} := \varpi^{\lambda}\dot{w}$.

We shall use $\dot{v}$ to relate the left and right I-action on $\mathrm{I}v\mathrm{I} \subset \mathrm{G}((\varpi))$. Namely, consider the $(\mathrm{I}\times\mathrm{I})$-action on $\mathrm{G}((\varpi))$ defined by $(b_1, b_2)\cdot g := b_1 g (b_2)^{-1}$. The stablizer of $\dot{v}$ is the subgroup $\mathrm{I}_{\dot{v}} \subset \mathrm{I}\times\mathrm{I}$ consisting of points (b_1, b_2) satisfying

$$b_1 = \dot{v} b_2 \dot{v}^{-1}.$$

In particular, the reduction $(\bar{b}_1, \bar{b}_2) \in \mathrm{T}\times\mathrm{T}$ of (b_1, b_2) satisfies

$$\bar{b}_1 = w(\bar{b}_2).$$

We thus obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{Hec}^{\mathrm{aff},v}_{\mathrm{G},x} \arrow[r, phantom, "\simeq"] & \mathsf{B}(\mathrm{I}_{\dot{v}}) \arrow[dl, "\overleftarrow{\pi}"'] \arrow[dr, "\overrightarrow{\pi}"] & \\
\mathsf{BT} \arrow[rr, "w^{-1}"] & & \mathsf{BT}
\end{tikzcd} \tag{3.30}$$

where $\overleftarrow{\pi}$, $\overrightarrow{\pi}$ stand for the compositions of the morphisms in (3.25) with the projection onto BT. It follows from (3.30) that we have an isomorphism of A-gerbes over $\mathrm{Hec}^{\mathrm{aff},v}_{\mathrm{G},x}$:

$$\overleftarrow{\pi}^*(w(\chi)\otimes\Psi) \simeq \overrightarrow{\pi}^*(\chi\otimes\Psi). \tag{3.31}$$

3.4.7. Using (3.31), we may view the pullback functor (3.29) as a functor

$$\pi^* : \mathsf{Lis}_{\mathcal{G}_{\varpi^\lambda},\zeta}(x) \to {}_{v(\chi)}(\mathsf{D}^{\mathrm{aff},v}_{\zeta})_{\chi} \tag{3.32}$$

defined for any $v = \mathsf{t}(\lambda)w \in \mathrm{W}^{\mathrm{aff}}$ and $\chi \in \mathrm{Hom}(\Lambda, \mathrm{A}(-1))$. Here, $v(\chi)$ refers to the $\mathrm{W}^{\mathrm{aff}}$-action on $\mathrm{Hom}(\Lambda, \mathrm{A}(-1))$ constructed in §3.4.4.

As in §3.3.2, we shall incorporate shifts and Tate twists: For any $\mathcal{E} \in \mathsf{Lis}_{\mathcal{G}_{\varpi^\lambda},\zeta}(x)$, we write $\widetilde{\mathcal{E}} := \pi^*\mathcal{E}(\frac{\ell(v)}{2})[\ell(v)]$. Then we define the *standard* and *costandard functors*

$$\Delta^v_\chi : \mathsf{Lis}_{\mathcal{G}_{\varpi^\lambda},\zeta}(x) \to {}_{v(\chi)}(\mathsf{D}^{\mathrm{aff}}_{\zeta})_{\chi}, \quad \mathcal{E} \mapsto (j^v)_!\widetilde{\mathcal{E}};$$
$$\nabla^v_\chi : \mathsf{Lis}_{\mathcal{G}_{\varpi^\lambda},\zeta}(x) \to {}_{v(\chi)}(\mathsf{D}^{\mathrm{aff}}_{\zeta})_{\chi}, \quad \mathcal{E} \mapsto (j^v)_*\widetilde{\mathcal{E}}.$$

Since j^v is affine, the images of Δ^v_χ and ∇^v_χ are perverse (after pulling back to the ind-scheme $\mathrm{Fl}_{\mathrm{G},x}$). Thus, we may also define the *intersection cohomology functor*

$$\mathrm{IC}^v_\chi : \mathsf{Lis}_{\mathcal{G}_{\varpi^\lambda},\zeta}(x) \to {}_{v(\chi)}(\mathsf{D}^{\mathrm{aff}}_{\zeta})_{\chi}, \tag{3.33}$$

sending $\mathcal{E}$ to the image of the natural morphism $(j^v)_!\widetilde{\mathcal{E}} \to (j^v)_*\widetilde{\mathcal{E}}$.

The key statement about IC^v_χ, which we shall prove, is the "parity vanishing" of its fiber cohomologies. It is an analogue of [LY20, Proposition 3.12] for étale levels.

Proposition 3.4.8. *Fix* $v \in \mathrm{W}^{\mathrm{aff}}$ *and* $\chi \in \mathrm{Hom}(\Lambda, \mathrm{A}(-1))$. *The fibers of* IC^v_χ *over* $\mathrm{Fl}_{\mathrm{G},x}$ *are concentrated in cohomological degrees* n *such that* $n + \ell(v)$ *is even.*

Remark 3.4.9. For our application of Proposition 3.4.8 (*cf.* §3.6), we will *only* need the case $\chi = 0$. However, in order to prove Proposition 3.4.8 for $\chi = 0$, we must work with all possible choices of χ. This feature is special to the twisted context.

3.5. Parity vanishing.

3.5.1. The goal of this subsection is to prove Proposition 3.4.8, so we retain the notation of §3.4. We begin by describing the restriction of $\mathcal{G}^{\mathrm{aff}}$ to dimension-1 Bruhat varieties.

Let $\mathrm{S} \subset \mathrm{W}^{\mathrm{aff}}$ be the subset of length-1 elements. By Matsumoto's formula, an element of S is one of the following kinds

1. a finite simple reflection $s_{\check{\alpha}} \in \mathrm{W}$ associated to $\check{\alpha} \in \check{\Delta}$, viewed as an element of $\mathrm{W}^{\mathrm{aff}}$ via the natural inclusion $\mathrm{W} \subset \mathrm{W}^{\mathrm{aff}}$;
2. a product $\mathsf{t}(\theta) s_{\check{\theta}}$, where $\check{\theta}$ is a highest root (which we call an *affine simple reflection*).

For $s \in \mathrm{S}$, write $\mathrm{P}_s \subset \mathrm{G}((\varpi))$ for the corresponding subminimal parahoric group scheme. The quotient P_s/I is isomorphic to $\mathbb{P}^1$, so isomorphism classes of A-gerbes over P_s/I are in bijection with $\mathrm{A}(-1)$, where $a \in \mathrm{A}(-1)$ corresponds to the class of $a \otimes \Psi(\mathcal{O}_{\mathbb{P}^1}(1))$.

3.5.2. Let us determine the restriction of $\mathcal{G}^{\mathrm{aff}}$ along the closed immersion

$$i_s : \mathrm{P}_s/\mathrm{I} \hookrightarrow \mathrm{G}((\varpi))/\mathrm{I} =: \mathrm{Fl}_{\mathrm{G},x}. \tag{3.34}$$

The following result is an analogue of the computation of the degrees of line bundles on $\mathrm{Gr}_{\mathrm{G},x}$, for G simple and simply connected, after pulling back to P_s/I (*cf.* [Fal03, Theorem 7]). It can be deduced from *loc.cit.*, but let us offer a proof internal to A-gerbes.

Lemma 3.5.3. *The isomorphism class $a(s) \in \mathrm{A}(-1)$ of $(i_s)^*\mathcal{G}^{\mathrm{aff}}$ is given by*

$$a(s) = \begin{cases} 0 & s = s_{\check{\alpha}} \text{ for } \check{\alpha} \in \check{\Delta} \\ \mathrm{Q}(\theta) & s = \mathsf{t}(\theta) s_{\check{\theta}} \text{ for } \check{\theta} \text{ a highest root} \end{cases}$$

Proof. If $s = s_{\check{\alpha}}$ is a finite simple reflection, then $\mathrm{P}_s \subset \mathrm{G}[[\varpi]]$ is the pre-image of the standard parabolic subgroup $\mathrm{P}_{\check{\alpha}} \subset \mathrm{G}$ corresponding to $\check{\alpha} \in \check{\Delta}$. This implies that (3.34) factors as

$$\mathrm{P}_s/\mathrm{I} \simeq \mathrm{P}_{\check{\alpha}}/\mathrm{B} \simeq \mathrm{G}[[\varpi]]/\mathrm{I} \hookrightarrow \mathrm{G}((\varpi))/\mathrm{I}. \tag{3.35}$$

Since $\mathcal{G}^{\mathrm{aff}}$ is trivial over $\mathrm{G}[[\varpi]]/\mathrm{I}$, we see that $(i_s)^*\mathcal{G}^{\mathrm{aff}}$ is trivial.

Suppose now that $s = \mathsf{t}(\theta) s_{\check{\theta}}$ is an affine simple reflection. Let us identify the composition of (3.34) with the projection onto $\mathrm{Gr}_{\mathrm{G},x}$:

$$\mathrm{P}_s/\mathrm{I} \to \mathrm{Gr}_{\mathrm{G},x}. \tag{3.36}$$

We first extend θ to a morphism $f_{\check{\theta}} : \mathrm{G}_{\check{\theta}} \to \mathrm{G}$, where $\mathrm{G}_{\check{\theta}}$ is simply connected of semisimple rank 1, covering the root subgroups $\mathrm{N}_{\check{\theta}}$, $\mathrm{N}_{-\check{\theta}}$ of G. The morphism (3.36) factors through the morphism $\mathrm{Gr}_{\mathrm{G}_{\check{\theta}},x} \to \mathrm{Gr}_{\mathrm{G},x}$ induced from $f_{\check{\theta}}$, so we may replace G by $\mathrm{G}_{\check{\theta}}$ and assume that G is simply connected with unique simple coroot θ.

Denote by ω the fundamental coweight of T_{ad} and by $\mathrm{K}_\lambda \subset \mathrm{L}_x\mathrm{G}$ the subgroup scheme $\varpi^\lambda \mathrm{L}_x^+\mathrm{G} \varpi^{-\lambda}$ for each $\lambda \in \Lambda_{\mathrm{ad}}$. Then $\mathrm{K}_\omega \subset \mathrm{P}_s$ and this inclusion induces an isomorphism

$$\mathrm{K}_\omega/\mathrm{K}_\omega \cap \mathrm{K}_0 \simeq \mathrm{P}_s/\mathrm{I}, \tag{3.37}$$

under which (3.36) corresponds to the natural inclusion

$$\mathrm{K}_\omega/\mathrm{K}_\omega \cap \mathrm{K}_0 \subset \mathrm{Gr}_{\mathrm{G},x}. \tag{3.38}$$

Let us form the group scheme $\widetilde{\mathrm{G}} := \mathrm{G} \rtimes \mathrm{T}_{\mathrm{ad}}$.[16] Its center is the anti-diagonal copy of T, so the natural embedding $\mathrm{G} \subset \widetilde{\mathrm{G}}$ induces an isomorphism of their adjoint forms. The inclusion

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[16] To be more explicit, one may fix an isomorphism $\mathrm{G} \simeq \mathrm{SL}_2$ and realize $\widetilde{\mathrm{G}}$ as GL_2.

(3.38) for G, $\widetilde{\mathrm{G}}$ are related by the commutative diagram

$$\begin{tikzcd}
\mathrm{K}_\omega/\mathrm{K}_\omega \cap \mathrm{K}_0 \arrow[r, phantom, "\subset"] \arrow[d] & \mathrm{Gr}_{\mathrm{G},x} \arrow[d] \\
\widetilde{\mathrm{K}}_\omega/\widetilde{\mathrm{K}}_\omega \cap \widetilde{\mathrm{K}}_0 \arrow[r, phantom, "\subset"] & \mathrm{Gr}_{\widetilde{\mathrm{G}},x}
\end{tikzcd} \tag{3.39}$$

Moreover, the lower inclusion in (3.39) coincides with the $\varpi^{(0,\omega)}$-translate of the Schubert cell in $\mathrm{Gr}_{\widetilde{\mathrm{G}},x}$ containing $\varpi^{(0,-\omega)}$, with respect to the maximal torus $\mathrm{T} \times \mathrm{T}_{\mathrm{ad}}$ of $\widetilde{\mathrm{G}}$.

Since G is simply connected, we may extend μ to an étale level $\widetilde{\mu}$ of $\widetilde{\mathrm{G}}$ (*cf.* §2.1.18). Evaluating the symmetric form (2.19) of $\widetilde{\mu}$ at $(0,-\omega)$, we obtain the character

$$\Lambda \oplus \Lambda_{\mathrm{ad}} \to \mathrm{A}(-1) \tag{3.40}$$

which annihilates the summand Λ_{ad} and sends $\theta \in \Lambda$ to $-\mathrm{Q}(\theta)$. It now follows from Proposition 3.1.4 (applied to $\widetilde{\mathrm{G}}$) that the restriction of $\mathcal{G}_{\mathrm{Gr}_{\mathrm{G}}}$ along (3.38) coincides with the pullback of $-\mathrm{Q}(\theta) \otimes \Psi$ along the composition

$$\begin{aligned}
\mathrm{K}_\omega/\mathrm{K}_\omega \cap \mathrm{K}_0 &\to \widetilde{\mathrm{K}}_\omega/\widetilde{\mathrm{K}}_\omega \cap \widetilde{\mathrm{K}}_0 \\
&\simeq \widetilde{\mathrm{K}}_0/\widetilde{\mathrm{K}}_0 \cap \widetilde{\mathrm{K}}_{-\omega} \to \mathsf{B}(\widetilde{\mathrm{K}}_0 \cap \widetilde{\mathrm{K}}_{-\omega}) \xrightarrow{\pi} \mathsf{B}(\mathrm{T} \times \mathrm{T}_{\mathrm{ad}}) \to \mathsf{B}\mathrm{T} \simeq \mathsf{B}\mathbb{G}_m
\end{aligned} \tag{3.41}$$

where $\pi := \pi^{(0,-\omega)}$ is the morphism defined in §3.1.2. The morphism (3.41) classifies the tautological line bundle $\mathcal{O}_{\mathbb{P}^1}(-1)$, upon identifying its source with $\mathbb{P}^1$. It follows that the restriction of $\mathcal{G}_{\mathrm{Gr}_{\mathrm{G}}}$ along (3.38) is classified by $\mathrm{Q}(\theta)$. ☐

3.5.4. For each $\chi \in \mathrm{Hom}(\Lambda, \mathrm{A}(-1))$, we define the subset

$$\mathrm{S}^\circ_\chi \subset \mathrm{S}$$

of *integral elements* (with respect to χ), which consists of

1. finite simple reflections $s_{\check{\alpha}}$ satisfying $\chi(\alpha) = 0$;
2. affine simple reflections $\mathsf{t}(\theta)s_{\check{\theta}}$ satisfying $\mathrm{Q}(\theta) + \chi(\theta) = 0$.

Remark 3.5.5. With respect to the $\mathrm{W}^{\mathrm{aff}}$-action on $\mathrm{Hom}(\Lambda, \mathrm{A}(-1))$ (*cf.* §3.4.4), each element $s \in \mathrm{S}^\circ_\chi$ fixes χ. This is clear when s is a finite simple reflection. When $s = \mathsf{t}(\theta)s_{\check{\theta}}$ is an affine simple reflection, this follows from

$$\begin{aligned}
s(\chi) = s_{\check{\theta}}(\chi) - b(\theta, \cdot) &= \chi - \langle \check{\theta}, \cdot \rangle \chi(\theta) - b(\theta, \cdot) \\
&= \chi - \langle \check{\theta}, \cdot \rangle (\chi(\theta) + \mathrm{Q}(\theta)) = \chi,
\end{aligned}$$

where we used (2.3) in the third equality.

Lemma 3.5.6. *Given $\chi \in \mathrm{Hom}(\Lambda, \mathrm{A}(-1))$, an element $s \in \mathrm{S}$ belongs to S°_χ if and only if $(i_s)^*(\mathcal{G}^{\mathrm{aff}} - \vec{\pi}^*(\chi \otimes \Psi))$ is trivial.*

Proof. The A-gerbe $(i_s)^*\vec{\pi}^*(-\chi \otimes \Psi)$ is classified by an element $a_\chi(s) \in \mathrm{A}(-1)$ (*cf.* §3.5.1). We claim that $a_\chi(s)$ is given by

$$a_\chi(s) = \begin{cases} \chi(\alpha) & s = s_{\check{\alpha}} \text{ for } \check{\alpha} \in \check{\Delta} \\ \chi(\theta) & s = \mathsf{t}(\theta)s_{\check{\theta}} \text{ for } \check{\theta} \text{ a highest root} \end{cases} \tag{3.42}$$

Indeed, for $s = s_{\check{\alpha}}$ a finite simple reflection, this follows from the factorization (3.35). For $s = \mathsf{t}(\theta) s_{\check{\theta}}$ an affine simple reflection, we argue as in the proof of Lemma 3.5.3, reducing to the fact that pulling back $\Psi^{-\chi}$ along (3.37) yields the A-gerbe classified by $\chi(\theta)$.

The assertion now follows by combining (3.42) with Lemma 3.5.3. □

3.5.7. We now study the functor IC^s_χ (*cf.* §3.4.7) for $s \in \mathrm{S}$.

Note that for $s \in \mathrm{S}^\circ_\chi$, the A-gerbe $(i_s)^*(\mathcal{G}^{\mathrm{aff}} - \vec{\pi}^*(\chi \otimes \Psi))$ is canonically trivial: It is trivial by Lemma 3.5.6 and its fiber along $e : \mathrm{I}/\mathrm{I} \to \mathrm{P}_s/\mathrm{I}$ admits a canonical trivialization. In this case, IC^s_χ may be understood as a functor

$$\mathrm{IC}^s_\chi : \mathsf{Lis}(x) \to {}_{\chi}(\mathsf{D}^{\mathrm{aff},s}_\zeta)_\chi,$$

as $s(\chi) = \chi$ by Remark 3.5.5. Write $\pi_s : \mathrm{P}_s/\mathrm{I} \to \operatorname{Spec} \mathsf{k} \simeq x$ for the structural map.

Proposition 3.5.8. *Given $\chi \in \mathrm{Hom}(\Lambda, \mathrm{A}(-1))$ and $s \in \mathrm{S}$, there holds*

(1) if $s \in \mathrm{S}^\circ_\chi$, then

$$\mathrm{IC}^s_\chi \simeq (i_s)_*(\pi_s)^*(\frac{1}{2})[1].$$

(2) if $s \notin \mathrm{S}^\circ_\chi$, then the canonical maps are isomorphisms:

$$\Delta^s_\chi \simeq \mathrm{IC}^s_\chi \simeq \nabla^s_\chi.$$

Proof. By construction, IC^s_χ is the intermediate extension of $((i_s)^*(\mathcal{G}^{\mathrm{aff}} - \vec{\pi}^*(\chi \otimes \Psi)), \zeta)$-twisted local systems along the inclusion $\mathrm{I}s\mathrm{I}/\mathrm{I} \subset \mathrm{P}_s/\mathrm{I}$. Identifying this inclusion with $\mathbb{A}^1 \subset \mathbb{P}^1$, we deduce the result from Lemma 3.5.6. □

3.5.9 *Proof of Proposition 3.4.8.* Finally, we are ready to prove the parity vanishing of IC^v_χ. The proof will span §3.5.10–§3.5.12 below.

To simplify the notation, we write $\Delta^v_\chi(\mathsf{e})$, $\mathrm{IC}^v_\chi(\mathsf{e})$, $\nabla^v_\chi(\mathsf{e})$ for the images of a rank-1 $(\mathcal{G}_{\varpi^\lambda}, \zeta)$-twisted local system over $x \simeq \operatorname{Spec} \mathsf{k}$ under Δ^v_χ, IC^v_χ, ∇^v_χ (where $v = \mathsf{t}(\lambda) w$). Since $\mathcal{G}_{\varpi^\lambda}$ is *non-canonically* trivial, these objects are only well-defined up to *non-unique* isomorphisms. The statement that we want to prove, *i.e.* the parity vanishing of fibers of $\mathrm{IC}^v_\chi(\mathsf{e})$, only concerns its isomorphism class.

3.5.10. We begin by observing that an object $\mathcal{A} \in {}_{v(\chi)}(\mathsf{D}^{\mathrm{aff}}_\zeta)_\chi$ satisfies the property

$$\mathrm{H}^n i^*_y \mathcal{A} \neq 0 \text{ for some } y \in \mathrm{Fl}_{\mathrm{G},x}(\mathsf{k}) \Rightarrow \ell(v) + n \text{ is even} \tag{3.43}$$

if and only if $\mathcal{A}$ belongs to the full subcategory of ${}_{v(\chi)}(\mathsf{D}^{\mathrm{aff}}_\zeta)_\chi$ generated under extension by objects of the form

$$\Delta^{v_1}_\chi(\mathsf{e})[d] \text{ where } v_1(\chi) = v(\chi) \text{ and } \ell(v) - \ell(v_1) - d \text{ is even.} \tag{3.44}$$

Indeed, any object of the form (3.44) satisfies (3.43) and this property is preserved under extensions. Conversely, any $\mathcal{A} \in {}_{v(\chi)}(\mathsf{D}^{\mathrm{aff}}_\zeta)_\chi$ is supported on Bruhat cells $\mathrm{I}v_1\mathrm{I}/\mathrm{I}$ satisfying $v_1(\chi) = v(\chi)$. If $\mathcal{A}$ satisfies (3.43), it belongs to the full subcategory generated by objects of the form (3.44) under extensions, by induction on support.

3.5.11. We now prove that $\mathrm{IC}^v_\chi(\mathsf{e})$ satisfies (3.43) by induction on $\ell(v)$.

For the base step $\ell(v) = 0$, the object $\mathrm{IC}^v_\chi(\mathsf{e})$ is a skyscraper, so $\mathrm{H}^n i^*_y \mathrm{IC}^v_\chi(\mathsf{e}) \neq 0$ for some $y \in \mathrm{Fl}_{\mathrm{G},x}(\mathsf{k})$ implies $n = 0$.

For the inductive step, we choose $s \in \mathrm{S}$ such that $\ell(v) = \ell(vs) + 1$. Applying the decomposition theorem to the Demazure resolution (*cf.* [BBD82, §6.2.4]), we see that $\mathrm{IC}^{v}_{\chi}(\mathsf{e})$ is a direct summand of the convolution product (*cf.* §3.4.3)

$$\mathrm{IC}^{vs}_{s(\chi)}(\mathsf{e}) \circ \mathrm{IC}^{s}_{\chi}(\mathsf{e}), \tag{3.45}$$

so it suffices to prove that the object (3.45) satisfies (3.43).

Applying the induction hypothesis to $\mathrm{IC}^{vs}_{s(\chi)}(\mathsf{e})$ and using the equivalence of §3.5.10, we may replace it by an object of the form $\Delta^{v_1}_{s(\chi)}(\mathsf{e})[d]$ where $v_1 s(\chi) = v(\chi)$ and $\ell(vs) - \ell(v_1) - d$ is even. It then suffices to prove the following *purity* statement:

$$\mathrm{H}^n i_y^*(\Delta^{v_1}_{s(\chi)}(\mathsf{e}) \circ \mathrm{IC}^{s}_{\chi}(\mathsf{e})) \neq 0 \text{ for some } y \in \mathrm{Fl}_{\mathrm{G},x}(\mathsf{k}) \Rightarrow \ell(v_1 s) + n = 0 \tag{3.46}$$

for any $s \in \mathrm{S}$ and $v_1 \in \mathrm{W}^{\mathrm{aff}}$.

3.5.12. Finally, we prove (3.46) via a case-by-case analysis.

(1) $s \notin \mathrm{S}^{\circ}_{\chi}$ *and* $\ell(v_1 s) = \ell(v_1) + 1$. In this case, we have

$$\Delta^{v_1}_{s(\chi)}(\mathsf{e}) \circ \mathrm{IC}^{s}_{\chi}(\mathsf{e}) \simeq \Delta^{v_1}_{s(\chi)}(\mathsf{e}) \circ \Delta^{s}_{\chi}(\mathsf{e}) \simeq \Delta^{v_1 s}_{\chi}(\mathsf{e}),$$

where we used the isomorphism $\Delta^{s}_{\chi}(\mathsf{e}) \simeq \mathrm{IC}^{s}_{\chi}(\mathsf{e})$ (*cf.* Proposition 3.5.8).

(2) $s \notin \mathrm{S}^{\circ}_{\chi}$ *and* $\ell(v_1 s) = \ell(v_1) - 1$. In this case, we have

$$\Delta^{v_1}_{s(\chi)}(\mathsf{e}) \simeq \Delta^{v_1 s}_{\chi}(\mathsf{e}) \circ \Delta^{s}_{s(\chi)}(\mathsf{e}). \tag{3.47}$$

On the other hand, Proposition 3.5.8 yields $\mathrm{IC}^{s}_{\chi}(\mathsf{e}) \simeq \nabla^{s}_{\chi}(\mathsf{e})$, and we have an isomorphism $\Delta^{s}_{s(\chi)}(\mathsf{e}) \circ \nabla^{s}_{\chi}(\mathsf{e}) \simeq \Delta^{e}_{\chi}(\mathsf{e})$ which reduces to a calculation for SL_2 (*cf.* [LY20, Lemma 3.5]). Putting these together gives

$$\begin{aligned}\Delta^{v_1}_{s(\chi)}(\mathsf{e}) \circ \mathrm{IC}^{s}_{\chi}(\mathsf{e}) &\simeq \Delta^{v_1 s}_{\chi}(\mathsf{e}) \circ \Delta^{s}_{s(\chi)}(\mathsf{e}) \circ \mathrm{IC}^{s}_{\chi}(\mathsf{e}) \\ &\simeq \Delta^{v_1 s}_{\chi}(\mathsf{e}) \circ \Delta^{s}_{s(\chi)}(\mathsf{e}) \circ \nabla^{s}_{\chi}(\mathsf{e}) \simeq \Delta^{v_1 s}_{\chi}(\mathsf{e}).\end{aligned}$$

(3) $s \in \mathrm{S}^{\circ}_{\chi}$ *and* $\ell(v_1 s) = \ell(v_1) + 1$. Consider the Cartesian square

$$\begin{tikzcd} \mathrm{G}((\varpi)) \times^{\mathrm{I}} \mathrm{P}_s/\mathrm{I} \arrow[r, "m"] \arrow[d, "p"] & \mathrm{G}((\varpi))/\mathrm{I} \arrow[d, "f"] \\ \mathrm{G}((\varpi))/\mathrm{I} \arrow[r, "f"] & \mathrm{G}((\varpi))/\mathrm{P}_s \end{tikzcd}$$

where f is induced from the inclusion $\mathrm{I} \subset \mathrm{P}_s$ and p, m are the projection, respectively multiplication maps. The condition $s \in \mathrm{S}^{\circ}_{\chi}$ allows us to identify $(\cdot) \circ \mathrm{IC}^{s}_{\chi}(\mathsf{e})$ with the functor $f^* f_!(\frac{1}{2})[1]$ (*cf.* Proposition 3.5.8).

On the other hand, the condition on v_1 implies that $\mathrm{I}v_1\mathrm{I} \times^{\mathrm{I}} \mathrm{P}_s \simeq \mathrm{I}v_1\mathrm{P}_s$ along the multiplication map. It induces an isomorphism $\mathrm{I}v_1\mathrm{I}/\mathrm{I} \simeq \mathrm{I}v_1\mathrm{P}_s/\mathrm{P}_s$, so

$$\Delta^{v_1}_{s(\chi)}(\mathsf{e}) \circ \mathrm{IC}^{s}_{\chi}(\mathsf{e}) \simeq f^* f_! \Delta^{v_1}_{\chi}(\mathsf{e})(\frac{1}{2})[1]$$

is isomorphic to the !-extension along $\mathrm{I}v_1\mathrm{P}_s/\mathrm{I} \to \mathrm{G}((\varpi))/\mathrm{I}$ of a constant sheaf placed in cohomological degree $-\ell(v_1 s)$.

(4) $s \in \mathrm{S}^{\circ}_{\chi}$ *and* $\ell(v_1 s) = \ell(v_1) - 1$. In this case, we again have (3.47). There is also an isomorphism $\Delta^{s}_{\chi}(\mathsf{e}) \circ \mathrm{IC}^{s}_{\chi}(\mathsf{e}) \simeq \mathrm{IC}^{s}_{\chi}(\mathsf{e})(-\frac{1}{2})[-1]$ as $\mathrm{IC}^{s}_{\chi}(\mathsf{e})$ is constant (*cf.* Proposition 3.5.8). Putting these together gives

$$\begin{aligned}\Delta^{v_1}_{s(\chi)}(\mathsf{e}) \circ \mathrm{IC}^{s}_{\chi}(\mathsf{e}) &\simeq \Delta^{v_1 s}_{\chi}(\mathsf{e}) \circ \Delta^{s}_{s(\chi)}(\mathsf{e}) \circ \mathrm{IC}^{s}_{\chi}(\mathsf{e}) \\ &\simeq \Delta^{v_1 s}_{\chi}(\mathsf{e}) \circ \mathrm{IC}^{s}_{\chi}(\mathsf{e})(-\frac{1}{2})[-1],\end{aligned}$$

whose fibers are concentrated in cohomological degree $-\ell(v_1 s)$ by step (3).

The proof of (3.46), hence of Proposition 3.4.8, is now complete. □

3.6. Applications to the Satake category.

3.6.1. The goal of this subsection is to deduce Theorem 3.3.5 from Proposition 3.4.8.

First, we note the analogue of Proposition 3.4.8 for the affine Grassmannian. We let x be a k-point of X and consider the functor IC^{λ} associated to $\lambda \in \Lambda^{\sharp,+}$ (*cf.* §3.3.4).

Lemma 3.6.2. *Fix* $\lambda \in \Lambda^{\sharp,+}$. *The fibers of* IC^{λ} *over* $\mathrm{Gr}_{\mathrm{G},x}$ *are concentrated in cohomological degrees* n *such that* $n + \langle 2\check{\rho}, \lambda\rangle$ *is even.*

Proof. The projection map $\pi : \mathrm{Fl}_{\mathrm{G},x} \to \mathrm{Gr}_{\mathrm{G},x}$ is smooth of relative dimension $\ell(w_0)$ and the pre-image of the Schubert stratification is refined by the Bruhat stratification (*cf.* §3.4.5). Thus, we have an isomorphism

$$\pi^* \mathrm{IC}^{\lambda}(\frac{\ell(w_0)}{2})[\ell(w_0)] \simeq \mathrm{IC}^{v}_{0},$$

where IC^{v}_{0} is the functor (3.33) associated to $\chi = 0$ and $v \in \mathrm{W}^{\mathrm{aff}}$ the unique element of $\mathrm{W}\lambda\mathrm{W}$ with length $\langle 2\check{\rho}, \lambda\rangle + \ell(w_0)$. The assertion now follows from Proposition 3.4.8. □

3.6.3. We shall now prove Theorem 3.3.5.

Proof of Theorem 3.3.5. Recall that $\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},x})$ may be realized as a full subcategory of $\mathsf{D}_{\mathcal{G},\zeta}(\mathrm{Gr}_{\mathrm{G},x})$ (*cf.* Remark 1.3.8).

Fix $\lambda \in \Lambda^{\sharp,+}$ and let $\mathcal{E}_1$, $\mathcal{E}_2$ be $(\mathcal{G}_{\varpi^{\lambda}}, \zeta)$-twisted local systems over S. Let us compute the space of maps from $\mathrm{IC}^{\lambda}(\mathcal{E}_1)$ to $\mathrm{IC}^{\lambda}(\mathcal{E}_2)[1]$ *as objects of* $\mathsf{D}_{\mathcal{G},\zeta}(\mathrm{Gr}_{\mathrm{G},x})$.

Denote by $j : \mathrm{Gr}^{\lambda}_{\mathrm{G},x} \hookrightarrow \overline{\mathrm{Gr}}^{\lambda}_{\mathrm{G},x}$ the open immersion of the Schubert cell into its closure and i its complement. Then $i^* \mathrm{IC}^{\lambda}(\mathcal{E}_1)$ is concentrated in perverse cohomological degrees ≤ -1, while $i^! \mathrm{IC}^{\lambda}(\mathcal{E}_2)[1]$ in degrees ≥ 0. Applying excision to $\mathrm{IC}^{\lambda}(\mathcal{E}_1)$, we obtain

$$\begin{aligned}\mathrm{Maps}_{\mathsf{D}_{\mathcal{G},\zeta}(\mathrm{Gr}_{\mathrm{G},x})}(\mathrm{IC}^{\lambda}(\mathcal{E}_1), \mathrm{IC}^{\lambda}(\mathcal{E}_2)[1]) &\simeq \mathrm{Maps}_{\mathsf{D}_{\mathcal{G},\zeta}(\mathrm{Gr}^{\lambda}_{\mathrm{G},x})}(\widetilde{\mathcal{E}}_1, \widetilde{\mathcal{E}}_2[1]) \\ &\simeq \mathrm{Maps}_{\mathsf{Lis}_{\mathcal{G}_{\varpi^{\lambda}},\zeta}(\mathrm{S})}(\mathcal{E}_1, \mathcal{E}_2[1]), \qquad (3.48)\end{aligned}$$

where the second isomorphism is due to the fact that the fibers of $\mathrm{Gr}^{\lambda}_{\mathrm{G},x} \to \mathrm{S}$ have vanishing cohomology in degree 1.

The isomorphism (3.48) shows that the functor IC^{λ} is fully faithful and its essential image is closed under extensions.

Next, we fix $\lambda_1 \neq \lambda_2 \in \Lambda^{\sharp,+}$ and $(\mathcal{G}_{\varpi^{\lambda_1}}, \zeta)$-, respectively $(\mathcal{G}_{\varpi^{\lambda_2}}, \zeta)$-twisted local systems $\mathcal{E}_1$, respectively $\mathcal{E}_2$ over S. We claim

$$\mathrm{Maps}_{\mathsf{D}_{\mathcal{G},\zeta}(\mathrm{Gr}_{\mathrm{G},x})}(\mathrm{IC}^{\lambda_1}(\mathcal{E}_1), \mathrm{IC}^{\lambda_2}(\mathcal{E}_2)[1]) \simeq 0. \qquad (3.49)$$

If S is a geometric point, this follows from Lemma 3.6.2 by the classical argument (*cf.* the proof of [Gai01, Proposition 1]). The case for general S follows from the base change property of Homs between ULA objects in $\mathsf{D}_{\mathcal{G},\zeta}(\mathrm{Gr}_{\mathrm{G},x})$ (*cf.* the proof of Proposition 3.2.5).

The vanishing (3.49) shows that the essential images of the functors IC^{λ_1} and IC^{λ_2} are mutually orthogonal and admit no nontrivial extensions.

The fully faithfulness of (3.23) follows from the fully faithfulness of IC^{λ}, for each $\lambda \in \Lambda^{\sharp,+}$, and the orthogonality of the essential images of IC^{λ_1}, IC^{λ_2}, for $\lambda_1 \neq \lambda_2$. The essential surjectivity of (3.23) follows from the fact that $\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},x})$ is generated under extensions by the essential images of IC^{λ} for $\lambda \in \Lambda^{\sharp,+}$ (*cf.* Corollary 3.1.10), the closure of the essential image of IC^{λ} under extensions, and the absence of nontrivial extensions between the essential images of IC^{λ_1} and IC^{λ_2}, for $\lambda_1 \neq \lambda_2$. □

4. TORI

In this section, we fix a smooth curve X over a field k and an X-torus T. Let e be a finite extension of $\mathbf{Q}_\ell$, for a prime ℓ invertible in k. Let $\zeta : \mathrm{A} \subset \mathsf{e}^\times$ be a finite subgroup and μ be an A-valued étale level of T.

The goal of this section is to construct the geometric Satake equivalence (*cf.* Theorem 2.3.2) for (T, μ), which will be supplied in §4.1.9. The key ingredient is Proposition 4.2.6, which relates $\mathcal{G}_{\mathrm{Gr}_\mathrm{T}}$ to the metaplectic dual datum ν.

4.1. Reduction to sharp tori.

4.1.1. We begin by specializing the metaplectic dual data (H, ν) (*cf.* §2.1) to (T, μ), whose construction is substantially simpler than the general case.

Indeed, writing Λ for the sheaf of cocharacters of T and $\Lambda^\sharp \subset \Lambda$ for the subsheaf defined in §2.1.6, we see that $\Lambda^\sharp \subset \Lambda$ corresponds to an isogeny of X-tori $\mathrm{T}^\sharp \to \mathrm{T}$ and H is the Langlands dual of $\mathrm{T}^\sharp$ (as a locally constant étale sheaf of $\mathbf{Z}$-tori).

On the other hand, the pullback $\mu^\sharp$ of the étale level μ to $\mathrm{T}^\sharp$ acquires an $\mathbb{E}_\infty$-monoidal structure, so it induces an $\mathbb{E}_\infty$-monoidal morphism $\nu : \Lambda^\sharp \to \mathsf{B}^2_{\mathrm{X}}\mathrm{A}$ by applying the functor $\mathcal{M}aps_*(\mathsf{B}_{\mathrm{X}}\mathbb{G}_m, \cdot)$ (*cf.* §2.1.10).

4.1.2. For each finite set I, we write

$$f_\mathrm{I} : \mathrm{Gr}_{\mathrm{T}^\sharp,\mathrm{I}} \to \mathrm{Gr}_{\mathrm{T},\mathrm{I}} \tag{4.1}$$

for the induced morphism on affine Grassmannians (*cf.* Remark 1.3.11).

Since (4.1) is ind-proper, the functor

$$(f_\mathrm{I})_! : \mathsf{D}_{\mathcal{G},\zeta}(\mathrm{Gr}_{\mathrm{T}^\sharp,\mathrm{I}}) \to \mathsf{D}_{\mathcal{G},\zeta}(\mathrm{Gr}_{\mathrm{T},\mathrm{I}}) \tag{4.2}$$

preserves universal local acyclicity relative to X^I. Moreoever, the base change of (4.1) to any geometric point of X^I is a closed immersion, so $(f_\mathrm{I})_!$ is also perverse t-exact. Finally, $\mathrm{L}^+_\mathrm{I}\mathrm{T}^\sharp \to \mathrm{L}^+_\mathrm{I}\mathrm{T}$ is surjective in the étale topology. By Remark 1.3.8, we see that (4.2) restricts to a functor on the Satake categories

$$(f_\mathrm{I})_! : \mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{T}^\sharp,\mathrm{I}}) \to \mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{T},\mathrm{I}}). \tag{4.3}$$

The functor (4.3) is naturally symmetric monoidal, as its formation is compatible with the fusion product (*cf.* §1.3.12).

Proposition 4.1.3. *The functor* (4.3) *is an equivalence of categories.*

4.1.4. Proposition 3.1.10 is an immediate consequence of Corollary 3.1.10 if $\mathrm{I} = \{1\}$. We shall reduce the general case to this one using factorization.

Suppose that T is split. For an I-tuple $\lambda^{\mathrm{I}} = (\lambda^i)_{i\in\mathrm{I}}$ of elements of Λ, we write $\varpi^{\lambda^{\mathrm{I}}} : \mathrm{X}^{\mathrm{I}} \to \mathrm{Gr}_{\mathrm{T},\mathrm{I}}$ for the closed immersion sending $x^{\mathrm{I}} = (x^i)_{i\in\mathrm{I}}$ to the modification of T-bundles

$$\mathcal{O} \overset{x^{\mathrm{I}}}{\sim} \mathcal{O}(\sum_{i\in\mathrm{I}} \lambda^i \Gamma_{x^i}).$$

Denote by $\mathrm{X}^{\mathrm{I},\mathrm{disj}} \subset \mathrm{X}^{\mathrm{I}}$ the open subscheme consisting of pairwise disjoint points x^i ($i \in \mathrm{I}$) of X. Write $j : \mathrm{Gr}_{\mathrm{T},\mathrm{I}}^{\mathrm{disj}} \to \mathrm{Gr}_{\mathrm{T},\mathrm{I}}$ for its base change. Recall (*cf.* Proposition 1.3.13) that the pullback functor is fully faithful:

$$j^* : \mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Gr}_{\mathrm{T},\mathrm{I}}) \subset \mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Gr}_{\mathrm{T},\mathrm{I}}^{\mathrm{disj}}), \tag{4.4}$$

where $\mathsf{Sat}_{\mathcal{G},\zeta}(\cdot)$, as usual, is the full subcategory of $\mathsf{D}_{\mathcal{G},\zeta}(\cdot)$ characterized by universal local acyclicity and perversity relative to X^{I}, respectively $\mathrm{X}^{\mathrm{I},\mathrm{disj}}$.

Lemma 4.1.5. *An object $\mathcal{A} \in \mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Gr}_{\mathrm{T},\mathrm{I}}^{\mathrm{disj}})$ belongs to the essential image of* (4.4) *if and only if $(\varpi^{\lambda^{\mathrm{I}}})^*\mathcal{A}$ extends as a twisted local system along $\mathrm{X}^{\mathrm{I},\mathrm{disj}} \subset \mathrm{X}^{\mathrm{I}}$ for every $\lambda^{\mathrm{I}} \in \Lambda^{\mathrm{I}}$.*

Proof. We note that the essential image of (4.4) is closed under direct summands. This is a twisted version of [HS23, Theorem 6.8(ii)] and follows from the same proof.

Suppose that $\mathcal{A} \simeq j^*\mathcal{B}$ for some $\mathcal{B} \in \mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Gr}_{\mathrm{T},\mathrm{I}})$. Since each $(\varpi^{\lambda^{\mathrm{I}}})_!(\varpi^{\lambda^{\mathrm{I}}})^*\mathcal{A}$ is a direct summand of $\mathcal{A}$, it extends to $\mathrm{Gr}_{\mathrm{T},\mathrm{I}}$ along (4.4), which must then be an object of the form $(\varpi^{\lambda^{\mathrm{I}}})_!\mathcal{B}^{\lambda^{\mathrm{I}}}$, where $\mathcal{B}^{\lambda^{\mathrm{I}}}$ is a twisted local system over X^{I} extending $(\varpi^{\lambda^{\mathrm{I}}})^*\mathcal{A}$.

Conversely, if each $(\varpi^{\lambda^{\mathrm{I}}})^*\mathcal{A}$ extends to some twisted local system $\mathcal{B}^{\lambda^{\mathrm{I}}}$ over X^{I}, then the sum $\bigoplus_{\lambda^{\mathrm{I}}}(\varpi^{\lambda^{\mathrm{I}}})_!\mathcal{B}^{\lambda^{\mathrm{I}}}$ belongs to $\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Gr}_{\mathrm{T},\mathrm{I}})$ and extends $\mathcal{A}$. ☐

4.1.6. We shall now prove Proposition 4.1.3.

Proof of Proposition 4.1.3. The statement is of étale local nature over X^{I}, so we may assume that T is split. Consider the commutative square

$$\begin{tikzcd} \mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{T}^\sharp,\mathrm{I}}) \arrow[r, phantom, "\subset"] \arrow[d, "(f_{\mathrm{I}})_!"] & \mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{T}^\sharp,\mathrm{I}}^{\mathrm{disj}}) \arrow[d, "(f_{\mathrm{I}})_!"] \\ \mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{T},\mathrm{I}}) \arrow[r, phantom, "\subset"] & \mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{T},\mathrm{I}}^{\mathrm{disj}}) \end{tikzcd} \tag{4.5}$$

where the horizontal embeddings are given by Proposition 1.3.13.

By the factorization structure on $(\mathrm{Hec}_{\mathrm{T}}, \mathcal{G}_{\mathrm{Hec}_{\mathrm{T}}})$ (*cf.* Proposition 1.2.11) and Corollary 3.1.10, the right vertical functor in (4.5) is an equivalence. This implies that the left vertical functor is fully faithful. Applying the characterization of Lemma 4.1.5 to both T and $\mathrm{T}^\sharp$, we see that it is also essentially surjective. ☐

4.1.7. In §4.2, we shall prove the following statement.

Proposition 4.1.8. *For a finite set* I*, there is a canonical equivalence of symmetric monoidal* e*-linear categories*

$$\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{T}^\sharp,\mathrm{I}}) \simeq \mathsf{Rep}_{\mathrm{H}^{\boxtimes\mathrm{I}},(\nu+\vartheta)^{\boxtimes\mathrm{I}}}, \tag{4.6}$$

which depends functorially on I *(*cf. *§2.3.5).*

4.1.9 *Theorem 2.3.2 for tori.* Using Proposition 4.1.3 and Proposition 4.1.8, we shall construct the geometric Satake equivalence (2.36) for (T, μ).

Namely, we set (2.36) to be the composition of (4.6) with the inverse of (4.3):

$$\begin{tikzcd}
\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{T}^\sharp,\mathrm{I}}) \arrow[r, phantom, "\simeq"] & \mathsf{Rep}_{\mathrm{H}^{\boxtimes \mathrm{I}},(\nu+\vartheta)^{\boxplus \mathrm{I}}} \\
\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{T},\mathrm{I}}) \arrow[u, "\simeq"] &
\end{tikzcd}$$

4.2. Equivalence for sharp tori.

4.2.1. The goal of this subsection is to prove Proposition 4.1.8. Note that (T, μ) and $(\mathrm{T}^\sharp, \mu^\sharp)$ share the same metaplectic dual data (H, ν), so we may replace (T, μ) by $(\mathrm{T}^\sharp, \mu^\sharp)$ and start with an $\mathbb{E}_\infty$-*monoidal* morphism

$$\mu : \mathsf{B}_{\mathrm{X}}\mathrm{T} \to \mathsf{B}^4_{\mathrm{X}}\mathrm{A}(1). \tag{4.7}$$

Recall that Gr_{T} is a factorization algebra in Stk (*cf.* Remark 1.2.8). Our first step is to lift $(\mathrm{Gr}_{\mathrm{T}}, \mathcal{G}_{\mathrm{Gr}_{\mathrm{T}}})$ to a factorization algebra in commutative multiplicative stacks endowed with a commutative multiplicative A-gerbe.

4.2.2. Denote by $\mathsf{Gp}^{\mathrm{com}}(\mathsf{Stk})$ the k-presheaf assigning to R the symmetric monoidal category of grouplike $\mathbb{E}_\infty$-monoids in étale R-stacks.

Recall that an R-point of Gr_{T} may be thought of as a morphism $\mathrm{D}_{\underline{x}} \to \mathsf{B}_{\mathrm{X}}\mathrm{T}$ over X (for $\underline{x} \in \mathrm{Ran}(\mathrm{R})$) with a trivialization over $\mathring{\mathrm{D}}_{\underline{x}}$. The $\mathbb{E}_\infty$-monoid structure on $\mathsf{B}_{\mathrm{X}}\mathrm{T}$ thus lifts $\mathrm{Gr}_{\mathrm{T},\underline{x}}$ to an object of $\mathsf{Gp}^{\mathrm{com}}(\mathsf{Stk})(\mathrm{R})$. Compatibility with the factorization structure then ensures that Gr_{T} lifts to a factorization algebra in $\mathsf{Gp}^{\mathrm{com}}(\mathsf{Stk})$.

Denote by $\mathsf{Gp}^{\mathrm{com}}(\mathsf{Stk})_{/\mathsf{B}^2\mathrm{A}}$ the k-presheaf assigning to R the symmetric monoidal category of objects $\mathcal{Z} \in \mathsf{Gp}^{\mathrm{com}}(\mathsf{Stk})(\mathrm{R})$ endowed with an $\mathbb{E}_\infty$-monoidal morphism $\mathcal{Z} \to \mathsf{B}^2\mathrm{A}$.

Proposition 4.2.3. *The pair* $(\mathrm{Gr}_{\mathrm{T}}, \mathcal{G}_{\mathrm{Gr}_{\mathrm{T}}})$ *canonically lifts to a factorization algebra in* $\mathsf{Gp}^{\mathrm{com}}(\mathsf{Stk})_{/\mathsf{B}^2\mathrm{A}}$*;*

Proof. This is analogous to the proof of Proposition 1.2.11, the only difference being that the morphism (1.22) for (T, μ):

$$\mathrm{Hec}_{\mathrm{T}}(\mathrm{R}) \to \Gamma(\mathrm{D}_{(\cdot)} \bmod \mathring{\mathrm{D}}_{(\cdot)}, \mathsf{B}^4\mathrm{A}(1))$$

lifts to a morphism of Ran(R)-algebras in $\mathrm{Span}(\mathsf{Gp}^{\mathrm{com}}(\mathsf{Spc}))$. ☐

Remark 4.2.4. The factorization structure supplied by Proposition 4.2.3 may be compared with that of $(\mathrm{Hec}_{\mathrm{T}}, \mathcal{G}_{\mathrm{Hec}_{\mathrm{T}}})$ (*cf.* Proposition 1.2.11) as follows.

Consider the morphism of Ran-stacks

$$\mathrm{Hec}_{\mathrm{T}} \to \mathrm{Gr}_{\mathrm{T}}, \quad (\mathrm{P}^0 \overset{\underline{x}}{\sim} \mathrm{P}^1) \mapsto (\mathcal{O} \overset{\underline{x}}{\sim} \mathrm{P}^1 \otimes (\mathrm{P}^0)^{-1}). \tag{4.8}$$

which lifts to a morphism of factorization algebras in $\mathrm{Span}(\mathsf{Gpd}(\mathsf{Stk}))$, under the forgetful functor $\mathsf{Gp}^{\mathrm{com}}(\mathsf{Stk}) \to \mathrm{Span}(\mathsf{Gpd}(\mathsf{Stk}))$.

One can show that (4.8) canonically lifts to a morphism $(\mathrm{Hec}_{\mathrm{T}}, \mathcal{G}_{\mathrm{Hec}_{\mathrm{T}}}) \to (\mathrm{Gr}_{\mathrm{T}}, \mathcal{G}_{\mathrm{Gr}_{\mathrm{T}}})$ of factorization algebras in $\mathrm{Span}(\mathsf{Gpd}(\mathsf{Stk}))_{/\mathsf{B}^2\mathrm{A}}$. (This structure will not be used in the sequel, so we omit its construction.) To the contrary, the structural morphism $(\mathrm{Gr}_{\mathrm{T}}, \mathcal{G}_{\mathrm{Gr}_{\mathrm{T}}}) \to (\mathrm{Hec}_{\mathrm{T}}, \mathcal{G}_{\mathrm{Hec}_{\mathrm{T}}})$ only lifts to a morphism of factorization algebras in $\mathrm{Span}(\mathsf{Stk})_{/\mathsf{B}^2\mathrm{A}}$, *i.e.* it is *incompatible* with the multiplicative structures.

4.2.5. Given a finite set I, the system of morphisms $\varpi^{\lambda^{\mathrm{I}}}$ (for $\lambda^{\mathrm{I}} \in \Lambda^{\mathrm{I}}$) of §4.1.4 organizes into a morphism of étale sheaves of abelian groups over X^{I}

$$\varpi : \Lambda^{\boxplus \mathrm{I}} \to \mathrm{Gr}_{\mathrm{T},\mathrm{I}}. \tag{4.9}$$

Indeed, étale locally over X, the X-torus T is split and the étale sheaf $\Lambda^{\boxplus \mathrm{I}}$ is represented by $\Lambda^{\mathrm{I}} \times \mathrm{X}^{\mathrm{I}}$, for Λ the abelian group of cocharacters of T. In this case, (4.9) is the sum of $\varpi^{\lambda^{\mathrm{I}}}$ over $\lambda^{\mathrm{I}} \in \Lambda^{\mathrm{I}}$. The general case follows from étale descent.

On the other hand, $\mathcal{G}_{\mathrm{Gr}_{\mathrm{T}}}$ defines an $\mathbb{E}_\infty$-monoidal morphism by Proposition 4.2.3:

$$\mathcal{G}_{\mathrm{Gr}_{\mathrm{T},\mathrm{I}}} : \mathrm{Gr}_{\mathrm{T},\mathrm{I}} \to \mathsf{B}^2_{\mathrm{X}^{\mathrm{I}}}\mathrm{A}. \tag{4.10}$$

Recall the $\mathbb{E}_\infty$-monoidal morphism $(\nu + \vartheta)^{\boxplus \mathrm{I}}$ (*cf.* §2.2.3) appearing on the dual side of the Satake equivalence (4.6). The following computation explains its geometric origin.

Proposition 4.2.6. *The composition of* (4.9) *and* (4.10) *is canonically identified with the* $\mathbb{E}_\infty$*-monoidal morphism* $(\nu + \vartheta)^{\boxplus \mathrm{I}}$.

Proof. Denote by $\nu_{\mathrm{I}}^{\mathrm{geom}}$ the composition of (4.9) and (4.10). Both $\nu_{\mathrm{I}}^{\mathrm{geom}}$ and $(\nu + \vartheta)^{\boxplus \mathrm{I}}$ are defined by étale descent from the case of a split X-torus T, so we may assume that T is split and view Λ as an abelian group.

Let us identify $\nu_{\mathrm{I}}^{\mathrm{geom}}$ with $(\nu_{\{1\}}^{\mathrm{geom}})^{\boxplus \mathrm{I}}$ and thereby reduce to the case $\mathrm{I} = \{1\}$. Indeed, given an R-point $x^{\mathrm{I}} = (x^i)_{i \in \mathrm{I}}$ of X^{I} (with induced R-point $\underline{x}$ of Ran), we have a commutative diagram of $\mathbb{E}_\infty$-monoids

$$\begin{tikzcd}
\Lambda^{\oplus \mathrm{I}} \arrow[r] & \bigoplus_{i \in \mathrm{I}} \Gamma(\mathrm{D}_{x^i} \bmod \mathring{\mathrm{D}}_{x^i}, \mathsf{B}\mathrm{T}) \arrow[r, "\Sigma"] \arrow[d, "\mu"] & \Gamma(\mathrm{D}_{\underline{x}} \bmod \mathring{\mathrm{D}}_{\underline{x}}, \mathsf{B}\mathrm{T}) \arrow[d, "\mu"] & \\
& \bigoplus_{i \in \mathrm{I}} \Gamma(\mathrm{D}_{x^i} \bmod \mathring{\mathrm{D}}_{x^i}, \mathsf{B}^4\mathrm{A}(1)) \arrow[r, "\Sigma"] & \Gamma(\mathrm{D}_{\underline{x}} \bmod \mathring{\mathrm{D}}_{\underline{x}}, \mathsf{B}^4\mathrm{A}(1)) \arrow[r, "\mathrm{tr}_{\underline{x}}"] & \Gamma(\operatorname{Spec} \mathrm{R}, \mathsf{B}^2\mathrm{A})
\end{tikzcd} \tag{4.11}$$

where the first map is the sum of the maps

$$\Lambda \to \Gamma(\mathrm{D}_{x^i} \bmod \mathring{\mathrm{D}}_{x^i}, \mathsf{B}\mathrm{T}), \quad \lambda \mapsto (\mathcal{O} \overset{x^i}{\sim} \mathcal{O}(\lambda x^i)),$$

and $\mathrm{tr}_{\underline{x}}$ is the (underlying $\mathbb{E}_\infty$-monoidal morphism of the) trace map (1.10). The upper circuit of (4.11) is the pullback of $\nu_{\mathrm{I}}^{\mathrm{geom}}$ along x^{I}, while the lower circuit is the pullback of $(\nu_{\{1\}}^{\mathrm{geom}})^{\boxplus \mathrm{I}}$ along x^{I} by additivity of the trace map.

We shall now assume $\mathrm{I} = \{1\}$ and omit it from the notation. It remains to identify the $\mathbb{E}_\infty$-monoidal morphism ν^{geom} with $\nu + \vartheta$. Fix an R-point x of X. The pullback of ν^{geom} along x may be expressed as the composition

$$\begin{aligned} \Lambda \to \mathrm{Maps}_*(\mathsf{B}_{\mathrm{X}}\mathbb{G}_m, \mathsf{B}_{\mathrm{X}}\mathrm{T}) &\xrightarrow{\mu} \mathrm{Maps}_*(\mathsf{B}_{\mathrm{X}}\mathbb{G}_m, \mathsf{B}^4_{\mathrm{X}}\mathrm{A}(1)) \\ &\xrightarrow{\mathcal{O}(x)^*} \Gamma(\mathrm{D}_x \bmod \mathring{\mathrm{D}}_x, \mathsf{B}^4\mathrm{A}(1)) \xrightarrow{\mathrm{tr}_x} \Gamma(\operatorname{Spec} \mathrm{R}, \mathsf{B}^2\mathrm{A}), \end{aligned} \tag{4.12}$$

where $\mathcal{O}(x)$ is viewed as a morphism $\mathrm{D}_x \to \mathsf{B}_{\mathrm{X}}\mathbb{G}_m$ trivialized over $\mathring{\mathrm{D}}_x$. Note that all but the morphism μ in (4.12) come from $\mathbf{HZ}$-linear morphisms.

Let us invoke the decomposition of Remark 2.1.4:

$$\mathrm{Maps}_*(\mathsf{B}_{\mathrm{X}}\mathbb{G}_m, \mathsf{B}^4_{\mathrm{X}}\mathrm{A}(1)) \simeq \Gamma(\mathrm{X}, \mathsf{B}^2\mathrm{A}) \oplus \Gamma(\mathrm{X}, \mathrm{A}(-1)). \tag{4.13}$$

Under this decomposition, the composition of the first two morphisms of (4.12) is the sum $\nu \oplus \epsilon$ (*cf.* §2.1.10, §2.1.21). Let us identify the composition $\mathrm{tr}_x \circ \mathcal{O}(x)^*$ of the last two morphisms of (4.12), individually for each summand in (4.13).

(1) $\Gamma(\mathrm{X}, \mathsf{B}^2\mathrm{A})$-*summand.* The inclusion $\Gamma(\mathrm{X}, \mathsf{B}^2\mathrm{A}) \to \mathrm{Maps}_*(\mathsf{B}_{\mathrm{X}}\mathbb{G}_m, \mathsf{B}^4_{\mathrm{X}}\mathrm{A}(1))$ is given by tensor product with the Kummer map $\Psi : \mathsf{B}_{\mathrm{X}}\mathbb{G}_m \to \mathsf{B}^2_{\mathrm{X}}\hat{\mathbf{Z}}(1)$.

By the natural isomorphism (*cf.* Remark 1.1.9, Remark 1.1.10)

$$\begin{aligned}(\mathrm{tr}_x \circ \mathcal{O}(x)^*)(f \otimes \Psi) &\simeq \mathrm{tr}_x(f \otimes \Psi(\mathcal{O}(x))) \\ &\simeq f \otimes \mathrm{tr}_x(\Psi(\mathcal{O}(x))) \simeq f\end{aligned}$$

linear in $f \in \Gamma(\mathrm{X}, \mathsf{B}^2\mathrm{A})$, we see that the restriction of $\mathrm{tr}_x \circ \mathcal{O}(x)^*$ to the $\Gamma(\mathrm{X}, \mathsf{B}^2\mathrm{A})$-summand is identified with the pullback along $x : \operatorname{Spec}\mathrm{R} \to \mathrm{X}$.

(2) $\Gamma(\mathrm{X}, \mathrm{A}(-1))$-*summand.* The inclusion $\Gamma(\mathrm{X}, \mathrm{A}(-1)) \to \mathrm{Maps}_*(\mathsf{B}_{\mathrm{X}}\mathbb{G}_m, \mathsf{B}^4_{\mathrm{X}}\mathrm{A}(1))$ is given by tensor product with $\Psi^{\otimes 2} : \mathsf{B}_{\mathrm{X}}\mathbb{G}_m \to \mathsf{B}^4_{\mathrm{X}}\hat{\mathbf{Z}}(2)$.

By the natural isomorphism (*cf.* Remark 1.1.9, Remark 1.1.10)

$$\begin{aligned}(\mathrm{tr}_x \circ \mathcal{O}(x)^*)(a \otimes \Psi^{\otimes 2}) &\simeq \mathrm{tr}_x(a \otimes \Psi(\mathcal{O}(x))^{\otimes 2}) \\ &\simeq a \otimes \Psi(\mathcal{O}(x))|_{\Gamma_x} \otimes \mathrm{tr}_x(\Psi(\mathcal{O}(x))) \simeq a \otimes \Psi(\mathcal{O}(x)|_{\Gamma_x})\end{aligned}$$

linear in $a \in \Gamma(\mathrm{X}, \mathrm{A}(-1))$, we see that the restriction of $\mathrm{tr}_x \circ \mathcal{O}(x)^*$ to the $\Gamma(\mathrm{X}, \mathrm{A}(-1))$ is the tensor product with $\Psi(\mathcal{O}(x)|_{\Gamma_x})$.

Since $\mathcal{O}(x)|_{\Gamma_x}$ is canonically isomorphic to the pullback of Ω^{-1}_{X} along $x : \operatorname{Spec}\mathrm{R} \to \mathrm{X}$. This is the tensor product with $\Psi(\Omega^{-1}_{\mathrm{X}})$, pulled back along x.

In conclusion, the decomposition (4.13) exhibits (4.12) as the sum $\nu + \epsilon \otimes \Psi(\Omega^{-1}_{\mathrm{X}})$, pulled back along $x : \operatorname{Spec}\mathrm{R} \to \mathrm{X}$. The latter is, by definition, the $\mathbb{E}_\infty$-monoidal morphism denoted by $\nu + \vartheta$ (*cf.* §2.1.21). □

4.2.7. We shall now use Proposition 4.2.3 and Proposition 4.2.6 to construct (4.6).

Proof of Proposition 4.1.8. As explained in §4.2.1, we may take as input an $\mathbb{E}_\infty$-monoidal morphism (4.7) and replace the left-hand-side of (4.6) by $\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{T},\mathrm{I}})$. Furthermore, the desired equivalence (4.6) is of étale local nature over X^{I}, so we may assume that T is split and view Λ as an abelian group.

Since (4.10) is $\mathbb{E}_\infty$-monoidal, we may endow $\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Gr}_{\mathrm{T},\mathrm{I}})$ with a convolution *symmetric* monoidal structure $\circ$. Pushforward along (4.9) yields a symmetric monoidal functor

$$\mathsf{Rep}_{\mathrm{H}^{\boxtimes \mathrm{I}}, (\nu+\vartheta)^{\boxplus \mathrm{I}}} \simeq \bigoplus_{\lambda^{\mathrm{I}} \in \Lambda^{\mathrm{I}}} \mathsf{Lis}_{(\nu+\vartheta)^{\boxplus \mathrm{I}}(\lambda^{\mathrm{I}})}(\mathrm{X}^{\mathrm{I}}) \xrightarrow{\varpi_!} \mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Gr}_{\mathrm{T},\mathrm{I}}), \tag{4.14}$$

where we used Proposition 4.2.6 to identify the pullback of (4.10) with $(\nu+\vartheta)^{\boxplus \mathrm{I}}$. By Lemma 4.1.5, the functor (4.14) is an equivalence.

On the other hand, pullback along $\mathrm{Gr}_{\mathrm{T},\mathrm{I}} \to \mathrm{Hec}_{\mathrm{T},\mathrm{I}}$ yields an equivalence

$$\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{T},\mathrm{I}}) \simeq \mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Gr}_{\mathrm{T},\mathrm{I}}) \tag{4.15}$$

as $\mathcal{G}_{\mathrm{Hec}_{\mathrm{T},\mathrm{I}}}$ descends along (4.8). Thus, the construction of §1.3.12 endows $\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Gr}_{\mathrm{T},\mathrm{I}})$ with a fusion symmetric monoidal structure $\star$. The equivalence (4.15) is symmetric monoidal when both sides are equipped with the fusion symmetric monoidal structure.

Finally, by Proposition 4.2.3 and the proof of Proposition 1.3.15, the convolution symmetric monoidal structure lifts $(\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Gr}_{\mathrm{T},\mathrm{I}}), \star)$ to a commutative monoid in the (2-)category

$\mathsf{CAlg}(\mathsf{Cat})$. Thus we obtain equivalences of *symmetric* monoidal categories

$$(\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Gr}_{\mathrm{T,I}}), \circ) \simeq (\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Gr}_{\mathrm{T,I}}), \star). \tag{4.16}$$

The desired equivalence (4.6) is the composition of (4.15), (4.16), and (4.14). It is functorial in I by construction. $\square$

Remark 4.2.8. We emphasize that the equivalence (4.15) would be *false* without the hypothesis that the étale level (4.7) is $\mathbb{E}_\infty$-monoidal.

Indeed, for general étale levels, objects on $\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{T,I}})$ are supported on the image of the morphism (4.1), by Proposition 4.1.3.

5. The $\hat{\mathrm{Z}}_{\mathrm{H}}$-grading

We put ourselves in the context of §2.3.1 and assume in addition that G is *split* reductive. Under this assumption, the dual group H is constant (*cf.* §2.1).

The goal of this section is to endow the Satake category for $\mathrm{I} = \{1\}$ with a $\hat{\mathrm{Z}}_{\mathrm{H}}$-grading compatible with all of its essential structures. This $\hat{\mathrm{Z}}_{\mathrm{H}}$-grading will correspond to the natural $\hat{\mathrm{Z}}_{\mathrm{H}}$-grading on $\mathsf{Rep}_{\mathrm{H},\nu+\vartheta}$ under the Satake equivalence (2.36). However, our proof of the latter makes critical use of this $\hat{\mathrm{Z}}_{\mathrm{H}}$-grading, so it must be constructed independently.

5.1. Central equivariance.

5.1.1. In this subsection, we work over one copy of the curve X, so we base change the local Hecke stack (*cf.* §1.1.3) along $\mathrm{X} \rightarrow \mathrm{Ran}$ and denote the result by $\mathrm{Hec}_{\mathrm{G}}$. The same convention applies to Gr_{G}, $\mathrm{L}^+\mathrm{G}$, LG and we abbreviate B_{X} to B.

Consider the $\mathsf{B}\mathrm{Z}$-action on $\mathsf{B}\mathrm{G}$ defined by multiplication (*cf.* §3.1.6). It induces an action of $\mathrm{L}^+(\mathsf{B}\mathrm{Z})$ on $\mathrm{Hec}_{\mathrm{G}}$. Explicitly, the action of an R-point P_{Z} of $\mathrm{L}^+(\mathsf{B}\mathrm{Z})$, viewed as Z-bundle over D_x (for x an R-point of X), carries $\mathrm{P}^0 \overset{x}{\sim} \mathrm{P}^1$ to the modification

$$\mathrm{P}^0 \otimes \mathrm{P}_{\mathrm{Z}} \overset{x}{\sim} \mathrm{P}^1 \otimes \mathrm{P}_{\mathrm{Z}}.$$

5.1.2. The inclusion $\Lambda^\sharp \subset \Lambda$ caries $\Lambda^\sharp_{\mathrm{sc}}$ into Λ_{sc}, so it induces a map $\hat{\mathrm{Z}}_{\mathrm{H}} \rightarrow \pi_1\mathrm{G}$ (*cf.* Remark 2.1.7). This map is neither injective nor surjective in general.

Given $\theta \in \hat{\mathrm{Z}}_{\mathrm{H}}$, we shall denote by $[\theta]$ its image in $\pi_1\mathrm{G}$ and by $\mathrm{Hec}_{\mathrm{G}}^{[\theta]} \subset \mathrm{Hec}_{\mathrm{G}}$ the corresponding connected component. Recall the A-gerbe $\mathcal{G}_{\mathrm{Hec}_{\mathrm{G}}}$ over $\mathrm{Hec}_{\mathrm{G}}$ (*cf.* §1.1.7), whose restriction to $\mathrm{Hec}_{\mathrm{G}}^{[\theta]}$ is denote by $\mathcal{G}_{\mathrm{Hec}_{\mathrm{G}}^{[\theta]}}$.

Proposition 5.1.3. *Given any $\theta \in \hat{\mathrm{Z}}_{\mathrm{H}}$ with image $[\theta] \in \pi_1\mathrm{G}$, the* A*-gerbe $\mathcal{G}_{\mathrm{Hec}_{\mathrm{G}}^{[\theta]}}$ admits a canonical* $\mathrm{L}^+(\mathsf{B}\mathrm{Z})$*-equivariance structure.*

Proof. Given R-points P_{Z} of $\mathrm{L}^+(\mathsf{B}\mathrm{Z})$, $(\mathrm{P}^0 \overset{x}{\sim} \mathrm{P}^1)$ of $\mathrm{Hec}_{\mathrm{G}}^{\theta}$ lying over the same R-point x of X, we shall construct a natural isomorphism of A-gerbes over Spec R:

$$\mathcal{G}_{\mathrm{Hec}_{\mathrm{G}}}(\mathrm{P}^0 \otimes \mathrm{P}_{\mathrm{Z}} \overset{x}{\sim} \mathrm{P}^1 \otimes \mathrm{P}_{\mathrm{Z}}) \simeq \mathcal{G}_{\mathrm{Hec}_{\mathrm{G}}}(\mathrm{P}^0 \overset{x}{\sim} \mathrm{P}^1) \tag{5.1}$$

along with cocycle data for multiple R-points of $\mathrm{L}^+(\mathsf{B}\mathrm{Z})$.

By construction, the left-hand-side of (5.1) is the image of

$$\mu(\mathrm{P}^0 \otimes \mathrm{P}_{\mathrm{Z}}) - \mu(\mathrm{P}^1 \otimes \mathrm{P}_{\mathrm{Z}}) \in \Gamma(\mathrm{D}_x \bmod \mathring{\mathrm{D}}_x, \mathsf{B}^4\mathrm{A}(1))$$

under the trace map (*cf.* §1.1.11). Using the canonical quadratic structure (3.6), we may identify this section as follows:

$$\mu(\mathrm{P}^0 \otimes \mathrm{P}_{\mathrm{Z}}) - \mu(\mathrm{P}^1 \otimes \mathrm{P}_{\mathrm{Z}}) \simeq \mu(\mathrm{P}^0) - \mu(\mathrm{P}^1) + (b_2 \otimes \Psi^{\otimes 2})(\mathrm{P}^0_{\mathrm{ab}} - \mathrm{P}^1_{\mathrm{ab}}, \mathrm{P}_{\mathrm{Z}}) \tag{5.2}$$

where $\mathrm{P}^0_{\mathrm{ab}}$, $\mathrm{P}^1_{\mathrm{ab}}$ are the images of P^0, P^1 under $\mathsf{B}\mathrm{G} \to \mathsf{B}\mathrm{G}_{\mathrm{ab}}$.

Consider the linear form $b_2([\theta], \cdot) : \mathrm{Fib}(\Lambda \to \Lambda_{\mathrm{ad}}) \to \mathrm{A}(-1)$, the induced $\mathbf{Z}$-linear morphism $b_2([\theta], \cdot) \otimes \Psi : \mathsf{B}\mathrm{Z} \to \mathsf{B}^2\mathrm{A}$, and its value $(b_2([\theta], \cdot) \otimes \Psi)(\mathrm{P}_{\mathrm{Z}}) \in \Gamma(\mathrm{D}_x, \mathsf{B}^2\mathrm{A})$. We shall construct a canonical isomorphism of A-gerbes over $\operatorname{Spec} \mathrm{R}$ linear in P_{Z}:

$$\mathrm{tr}_x((b_2 \otimes \Psi^{\otimes 2})(\mathrm{P}^0_{\mathrm{ab}} - \mathrm{P}^1_{\mathrm{ab}}, \mathrm{P}_{\mathrm{Z}})) \simeq (b_2([\theta], \cdot) \otimes \Psi)(\mathrm{P}_{\mathrm{Z}})|_{\Gamma_x}. \tag{5.3}$$

This will give rise to (5.1) for the following reason: The lift $\theta \in \hat{\mathrm{Z}}_{\mathrm{H}}$ of $[\theta]$ trivializes $b_2([\theta], \cdot)$ as a linear form, because b vanishes over $\Lambda^\sharp \otimes \Lambda$ and b_1 vanishes over $\Lambda^\sharp_{\mathrm{sc}} \otimes \Lambda_{\mathrm{ad}}$ (*cf.* §3.1.5, Remark 2.1.9). This then induces a trivialization of the left-hand-side of (5.3), so (5.2) gives rise to (5.1) under tr_x. The cocycle data are induced from those of the canonical quadratic structure (*cf.* §3.1.6) and the linearity of (5.3) in P_{Z}.

To construct (5.3), we note that for any R-point $(\mathrm{P}^0 \overset{x}{\sim} \mathrm{P}^1)$ of $\mathrm{Hec}_{\mathrm{G}}$ and any $\mathbf{Z}$-linear morphism $f : \pi_1\mathrm{G} \to \mathsf{B}^2\mathrm{A}$, we have a canonical isomorphism in $\Gamma(\operatorname{Spec} \mathrm{R}, \mathsf{B}^2\mathrm{A})$:

$$\mathrm{tr}_x(\Psi^f(\mathrm{P}^1_{\mathrm{ab}} - \mathrm{P}^0_{\mathrm{ab}})) \simeq f(\deg(\mathrm{P}^0 \overset{x}{\sim} \mathrm{P}^1)), \tag{5.4}$$

where $\Psi^f : \mathsf{B}\mathrm{G}_{\mathrm{ab}} \to \mathsf{B}^4\mathrm{A}(1)$ is the tensor product f with Ψ and $\deg : \mathrm{Hec}_{\mathrm{G}} \to \pi_1\mathrm{G}$ is the degree map classifying connected components. Indeed, (5.4) follows from the universal case with f the reduction of $\pi_1\mathrm{G}$ along $\mathbf{Z} \to \hat{\mathbf{Z}}$, where it amounts to the compatibility between degree and étale Chern classes. The isomorphism (5.4) depends linearly on f.

The isomorphism (5.3) is obtained by specializing (5.4) to $f := (b_2 \otimes \Psi)(\cdot, \mathrm{P}_{\mathrm{Z}})$, where $b_2 \otimes \Psi$ is the tensor product of b_2 with Ψ along the second factor. ☐

Remark 5.1.4. Proposition 5.1.3 implies that given $\theta \in \hat{\mathrm{Z}}_{\mathrm{H}}$, the restriction $\mathcal{G}_{\mathrm{Gr}_{\mathrm{G}}^{[\theta]}}$ of $\mathcal{G}_{\mathrm{Gr}_{\mathrm{G}}}$ to the connected component $\mathrm{Gr}_{\mathrm{G}}^{[\theta]}$ admits a canonical $\mathrm{L}^+\mathrm{G}_{\mathrm{ad}}$-equivariance structure.

Indeed, this follows from the canonical map of étale X-stacks

$$\mathrm{L}^+\mathrm{G}_{\mathrm{ad}} \backslash \mathrm{Gr}_{\mathrm{G}}^{[\theta]} \to \mathrm{Hec}_{\mathrm{G}}^{[\theta]} / \mathrm{L}^+(\mathsf{B}\mathrm{Z})$$

induced from the morphism $\mathrm{L}^+(\mathsf{B}\mathrm{Z}) \to \mathsf{B}\mathrm{L}^+\mathrm{Z} \simeq \mathrm{L}^+\mathrm{G}_{\mathrm{ad}}/\mathrm{L}^+\mathrm{G}$.

5.1.5. It is critical in Proposition 5.1.3 that we take as input $\theta \in \hat{\mathrm{Z}}_{\mathrm{H}}$ rather than an element of $\pi_1\mathrm{G}$. Namely, different lifts of the same element of $\pi_1\mathrm{G}$ to $\hat{\mathrm{Z}}_{\mathrm{H}}$ provide different $\mathrm{L}^+(\mathsf{B}\mathrm{Z})$-equivariance structures.

Let us be more precise. Define a linear map

$$\chi : \ker(\hat{\mathrm{Z}}_{\mathrm{H}} \to \pi_1\mathrm{G}) \to \mathcal{H}om(\mathsf{B}\mathrm{Z}, \mathsf{B}\mathrm{A}) (\simeq \mathcal{H}om(\mathrm{Z}, \mathrm{A})) \tag{5.5}$$

as follows: Given $\lambda \in \Lambda^\sharp \cap \Lambda_{\mathrm{sc}}$, the morphism $b_1(\lambda, \cdot) \otimes \Psi : \mathrm{T}_{\mathrm{ad}} \to \mathsf{B}\mathrm{A}$ is canonically trivialized along $\mathrm{T} \to \mathrm{T}_{\mathrm{ad}}$, so it factors through a map $\mathsf{B}\mathrm{Z} \to \mathsf{B}\mathrm{A}$. This map vanishes for $\lambda \in \Lambda^\sharp_{\mathrm{sc}}$, so it induces (5.5) after identifying the source with $(\Lambda^\sharp \cap \Lambda_{\mathrm{sc}})/\Lambda^\sharp_{\mathrm{sc}}$.

Lemma 5.1.6. *The map* (5.5) *is injective.*

Proof. If $\lambda \in \Lambda^\sharp \cap \Lambda_{\mathrm{sc}}$ satisfies $\chi(\lambda) = 0$, then $b_1(\lambda, \cdot) \otimes \Psi : \mathrm{T}_{\mathrm{ad}} \to \mathsf{B}\mathrm{A}$ must also vanish because it is the composition of $\mathrm{T}_{\mathrm{ad}} \to \mathsf{B}\mathrm{Z}$ with $\chi(\lambda)$. This means that $b_1(\lambda, \cdot)$ is the zero form on Λ_{ad}. We conclude because $\Lambda^\sharp_{\mathrm{sc}} \subset \Lambda_{\mathrm{sc}}$ is precisely the kernel of b_1 (*cf.* Remark 2.1.9). ☐

5.1.7. Given $\theta_1, \theta_2 \in \hat{\mathrm{Z}}_{\mathrm{H}}$ with equal image $[\theta] \in \pi_1\mathrm{G}$, Proposition 5.1.3 supplies two $\mathrm{L}^+(\mathsf{B}\mathrm{Z})$-equivariance structures on $\mathcal{G}_{\mathrm{Hec}_{\mathrm{G}}^{[\theta]}}$, or equivalently two A-gerbes $\mathcal{G}^{\theta_1}_{\mathrm{Hec}_{\mathrm{G}}}$, $\mathcal{G}^{\theta_2}_{\mathrm{Hec}_{\mathrm{G}}}$ over the quotient stack $\mathrm{Hec}_{\mathrm{G}}^{[\theta]}/\mathrm{L}^+(\mathsf{B}\mathrm{Z})$ whose pullbacks to $\mathrm{Hec}_{\mathrm{G}}^{[\theta]}$ are identified with $\mathcal{G}_{\mathrm{Hec}_{\mathrm{G}}}$.

The difference

$$\mathcal{G}^{\theta_2}_{\mathrm{Hec}_{\mathrm{G}}} - \mathcal{G}^{\theta_1}_{\mathrm{Hec}_{\mathrm{G}}} \tag{5.6}$$

may thus be viewed as an A-gerbe over $\mathrm{Hec}_{\mathrm{G}}^{[\theta]}/\mathrm{L}^+(\mathsf{B}\mathrm{Z})$ neutralized over $\mathrm{Hec}_{\mathrm{G}}^{[\theta]}$.

Proposition 5.1.8. *Given $\theta_1, \theta_2 \in \hat{\mathrm{Z}}_{\mathrm{H}}$ with equal image $[\theta] \in \pi_1\mathrm{G}$, the* A*-gerbe* (5.6) *is canonically identified with the pullback of the deloop of $\chi(\theta_2 - \theta_1)$ along the map*

$$\mathrm{Hec}_{\mathrm{G}}^{[\theta]}/\mathrm{L}^+(\mathsf{B}\mathrm{Z}) \to \mathrm{X}/\mathrm{L}^+(\mathsf{B}\mathrm{Z}) \to \mathsf{B}^2\mathrm{Z}.$$

Proof. Let us interpret (5.5) in terms of the form b_2 (*cf.* §3.1.5). Namely, tensoring b_2 with Ψ along its second factor, we obtain the pairing

$$b_2 \otimes \Psi : \pi_1\mathrm{G} \otimes \mathsf{B}\mathrm{Z} \to \mathsf{B}^2\mathrm{A}$$

which is canonically trivialized over $\hat{\mathrm{Z}}_{\mathrm{H}} \otimes \mathsf{B}\mathrm{Z}$ (*cf.* §3.1.5, Remark 2.1.9). It then induces the pairing adjoint to (5.5):

$$\ker(\hat{\mathrm{Z}}_{\mathrm{H}} \to \pi_1\mathrm{G}) \otimes \mathsf{B}\mathrm{Z} \to \mathsf{B}\mathrm{A}. \tag{5.7}$$

On the other hand, the choices $\theta_1, \theta_2 \in \hat{\mathrm{Z}}_{\mathrm{H}}$ intervene in the construction of the $\mathrm{L}^+(\mathsf{B}\mathrm{Z})$-equivariance of $\mathcal{G}_{\mathrm{Hec}_{\mathrm{G}}}$ over $\mathrm{Hec}_{\mathrm{G}}^{[\theta]}$ by providing different trivializations of (5.3), for any R-point P_{Z} of $\mathrm{L}^+(\mathsf{B}\mathrm{Z})$ lying over $x \in \mathrm{X}(\mathrm{R})$. The difference in the trivializations is precisely the image of $(\theta_2 - \theta_1) \otimes \mathrm{P}_{\mathrm{Z}}|_{\Gamma_x}$ under (5.7). □

5.2. Compatibility lemmas.

5.2.1. We keep the notation of §5.1.1.

Given an element $\theta \in \hat{\mathrm{Z}}_{\mathrm{H}}$ with image $[\theta] \in \pi_1\mathrm{G}$, Proposition 5.1.3 allows us to descend $\mathcal{G}_{\mathrm{Hec}_{\mathrm{G}}^{[\theta]}}$ to an A-gerbe $\mathcal{G}^{\theta}_{\mathrm{Hec}_{\mathrm{G}}}$ along the quotient map

$$\mathrm{Hec}_{\mathrm{G}}^{[\theta]} \to \mathrm{Hec}_{\mathrm{G}}^{[\theta]}/\mathrm{L}^+(\mathsf{B}\mathrm{Z}).$$

In this subsection, we show that the association $\theta \mapsto \mathcal{G}^{\theta}_{\mathrm{Hec}_{\mathrm{G}}}$ is compatible with the convolution product on $\mathrm{Hec}_{\mathrm{G}}$, passage to Levis, and restrictions to Schubert cells.

5.2.2 *Convolution product*. Given $\theta_1, \cdots, \theta_n \in \hat{\mathrm{Z}}_{\mathrm{H}}$ with images $[\theta_1], \cdots, [\theta_n] \in \pi_1\mathrm{G}$, we denote by $\mathrm{Hec}_{\mathrm{G}}^{[\theta_1],\cdots,[\theta_n]}$ the substack of $\mathrm{Hec}_{\mathrm{G}}^{[n]}$ (*cf.* §1.1.6) corresponding to modifications

$$\mathrm{P}^0 \overset{x}{\sim} \mathrm{P}^1 \overset{x}{\sim} \cdots \overset{x}{\sim} \mathrm{P}^n \tag{5.8}$$

where $\mathrm{P}^{j-1} \overset{x}{\sim} \mathrm{P}^j$ belongs to $\mathrm{Hec}_{\mathrm{G}}^{[\theta_j]}$ for each $j = 1, \cdots, n$.

There is an $\mathrm{L}^+\mathsf{B}\mathrm{Z}$-action on $\mathrm{Hec}_{\mathrm{G}}^{[\theta_1],\cdots,[\theta_n]}$, where P_{Z} carries (5.8) to its termwise tensor product with P_{Z}. The structural morphisms (1.25) thus induce morphisms

$$\begin{tikzcd}
\mathrm{Hec}_{\mathrm{G}}^{[\theta_1],\cdots,[\theta_n]}/\mathrm{L}^+\mathsf{B}\mathrm{Z} \arrow[r, "\prod p_j"] \arrow[d, "m"] & \prod_{j=1}^n \mathrm{Hec}_{\mathrm{G}}^{[\theta_j]}/\mathrm{L}^+\mathsf{B}\mathrm{Z} \\
\mathrm{Hec}_{\mathrm{G}}^{[\theta_1+\cdots+\theta_n]}/\mathrm{L}^+\mathsf{B}\mathrm{Z} &
\end{tikzcd}$$

Lemma 5.2.3. *The isomorphism* (1.26) *over* $\mathrm{Hec}_{\mathrm{G}}^{[\theta_1],\cdots,[\theta_n]}$ *canonically extends to an isomorphism of* A*-gerbes over* $\mathrm{Hec}_{\mathrm{G}}^{[\theta_1],\cdots,[\theta_n]}/\mathrm{L}^+\mathsf{B}\mathrm{Z}$*:*

$$m^*\mathcal{G}^{\theta_1+\cdots+\theta_n}_{\mathrm{Hec}_{\mathrm{G}}} \simeq p_1^*\mathcal{G}^{\theta_1}_{\mathrm{Hec}_{\mathrm{G}}} + \cdots + p_n^*\mathcal{G}^{\theta_n}_{\mathrm{Hec}_{\mathrm{G}}}.$$

Proof. By construction, (1.26) arises as the image of the isomorphism

$$\mu(\mathrm{P}^n) - \mu(\mathrm{P}^0) \simeq \sum_{j=1}^n (\mu(\mathrm{P}^j) - \mu(\mathrm{P}^{j-1})) \tag{5.9}$$

in $\Gamma(\mathrm{D}_x \text{ mod } \mathring{\mathrm{D}}_x, \mathsf{B}^4\mathrm{A}(1))$ under the trace map (*cf.* the proof of Lemma 1.1.14). By inspecting the proof of Proposition 5.1.3, it suffices to identify the trivializations of

$$b_2([\theta_1+\cdots+\theta_n],\cdot)\otimes\Psi \simeq \sum_{j=1}^n b_2([\theta_j],\cdot)\otimes\Psi$$

supplied by $\theta_1+\cdots+\theta_n \in \hat{\mathrm{Z}}_{\mathrm{H}}$, respectively the product of those supplied by $\theta_1,\cdots,\theta_n \in \hat{\mathrm{Z}}_{\mathrm{H}}$. This holds because b_2 is trivialized over $\hat{\mathrm{Z}}_{\mathrm{H}} \otimes \mathrm{Fib}(\Lambda \to \Lambda_{\mathrm{ad}})$ as a *bilinear* form. ☐

5.2.4 *Passage to Levis***.** Let $\mathrm{P} \subset \mathrm{G}$ be a parabolic subgroup with Levi quotient $\mathrm{P} \twoheadrightarrow \mathrm{M}$. We have natural maps $\mathrm{Z} \to \mathrm{Z}_{\mathrm{M}}$ and $\pi_1\mathrm{M} \to \pi_1\mathrm{G}$. Moreover, the pullback of μ to P canonically descends to an étale level μ_{M} for M and we obtain the dual group H_{M} for $(\mathrm{M},\mu_{\mathrm{M}})$. The map $\pi_1\mathrm{M} \to \pi_1\mathrm{G}$ lifts to a map $\hat{\mathrm{Z}}_{\mathrm{H}_{\mathrm{M}}} \to \hat{\mathrm{Z}}_{\mathrm{H}}$.

Let λ be an element of $\hat{\mathrm{Z}}_{\mathrm{H}_{\mathrm{M}}}$. Its images in $\pi_1\mathrm{M}$, $\hat{\mathrm{Z}}_{\mathrm{H}}$, and $\pi_1\mathrm{G}$ are denoted by $[\lambda]$, θ, and $[\theta]$, respectively. Consider the following quotients of the structural morphisms:

$$\begin{tikzcd} & \mathrm{Hec}_{\mathrm{P}}^{[\lambda]}/\mathrm{L}^+\mathsf{B}\mathrm{Z} \arrow[dl, "q"'] \arrow[dr, "p"] & \\ \mathrm{Hec}_{\mathrm{G}}^{[\theta]}/\mathrm{L}^+\mathsf{B}\mathrm{Z} & & \mathrm{Hec}_{\mathrm{M}}^{[\lambda]}/\mathrm{L}^+\mathsf{B}\mathrm{Z}_{\mathrm{M}} \end{tikzcd}$$

where $\mathrm{Hec}_{\mathrm{P}}^{[\lambda]}$ denotes the pre-image of $\mathrm{Hec}_{\mathrm{M}}^{[\lambda]}$ under $\mathrm{Hec}_{\mathrm{P}} \to \mathrm{Hec}_{\mathrm{M}}$.

Lemma 5.2.5. *The isomorphism* (1.46) *over* $\mathrm{Hec}_{\mathrm{P}}^{[\lambda]}$ *canonically extends to an isomorphism of* A*-gerbes over* $\mathrm{Hec}_{\mathrm{P}}^{[\lambda]}/\mathrm{L}^+\mathsf{B}\mathrm{Z}$*:*

$$q^*\mathcal{G}^{\theta}_{\mathrm{Hec}_{\mathrm{G}}} \simeq p^*\mathcal{G}^{\lambda}_{\mathrm{Hec}_{\mathrm{M}}}.$$

Proof. By inspecting the proof of Proposition 5.1.3, this follows from the compatibility of the canonical quadratic structure (3.6) with passage to Levis.

Namely, we consider the $\mathsf{B}\mathrm{Z}$-action on $\mathsf{B}\mathrm{P}$:

$$a : \mathsf{B}\mathrm{Z} \times \mathsf{B}\mathrm{P} \to \mathsf{B}\mathrm{P},$$

with projection maps p_1, p_2 from $\mathsf{B}\mathrm{Z} \times \mathsf{B}\mathrm{P}$ onto $\mathsf{B}\mathrm{Z}$, respectively $\mathsf{B}\mathrm{P}$. Then the canonical quadratic structures (3.6) for (G,μ) and $(\mathrm{M},\mu_{\mathrm{M}})$ restrict to the same isomorphism

$$a^*\mu_{\mathrm{P}} \simeq (p_1)^*\mu_{\mathrm{P}} + (p_2)^*\mu_{\mathrm{Z}} + b_2 \otimes \Psi^{\otimes 2},$$

where μ_{P}, μ_{Z} denote the restrictions of μ to P, respectively Z. ☐

5.2.6 *Schubert cells.* Let us now fix a maximal torus and a Borel subgroup $\mathrm{T} \subset \mathrm{B} \subset \mathrm{G}$.

Given $\lambda \in \Lambda^\sharp$, the restriction of $\mathcal{G}_{\mathrm{Hec_G}}$ to the Schubert cell $\mathrm{Hec}_{\mathrm{G}}^{\lambda}$ canonically descends to the A-gerbe $\mathcal{G}_{\varpi^\lambda}$ over X (*cf.* Corollary 3.1.12). On the other hand, λ has class $\theta \in \hat{\mathrm{Z}}_{\mathrm{H}}$, which gives rise to the A-gerbe $\mathcal{G}^{\theta}_{\mathrm{Hec_G}}$.

Consider the structural morphisms

$$\begin{tikzcd}
\mathrm{Hec}_{\mathrm{G}}^{\lambda}/\mathrm{L}^+\mathsf{BZ} \arrow[r, "j"] \arrow[d, "\pi"] & \mathrm{Hec}_{\mathrm{G}}^{[\theta]}/\mathrm{L}^+\mathsf{BZ} \\
\mathrm{X} &
\end{tikzcd}$$

Lemma 5.2.7. *The isomorphism* (3.8) *over* $\mathrm{Hec}_{\mathrm{G}}^{\lambda}$ *canonically extends to an isomorphism of* A*-gerbes over* $\mathrm{Hec}_{\mathrm{G}}^{\lambda}/\mathrm{L}^+\mathsf{BZ}$*:*

$$j^*\mathcal{G}^{\theta}_{\mathrm{Hec_G}} \simeq \pi^*\mathcal{G}_{\varpi^\lambda}. \tag{5.10}$$

Proof. Note that A-gerbes over $\mathrm{Hec}_{\mathrm{G}}^{\lambda}/\mathrm{L}^+\mathsf{BZ}$ canonically descend to $\mathsf{B}(\mathrm{M}^\lambda/\mathrm{Z})$ along the quotient of (3.2). Moreover, T_{ad} may be thought of as a maximal torus of $\mathrm{M}^\lambda/\mathrm{Z}$, so A-gerbes over $\mathsf{B}(\mathrm{M}^\lambda/\mathrm{Z})$ are uniquely determined after pulling back to $\mathsf{B}\mathrm{T}_{\mathrm{ad}}$.

It therefore suffices to construct (5.10) after pulling back to $\mathrm{Hec}_{\mathrm{T}}^{\lambda}/\mathrm{L}^+\mathsf{BZ}$. The pullback of $\mathcal{G}^{\theta}_{\mathrm{Hec_G}}$ to $\mathrm{Hec}_{\mathrm{T}}^{\lambda}/\mathrm{L}^+\mathsf{BZ}$ extends to the A-gerbe $\mathcal{G}^{\lambda}_{\mathrm{Hec_T}}$ over $\mathrm{Hec}_{\mathrm{T}}^{\lambda}/\mathrm{L}^+\mathsf{B}\mathrm{T}$ (*cf.* Lemma 5.2.5). It thus remains to compare $\mathcal{G}^{\lambda}_{\mathrm{Hec_T}}$ with the descent of $\mathcal{G}_{\mathrm{Hec}_{\mathrm{T}}^{\lambda}}$ to X supplied by (3.8) for the torus T. In the case of tori, the equivariance structure constructed in Proposition 5.1.3 is identical to one constructed in Corollary 3.1.12. ☐

5.3. Construction of the $\hat{\mathrm{Z}}_{\mathrm{H}}$-grading.

5.3.1. Recall that for any S-point x of X (with $\mathrm{S} \in \mathsf{Sch}$), we have constructed the Satake category $\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},x})$ (*cf.* §1.3.7), which is an e-linear category equipped with the convolution monoidal structure (*cf.* §1.3.10).

The goal of this subsection is to equip $\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},x})$ with a $\hat{\mathrm{Z}}_{\mathrm{H}}$-grading compatible with the monoidal structure and constant term functors (*cf.* §1.5.7).

5.3.2. For each $\theta \in \hat{\mathrm{Z}}_{\mathrm{H}}$ with image $[\theta] \in \pi_1\mathrm{G}$, we shall consider the quotient map

$$\mathrm{Hec}_{\mathrm{G},x}^{[\theta]} \to \mathrm{L}_x^+\mathrm{G}_{\mathrm{ad}}\backslash\mathrm{Gr}_{\mathrm{G},x}^{[\theta]} \tag{5.11}$$

and view (the pulback of) $\mathcal{G}^{\theta}_{\mathrm{Hec_G}}$ as an $\mathrm{L}_x^+\mathrm{G}_{\mathrm{ad}}$-equivariant A-gerbe over $\mathrm{Gr}_{\mathrm{G},x}$ (*cf.* Remark 5.1.4). Consider the full subcategory $\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Gr}_{\mathrm{G},x}^{[\theta]}) \subset \mathsf{D}_{\mathcal{G},\zeta}(\mathrm{Gr}_{\mathrm{G},x}^{[\theta]})$ of perverse ULA sheaves relative to S and the e-linear category

$$\mathsf{Sat}^{\theta}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},x}) \coloneqq \mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Gr}_{\mathrm{G},x}^{[\theta]})^{\mathrm{L}_x^+\mathrm{G}_{\mathrm{ad}}}$$

of its $\mathrm{L}_x^+\mathrm{G}_{\mathrm{ad}}$-equivariant objects defined with respect to $\mathcal{G}^{\theta}_{\mathrm{Hec_G}}$.

Pullback along (5.11) yields a functor

$$\mathsf{Sat}^{\theta}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},x}) \to \mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},x}^{[\theta]}), \tag{5.12}$$

whose target is naturally a full subcategory of $\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},x})$. Furthermore, (5.12) is fully faithful and its essential image is closed under extensions (*cf.* Remark 1.3.8).

Proposition 5.3.3. *The sum of* (5.12) *over* $\theta \in \hat{\mathrm{Z}}_{\mathrm{H}}$ *yields an equivalence*

$$\bigoplus_{\theta \in \hat{\mathrm{Z}}_{\mathrm{H}}} \mathsf{Sat}^{\theta}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},x}) \simeq \mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},x}). \tag{5.13}$$

Proof. We first prove that the essential images of distinct summands in (5.13) are *derived* orthogonal in $\mathsf{D}_{\mathcal{G},\zeta}(\mathrm{Gr}_{\mathrm{G},x})$. More precisely, given objects $\mathcal{A}_1, \mathcal{A}_2 \in \mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},x})$ coming from summands corresponding to $\theta_1 \neq \theta_2 \in \hat{\mathrm{Z}}_{\mathrm{H}}$, we shall prove

$$\mathrm{Hom}(\mathcal{A}_1, \mathcal{A}_2) \simeq 0, \tag{5.14}$$

where $\mathrm{Hom}(\cdot,\cdot)$ is the derived Hom functor of $\mathsf{D}_{\mathcal{G},\zeta}(\mathrm{Gr}_{\mathrm{G},x})$.

The statement is clear if θ_1, θ_2 have distinct images in $\pi_1\mathrm{G}$, as then $\mathcal{A}_1$, $\mathcal{A}_2$ would be supported on distinct connected components of $\mathrm{Hec}_{\mathrm{G},x}$, so let us assume that θ_1, θ_2 have equal image $[\theta] \in \pi_1\mathrm{G}$. Note that $\mathrm{Hom}(\mathcal{A}_1, \mathcal{A}_2)$ is the global section of $\mathcal{H}om(\mathcal{A}_1, \mathcal{A}_2)$, which admits an $\mathrm{L}^+_x\mathrm{G}_{\mathrm{ad}}$-equivariance structure with respect to

$$\mathcal{G}^{\theta_2}_{\mathrm{Hec}_{\mathrm{G}}} - \mathcal{G}^{\theta_1}_{\mathrm{Hec}_{\mathrm{G}}}. \tag{5.15}$$

By Proposition 5.1.8, (5.15) coincides with the pullback of the A-gerbe over $\mathsf{B}\mathrm{G}_{\mathrm{ad}}$ classified by the character $b_1(\theta_2 - \theta_1, \cdot) : \pi_1\mathrm{G}_{\mathrm{ad}} \to \mathrm{A}(-1)$ along the projection

$$\mathrm{L}^+_x\mathrm{G}_{\mathrm{ad}}\backslash\mathrm{Gr}^{[\theta]}_{\mathrm{G},x} \to \mathsf{B}\mathrm{L}^+_x\mathrm{G}_{\mathrm{ad}} \to \mathsf{B}\mathrm{G}_{\mathrm{ad}}.$$

The assumption $\theta_1 \neq \theta_2$ implies that $b_1(\theta_2 - \theta_1, \cdot)$ is nontrivial (*cf.* Remark 2.1.9). The global section of $\mathcal{H}om(\mathcal{A}_1, \mathcal{A}_2)$ thus vanishes by Proposition B.2.4.

The vanishing (5.14) implies that the essential images of distinct summands in (5.13) are orthogonal and have no nontrivial extensions in $\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},x})$. It remains to prove that they generate $\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},x})$ under extensions. For this, it suffices to prove that the standard functor Δ^{λ} for $\lambda \in \Lambda^{\sharp,+}$ (*cf.* §3.3.2) factors through (5.12), for θ the class of λ.

Consider the commutative square

$$\begin{tikzcd}
\mathrm{Hec}^{\lambda}_{\mathrm{G},x} \arrow[r, phantom, "\subset"] \arrow[d] & \mathrm{Hec}^{[\theta]}_{\mathrm{G},x} \arrow[d] \\
\mathrm{L}^+_x\mathrm{G}_{\mathrm{ad}}\backslash\mathrm{Gr}^{\lambda}_{\mathrm{G},x} \arrow[r, phantom, "\subset"] & \mathrm{L}^+_x\mathrm{G}_{\mathrm{ad}}\backslash\mathrm{Gr}^{[\theta]}_{\mathrm{G},x}
\end{tikzcd} \tag{5.16}$$

defined by inclusions of the Schubert cell. Using Lemma 5.2.7, we may factor the standard functor Δ^{λ} through extension along $\mathrm{Gr}^{\lambda}_{\mathrm{G},x} \subset \mathrm{Gr}^{[\theta]}_{\mathrm{G},x}$ as $\mathrm{L}^+_x\mathrm{G}_{\mathrm{ad}}$-*equivariant* perverse sheaves, followed by the functor (5.12). ☐

Corollary 5.3.4. *The* $\hat{\mathrm{Z}}_{\mathrm{H}}$*-grading* (5.13) *on* $\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},x})$ *is compatible with the convolution monoidal structure,* i.e.

1. *the monoidal unit is homogeneous of degree* 0*;*
2. *given* $\mathcal{A}_1, \mathcal{A}_2 \in \mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},x})$ *which are homogeneous of degrees* θ_1, θ_2*, the convolution product* $\mathcal{A}_1 \circ \mathcal{A}_2$ *is homogeneous of degree* $\theta_1 + \theta_2$*.*

Proof. Statement (1) is clear.

Statement (2) follows from Lemma 5.2.3 and the definition of the convolution product (*cf.* §1.3.5). ☐

5.3.5. Next, we formulate the compatibility between the $\hat{\mathrm{Z}}_{\mathrm{H}}$-grading (5.13) with constant term functors. We fix a parabolic subgroup $\mathrm{P} \subset \mathrm{G}$ with Levi quotient $\mathrm{P} \twoheadrightarrow \mathrm{M}$.

Recall the constant term functor CT_{P} on the Satake categories (*cf.* §1.5.7) and the natural map $\hat{\mathrm{Z}}_{\mathrm{H}_{\mathrm{M}}} \to \hat{\mathrm{Z}}_{\mathrm{H}}$ (*cf.* §5.2.4).

Corollary 5.3.6. *Given an object* $\mathcal{A} \in \mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},x})$ *homogeneous of degree* $\theta \in \hat{\mathrm{Z}}_{\mathrm{H}}$*, its image* $\mathrm{CT}_{\mathrm{P}}(\mathcal{A}) \in \mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{M},x})$ *belongs to the sum of* λ*-graded components, where* $\lambda \in \hat{\mathrm{Z}}_{\mathrm{H}_{\mathrm{M}}}$ *lies over* θ*.*

Proof. Let $\lambda \in \hat{\mathrm{Z}}_{\mathrm{H}_{\mathrm{M}}}$ be an element whose image in $\hat{\mathrm{Z}}_{\mathrm{H}}$ is $\theta_1 \neq \theta$. It suffices to prove that $\mathrm{CT}_{\mathrm{P}}(\mathcal{A})$ is orthogonal to any object $\mathcal{B} \in \mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{M},x})$ homogeneous of degree λ. Since $\mathrm{CT}_{\mathrm{P}}(\mathcal{A})$ is supported on the connected components of $\mathrm{Hec}_{\mathrm{M},x}$ corresponding to elements of $\pi_1\mathrm{M}$ lying over $[\theta] \in \pi_1\mathrm{G}$, we may assume that $[\theta_1] = [\theta]$ in this proof.

We shall prove the vanishing of the derived Hom

$$\mathrm{Hom}(\mathrm{CT}_{\mathrm{P}}^{[\lambda]}(\mathcal{A}), \mathcal{B}) \simeq 0 \tag{5.17}$$

in $\mathsf{D}_{\mathcal{G},\zeta}(\mathrm{Gr}_{\mathrm{M},x}^{[\lambda]})$, where $\mathrm{CT}_{\mathrm{P}}^{[\lambda]}(\mathcal{A})$ denotes the restriction of $\mathrm{CT}_{\mathrm{P}}(\mathcal{A})$ to the connected component corresponding to $[\lambda] \in \pi_1\mathrm{M}$.

Note that $\mathrm{CT}_{\mathrm{P}}^{[\lambda]}(\mathcal{A})$ and $\mathcal{B}$ admit $\mathrm{L}_x^+(\mathrm{M}/\mathrm{Z})$-equivariance structures with respect to *different* $\mathrm{L}_x^+(\mathrm{M}/\mathrm{Z})$-equivariance structures on $\mathcal{G}_{\mathrm{Gr}_{\mathrm{M},x}}$, induced from $\mathcal{G}_{\mathrm{Hec}_{\mathrm{G}}}^{\theta}$, respectively $\mathcal{G}_{\mathrm{Hec}_{\mathrm{G}}}^{\theta_1}$ (*cf.* Lemma 5.2.5). The difference $\mathcal{G}_{\mathrm{Hec}_{\mathrm{G}}}^{\theta_1} - \mathcal{G}_{\mathrm{Hec}_{\mathrm{G}}}^{\theta}$ is pulled back from the A-gerbe over $\mathsf{B}\mathrm{G}_{\mathrm{ad}}$ classified by the character $b_1(\theta - \theta_1, \cdot) : \pi_1\mathrm{G}_{\mathrm{ad}} \to \mathrm{A}(-1)$ (*cf.* Proposition 5.1.8). Since $\theta \neq \theta_1$, this character is nonzero, so it defines a nontrivial A-gerbe over $\mathsf{B}(\mathrm{M}/\mathrm{Z})$. The vanishing (5.17) thus follows from Proposition B.2.4 as in the proof of Proposition 5.3.3. ☐

Remark 5.3.7. It follows from the proof of Proposition 5.3.3 that (5.13) is a *coarsening* of the $\Lambda^{\sharp,+}$-grading provided by Theorem 3.3.5. More precisely, for each $\lambda \in \Lambda^{\sharp,+}$, the λ-graded component of Theorem 3.3.5 is contained in the θ-graded component of (5.13) for $\theta \in \hat{\mathrm{Z}}_{\mathrm{H}}$ the class of λ. However, there are *no* analogues of Corollary 5.3.4 and Corollary 5.3.6 for the $\Lambda^{\sharp,+}$-grading. If one *defines* the $\hat{\mathrm{Z}}_{\mathrm{H}}$-grading by coarsening the $\Lambda^{\sharp,+}$-grading, there also seems to be no convenient way of proving these statements.

We find it more profitable to view (5.13) as a *refinement* of the $\pi_1\mathrm{G}$-grading given by connected components of $\mathrm{Hec}_{\mathrm{G},x}$. Unlike Theorem 3.3.5, the construction of (5.13) can be extended to the derived category (replacing $\mathrm{L}_x^+\mathrm{G}_{\mathrm{ad}}$-equivariant objects by $\mathrm{L}_x^+\mathrm{G}_{\mathrm{ad}}$-monodromic objects), as well as to more general coefficients.

6. TANNAKIAN RECONSTRUCTION

In this section, we complete the construction of the geometric Satake equivalence (*cf.* Theorem 2.3.2). We shall begin by lifting the Satake category ${}^{+}\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\mathrm{I}})$, for split G, to a $(\hat{\mathrm{Z}}_{\mathrm{H}})^{\oplus \mathrm{I}}$-graded étale sheaf of symmetric monoidal e-linear categories over X^{I} in the sense of §A.2.4. This will allow us to construct an "untwisted" Satake category, for which it is possible to define a fiber functor and apply the Tannakian formalism.

We remain in the context of §2.3.1 throughout this subsection.

6.1. The fiber functor.

6.1.1. We begin with a mild generalization of some constructions of §4 and §5.3, adapted to a sheaf-theoretic version of the Satake category.

Let us return to the context of §1.3. Given a finite set I, the construction of the symmetric monoidal e-linear category ${}^{+}\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\mathrm{I}})$ extends to the case where X^{I} is replaced by an étale X^{I}-scheme U: The only change is that the fusion product is defined using the base change of $\mathrm{X}^{\sqcup_n \mathrm{I},\mathrm{disj}} \subset \mathrm{X}^{\sqcup_n \mathrm{I}}$ to U (*cf.* §1.3.12).

This construction is functorial in U, defining an étale sheaf of symmetric monoidal e-linear categories ${}^{+}\mathcal{S}at_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\mathrm{I}})$ over X^{I}.

6.1.2 *The case for tori***.** For G = T a torus, the construction of the geometric Satake equivalence for (T,μ) (*cf.* §4.1.9) yields an equivalence of étale sheaves of symmetric monoidal e-linear categories over X^{I}:

$$
{}^{+}\mathcal{S}at_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{T},\mathrm{I}}) \simeq \mathcal{R}ep_{\mathrm{T}_{\mathrm{H}}^{\boxtimes \mathrm{I}},(\nu+\vartheta)^{\boxplus \mathrm{I}}}, \tag{6.1}
$$

by the étale local nature of the construction.

Here, the normalization of the commutativity constraint has no effect (*cf.* §1.3.16).

6.1.3. Next, we assume that G is split. Our goal is to generalize the $\hat{\mathrm{Z}}_{\mathrm{H}}$-grading (5.13) to ${}^{+}\mathcal{S}at_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\mathrm{I}})$ for any finite set I.

The pair $(\mathrm{Hec}_{\mathrm{G}},\mathcal{G}_{\mathrm{Hec}_{\mathrm{G}}})$ factorizes over the pairwise disjoint locus $\mathrm{X}^{\mathrm{I},\mathrm{disj}} \subset \mathrm{X}^{\mathrm{I}}$ (*cf.* Proposition 1.2.11), so Proposition 5.1.3 supplies an $\mathrm{L}^{+}_{\mathrm{I}}(\mathsf{BZ})$-equivariance on the pullback of $\mathcal{G}_{\mathrm{Hec}_{\mathrm{G}}}$ to the component

$$
\mathrm{Hec}^{\mathrm{disj},[\theta^{\mathrm{I}}]}_{\mathrm{G},\mathrm{I}} := \Big(\prod_{i\in\mathrm{I}} \mathrm{Hec}^{[\theta^{i}]}_{\mathrm{G},\{i\}}\Big) \times_{\mathrm{X}^{\mathrm{I}}} \mathrm{X}^{\mathrm{I},\mathrm{disj}}
$$

associated to the each tuple $\theta^{\mathrm{I}} \in (\hat{\mathrm{Z}}_{\mathrm{H}})^{\oplus \mathrm{I}}$ with image $[\theta^{\mathrm{I}}] \in (\pi_1\mathrm{G})^{\oplus\mathrm{I}}$. As in Remark 5.1.4, this leads to an $\mathrm{L}^{+}_{\mathrm{I}}\mathrm{G}_{\mathrm{ad}}$-equivariance on the pullback of $\mathcal{G}_{\mathrm{Gr}_{\mathrm{G}}}$ to the corresponding component $\mathrm{Gr}^{\mathrm{disj},[\theta^{\mathrm{I}}]}_{\mathrm{G},\mathrm{I}}$ of the affine Grassmannian over $\mathrm{X}^{\mathrm{I},\mathrm{disj}}$.

6.1.4. For each $\theta^{\mathrm{I}} \in (\hat{\mathrm{Z}}_{\mathrm{H}})^{\oplus \mathrm{I}}$, we define a full subsheaf

$$
\mathcal{S}at^{\theta^{\mathrm{I}}}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\mathrm{I}}) \subset {}^{+}\mathcal{S}at_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\mathrm{I}}) \tag{6.2}
$$

as follows: Over each étale X^{I}-scheme U, it consists of objects of ${}^{+}\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\mathrm{U}})$ whose restriction to the pairwise disjoint locus comes from $\mathrm{L}^{+}_{\mathrm{I}}\mathrm{G}_{\mathrm{ad}}$-equivariant objects of the category $\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Gr}^{\mathrm{disj},[\theta^{\mathrm{I}}]}_{\mathrm{G},\mathrm{U}})$, where the $\mathrm{L}^{+}_{\mathrm{I}}\mathrm{G}_{\mathrm{ad}}$-equivariance on the restriction of $\mathcal{G}_{\mathrm{Gr}_{\mathrm{G}}}$ is supplied by θ^{I}, and the subscript U means base change along $\mathrm{U} \to \mathrm{X}^{\mathrm{I}}$.

The following results are straightforward adaptations of those of §5.3.

Proposition 6.1.5. *The functors* (6.2) *induce an equivalence*

$$
\bigoplus_{\theta^{\mathrm{I}}\in(\hat{\mathrm{Z}}_{\mathrm{H}})^{\oplus\mathrm{I}}} \mathcal{S}at^{\theta^{\mathrm{I}}}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\mathrm{I}}) \simeq {}^{+}\mathcal{S}at_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\mathrm{I}}). \tag{6.3}
$$

Proof. This follows from Proposition 5.3.3 together with the fact the restriction functor realizes $\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Gr}_{\mathrm{G},\mathrm{I}} \times_{\mathrm{X}^{\mathrm{I}}} \mathrm{U})$ as a full subcategory of $\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Gr}^{\mathrm{disj}}_{\mathrm{G},\mathrm{I}} \times_{\mathrm{X}^{\mathrm{I},\mathrm{disj}}} \mathrm{U}^{\mathrm{disj}})$ closed under direct summands (*cf.* [HS23, Theorem 6.8]). □

Corollary 6.1.6. *The* $(\hat{\mathrm{Z}}_{\mathrm{H}})^{\oplus\mathrm{I}}$*-grading* (6.3) *on* $\mathcal{S}at_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\mathrm{I}})$ *is compatible with the convolution monoidal structure,* i.e.

(1) *the monoidal unit is homogeneous of degree* 0*;*
(2) *given* $\mathcal{A}_1, \mathcal{A}_2 \in \mathcal{S}at_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\mathrm{I}})$ *which are homogeneous of degrees* $\theta^{\mathrm{I}}_1, \theta^{\mathrm{I}}_2$*, the convolution product* $\mathcal{A}_1 \circ \mathcal{A}_2$ *is homogeneous of degree* $\theta^{\mathrm{I}}_1 + \theta^{\mathrm{I}}_2$*.*

Proof. This follows from Corollary 5.3.4 and the fact that the convolution product factorizes over $\mathrm{X}^{\mathrm{I}}_{\mathrm{disj}}$ (*cf.* the proof of Proposition 1.3.15). □

Corollary 6.1.7. *The* $(\hat{\mathrm{Z}}_{\mathrm{H}})^{\oplus \mathrm{I}}$*-grading* (6.3) *on* $\mathcal{S}at_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\mathrm{I}})$ *is compatible with constant term functors,* i.e. *given a parabolic subgroup* $\mathrm{P} \subset \mathrm{G}$ *with Levi quotient* $\mathrm{P} \twoheadrightarrow \mathrm{M}$*, the functor* CT_{P} *carries the* θ^{I}*-graded component into the sum of the* λ^{I}*-graded components where* $\lambda^{\mathrm{I}} \in (\hat{\mathrm{Z}}_{\mathrm{H}_{\mathrm{M}}})^{\oplus \mathrm{I}}$ *lies over* θ^{I}*.*

Proof. This follows from Corollary 5.3.6 and the fact that CT_{P} factorizes over $\mathrm{X}^{\mathrm{I}}_{\mathrm{disj}}$. □

6.1.8. It follows from Corollary 6.1.6 that the assignment of the θ^{I}-graded component (6.2) to each $\theta^{\mathrm{I}} \in (\hat{\mathrm{Z}}_{\mathrm{H}})^{\oplus \mathrm{I}}$ is lax monoidal. By equipping it with the commutativity constraint inherited from ${}^{+}\mathcal{S}at_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\mathrm{I}})$, we lift the latter to a $(\hat{\mathrm{Z}}_{\mathrm{H}})^{\oplus \mathrm{I}}$-graded étale sheaf of *symmetric* monoidal e-linear categories in the sense of §A.2.4.

In particular, we may use any $\mathbb{E}_{\infty}$-monoidal morphism $(\hat{\mathrm{Z}}_{\mathrm{H}})^{\oplus \mathrm{I}} \to \mathsf{B}^2\mathsf{e}^{\times}$ to form the symmetric monoidal twist of ${}^{+}\mathcal{S}at_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\mathrm{I}})$ (*cf.* §A.2.7). We apply this construction to the *opposite* of (2.28) to obtain a sheaf of symmetric monoidal e-linear categories

$$
{}^{+}\mathcal{S}at_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\mathrm{I}})_{-(\nu+\vartheta)^{\boxplus \mathrm{I}}}, \tag{6.4}
$$

which inherits a $(\hat{\mathrm{Z}}_{\mathrm{H}})^{\oplus \mathrm{I}}$-grading. We refer to its global section ${}^{+}\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\mathrm{I}})_{-(\nu+\vartheta)^{\boxplus \mathrm{I}}}$ as the *untwisted Satake category* and to (6.4) as the *sheaf of untwisted Satake categories.*

6.1.9. Let $\mathrm{B} \subset \mathrm{G}$ be a Borel subgroup. Using the compatibility between the $(\hat{\mathrm{Z}}_{\mathrm{H}})^{\oplus \mathrm{I}}$-grading (6.3) with constant term functors (*cf.* Corollary 6.1.7), we obtain a constant term functor between the sheaves of untwisted Satake categories for G and T:

$$
{}^{+}\mathcal{S}at_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\mathrm{I}})_{-(\nu+\vartheta)^{\boxplus \mathrm{I}}} \to {}^{+}\mathcal{S}at_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{T},\mathrm{I}})_{-(\nu+\vartheta)^{\boxplus \mathrm{I}}} \tag{6.5}
$$

We shall argue that (6.5) is "independent of the choice of B." To make sense of this statement, we need to justify why the target of (6.5) is indepedent of B. (Note that the induced étale level μ_{T} of T *depends* on B.) By Proposition 4.1.3, we may replace the target of (6.5) by ${}^{+}\mathcal{S}at_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{T}^{\sharp},\mathrm{I}})_{-(\nu+\vartheta)^{\boxplus \mathrm{I}}}$, but the induced étale level $\mu_{\mathrm{T}^{\sharp}}$ of $\mathrm{T}^{\sharp}$ is canonically independent of B (*cf.* Remark 2.1.15).

It thus suffices to show that the constant term functor CT_{B} is independent of the choice of B upon replacing its target by ${}^{+}\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{T}^{\sharp},\mathrm{I}})$, as we do now.

Lemma 6.1.10. *The symmetric monoidal functor*

$$
\mathrm{CT}_{\mathrm{B}} : {}^{+}\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\mathrm{I}}) \to {}^{+}\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{T}^{\sharp},\mathrm{I}}) \tag{6.6}
$$

is canonically independent of the Borel subgroup $\mathrm{B} \subset \mathrm{G}$*.*

Proof. Recall that the target of (6.6) embeds fully faithfully into ${}^{+}\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{T}^{\sharp},\mathrm{I}} \times_{\mathrm{X}^{\mathrm{I}}} \mathrm{X}^{\mathrm{I},\mathrm{disj}})$, where $\mathrm{X}^{\mathrm{I},\mathrm{disj}} \subset \mathrm{X}^{\mathrm{I}}$ is the pairwise disjoint locus (*cf.* Proposition 1.3.13(1)). By compatibility of CT_{B} with factorization, we reduce to $\mathrm{I} = \{1\}$ and suppress it from the notation. It remains to prove that the functor

$$
\mathrm{CT}_{\mathrm{B}} : \mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G}}) \to \mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{T}^{\sharp}}) \tag{6.7}
$$

is independent of B.

To do so, we construct (6.7) using the universal Borel subgroup of G over Torel and verify that the result descends along Torel → X (*cf.* Remark 2.1.15). However, since the geometric fibers of Torel are simply connected, every e-local system over Torel descends to X. ☐

6.1.11 *The fiber functor***.** Define the *fiber functor* to be the composition

$$\begin{aligned}\omega^{\mathrm{I}} : {}^{+}\mathcal{S}at_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G,I}})_{-(\nu+\vartheta)^{\boxplus \mathrm{I}}} &\xrightarrow{(6.5)} {}^{+}\mathcal{S}at_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{T,I}})_{-(\nu+\vartheta)^{\boxplus \mathrm{I}}} \\ &\simeq \mathcal{R}ep_{\mathrm{T}_{\mathrm{H}}^{\boxtimes \mathrm{I}}} \to \mathcal{L}is(\mathrm{X}^{\mathrm{I}}),\end{aligned} \tag{6.8}$$

where the equivalence is the untwist of (6.1) and the last functor is the forgetful functor, for $\mathrm{T}_{\mathrm{H}} \subset \mathrm{H}$ the canonical maximal torus.

The basic properties of (6.8) are summarized as follows.

Proposition 6.1.12. *The functor* (6.8) *is symmetric monoidal, exact, conservative, and* $\mathcal{L}is(\mathrm{X}^{\mathrm{I}})$*-linear.*

Proof. The symmetric monoidal structure on (6.8) comes from the symmetric monoidal structure on CT_{B} (*cf.* Lemma 1.5.8) and the functoriality of symmetric monoidal twists. The exactness and conservativity of (6.8) are consequences of Proposition 1.5.6.

The $\mathcal{L}is(\mathrm{X}^{\mathrm{I}})$-linearity is immediate, as all constructions we perform respect the $\mathcal{L}is(\mathrm{X}^{\mathrm{I}})$-module structure of the Satake category. ☐

6.2. The Tannaka dual $\check{\mathrm{G}}$.

6.2.1. We continue to assume that G is split.

For any finite set I and any étale X^{I}-scheme U, the value of (6.8) at U yields a symmetric monoidal, exact, conservative Lis(U)-linear functor

$$\omega^{\mathrm{I}} : {}^{+}\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G,U}})_{-(\nu+\vartheta)^{\boxplus \mathrm{I}}} \to \mathsf{Lis}(\mathrm{U}), \tag{6.9}$$

where $\mathrm{Hec}_{\mathrm{G,U}}$ is the base change of $\mathrm{Hec}_{\mathrm{G,I}}$ to U (*cf.* Proposition 6.8).

We wish to apply the relative Tannakian formalism (*cf.* [FS24, Proposition VI.10.2]) to (6.9) and identify the resulting bi-algebra object $\mathcal{A}_{\mathrm{U}} \in \mathrm{Ind}\,\mathsf{Lis}(\mathrm{U})$ with (the restriction of) the external tensor product of $\mathcal{A}_{\mathrm{X}}$, associated to $\mathrm{I} = \{1\}$ and $\mathrm{U} = \mathrm{X}$. For this purpose, we need to filter the source of (6.9) by full subcategories on which it is co-representable.

6.2.2 *Weight functors***.** Let us begin with the case $\mathrm{I} = \{1\}$.

For each $\lambda \in \Lambda^{\sharp}$, we consider the post-composition of (6.5) with the restriction functor along $\varpi^{\lambda} : \mathrm{X} \to \mathrm{Hec}_{\mathrm{T}}$ (*cf.* §3.1.2):

$$\omega^{\lambda} : \mathcal{S}at_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\{1\}})_{-(\nu+\vartheta)} \to \mathcal{L}is(\mathrm{X}), \tag{6.10}$$

where we used the canonical identification of $(\varpi^{\lambda})^{*}\mathcal{G}_{\mathrm{Hec}_{\mathrm{T}}}$ with $(\nu + \vartheta)(\lambda)$ to identify the target with $\mathcal{L}is(\mathrm{X})$ (*cf.* Proposition 4.2.6).

Note that the functor (6.10) vanishes on all but one component of the source with respect to the $\hat{\mathrm{Z}}_{\mathrm{H}}$-grading, namely the one corresponding to the image $\theta \in \hat{\mathrm{Z}}_{\mathrm{H}}$ of λ (*cf.* Corollary 5.3.6). It may thus be viewed as a functor

$$\omega^{\lambda} : \mathcal{S}at^{\theta}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\{1\}})_{-(\nu+\vartheta)(\lambda)} \to \mathcal{L}is(\mathrm{X}). \tag{6.11}$$

By construction, (6.11) is the $-(\nu+\vartheta)(\lambda)$-twist of the λ-graded component of the constant term functor CT_{B}. We refer to (6.11) as the λ*-weight functor*.

6.2.3. Given an $\mathrm{L}^+_{\{1\}}\mathrm{G}$-stable closed subscheme Z of $\mathrm{Gr}_{\mathrm{G},\{1\}}$, we may consider the full subsheaf of categories

$$\mathcal{S}at_{\mathcal{G},\zeta}(\mathrm{L}^+_{\{1\}}\mathrm{G}\backslash\mathrm{Z})_{-(\nu+\vartheta)} \subset {}^+\mathcal{S}at_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\{1\}})_{-(\nu+\vartheta)} \tag{6.12}$$

consisting of objects supported on (the base change of) Z.

Restricting the functor (6.9) (for $\mathrm{I} = \{1\}$) along (6.12), we obtain a functor

$$\omega^{\{1\}} : \mathcal{S}at_{\mathcal{G},\zeta}(\mathrm{L}^+_{\{1\}}\mathrm{G}\backslash\mathrm{Z})_{-(\nu+\vartheta)} \to \mathcal{L}is(\mathrm{X}). \tag{6.13}$$

Lemma 6.2.4. *The functor* (6.13) *is co-representable (over any étale* X*-scheme).*

Proof. It suffices to prove that for each $\lambda \in \Lambda^\sharp$, the restriction of the λ-weight functor (6.11) to objects supported on Z is co-representable. More precisely, writing $\theta \in \hat{\mathrm{Z}}_{\mathrm{H}}$ for the image of λ, we need to construct an object

$$\mathcal{P}^\lambda_{\mathrm{Z}} \in \mathsf{Sat}^\theta_{\mathcal{G},\zeta}(\mathrm{L}^+_{\{1\}}\mathrm{G}\backslash\mathrm{Z})_{-(\nu+\vartheta)}, \tag{6.14}$$

together with an isomorphism of e-local systems over X:

$$\pi_*\mathcal{H}om(\mathcal{P}^\lambda_{\mathrm{Z}}, \mathcal{A}) \simeq \omega^\lambda(\mathcal{A}) \tag{6.15}$$

natural in $\mathcal{A} \in \mathsf{Sat}^\theta_{\mathcal{G},\zeta}(\mathrm{L}^+_{\{1\}}\mathrm{G}\backslash\mathrm{Z})_{-(\nu+\vartheta)}$. Here, $\pi : \mathrm{L}^+_{\{1\}}\mathrm{G}\backslash\mathrm{Z} \to \mathrm{X}$ denotes the projection, and the formation of (6.14), (6.15) must commute with étale base change along X.

Using hyperbolic localization (*cf.* the proof of Proposition 1.5.6), we may present the restriction of (6.11) to objects supported on Z as the functor $(p^-)_*(q^-)^!(\frac{d_{\mathrm{G},\mathrm{T}}}{2})[d_{\mathrm{G},\mathrm{T}}]$, formed with respect to the morphisms

$$\begin{tikzcd} & \mathrm{Z} \cap \mathrm{Gr}^{-,\lambda}_{\mathrm{B},\{1\}} \arrow[dl, "q^-"'] \arrow[dr, "p^-"] & \\ \mathrm{Z} & & \mathrm{X} \end{tikzcd}$$

and the trivialization of $\mathcal{G}_{\mathrm{Gr}_{\mathrm{G}}} - (\nu+\vartheta)(\theta)$ over $\mathrm{Z} \cap \mathrm{Gr}^{-,\lambda}_{\mathrm{B},\{1\}}$ specified in §6.2.2.

On the other hand, the functor of forgetting the $\mathrm{L}^+_{\{1\}}\mathrm{G}_{\mathrm{ad}}$-equivariance

$$\mathsf{D}_{\mathcal{G},\zeta}(\mathrm{L}^+_{\{1\}}\mathrm{G}_{\mathrm{ad}}\backslash\mathrm{Z}) \to \mathsf{D}_{\mathcal{G},\zeta}(\mathrm{Z}),$$

where $\mathcal{G}_{\mathrm{Gr}_{\mathrm{G}}}$ is endowed with the $\mathrm{L}^+_{\{1\}}\mathrm{G}_{\mathrm{ad}}$-equivariance structure corresponding to θ, admits a left adjoint $\mathrm{Av}^\theta_!$ (*cf.* [BR18, §A.2]). Since the formation of $\mathrm{Av}^\theta_!$ is of étale local nature over X, we may twist it by $-(\nu+\vartheta)(\theta)$ and still write $\mathrm{Av}^\theta_!$ for the resulting functor. By adjunction, we have a natural isomorphism

$$\omega^\lambda(\mathcal{A}) \simeq \pi_*\mathcal{H}om({}^p\mathrm{H}^0\,\mathrm{Av}^\theta_!(q^-)_!\mathsf{e}(-\frac{d_{\mathrm{G},\mathrm{T}}}{2})[-d_{\mathrm{G},\mathrm{T}}], \mathcal{A}).$$

It follows that the following $\mathrm{L}^+_{\{1\}}\mathrm{G}_{\mathrm{ad}}$-equivariant perverse sheaf over Z:

$$\mathcal{P}^\lambda_{\mathrm{Z}} := {}^p\mathrm{H}^0\,\mathrm{Av}^\theta_!(q^-)_!\mathsf{e}(-\frac{d_{\mathrm{G},\mathrm{T}}}{2})[-d_{\mathrm{G},\mathrm{T}}]$$

co-represents the functor ω^λ. We need to show that it belongs to the Satake category.

The only remaning property to check is that $\mathcal{P}^\lambda_{\mathrm{Z}}$ is ULA relative to X. To prove this, we induct on the number of $\mathrm{L}^+_{\{1\}}\mathrm{G}$-orbits in Z. Let $j : \mathrm{Gr}^{\lambda'}_{\mathrm{G},\{1\}} \hookrightarrow \mathrm{Z}$ be an open Schubert cell, with complementary (reduced) closed subscheme $i : \mathrm{Z}' \hookrightarrow \mathrm{Z}$. Then $j^*\mathcal{P}^\lambda_{\mathrm{Z}}$ is locally constant perverse, hence of the form $\widetilde{\mathcal{E}}$ for some e-local system $\mathcal{E}$ over X (*cf.* notation of §3.3.2). We claim that the co-unit of the adjunction

$$ {}^p\mathrm{H}^0 j_! \widetilde{\mathcal{E}} \to \mathcal{P}^\lambda_{\mathrm{Z}} \tag{6.16}$$

is *injective*: This holds because of Corollary 3.3.8, which exhibits $\mathcal{P}^\lambda_{\mathrm{Z}} \to {}^p\mathrm{H}^0 j_* \widetilde{\mathcal{E}}$ as a retraction of (6.16). By the Cousin triangle, the cokernel of (6.16) is identified with $i_* {}^p\mathrm{H}^0 i^* \mathcal{P}^\lambda_{\mathrm{Z}}$, but we have an isomorphism

$$ {}^p\mathrm{H}^0 i^* \mathcal{P}^\lambda_{\mathrm{Z}} \simeq \mathcal{P}^\lambda_{\mathrm{Z}'}$$

because they co-represent the same functor.

It now follows from the ULA property of $\Delta^{\lambda'}(\mathcal{E})$ (*cf.* Proposition 3.3.3) and the induction hypothesis that $\mathcal{P}^\lambda_{\mathrm{Z}}$ is an extension of ULA objects, hence ULA. ☐

6.2.5. In the context of §6.2.3, we write

$$\mathcal{P}_{\mathrm{Z}} \in \mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{L}^+_{\{1\}}\mathrm{G}\backslash\mathrm{Z})_{-(\nu+\vartheta)}$$

for the object co-representing (6.13), which exists by Lemma 6.2.4. (More precisely, for any étale X-scheme U, the pullback $\mathcal{P}_{\mathrm{Z}}|_{\mathrm{U}}$ co-represents the section of (6.13) over U.)

More generally, given a finite set I and an I-tuple $\{\mathrm{Z}_i\}_{i\in\mathrm{I}}$ of $\mathrm{L}^+_{\{1\}}\mathrm{G}$-stable closed subschemes of $\mathrm{Gr}_{\mathrm{G},\{1\}}$, we may consider the full subsheaf of categories

$$\mathcal{C}_{\{\mathrm{Z}_i\}} \subset {}^+\mathcal{S}at_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G}})_{-(\nu+\vartheta)^{\boxplus \mathrm{I}}} \tag{6.17}$$

consisting of objects whose restrictions along the pairwise disjoint locus $\mathrm{X}^{\mathrm{I},\mathrm{disj}} \subset \mathrm{X}^{\mathrm{I}}$ are supported on (the base change of) $\prod_{i\in\mathrm{I}} \mathrm{Z}_i$ under the factorization isomorphism.

The I-tuples $\{\mathrm{Z}_i\}_{i\in\mathrm{I}}$ form a poset under inclusion and (6.17) exhibits the target as the filtered colimit of $\mathcal{C}_{\{\mathrm{Z}_i\}}$ over $\{\mathrm{Z}_i\}_{i\in\mathrm{I}}$.

Lemma 6.2.6. *The restriction of* (6.9) *to* $\mathcal{C}_{\{\mathrm{Z}_i\}}$ *is co-represented by the external fusion product of* $\mathcal{P}_{\mathrm{Z}_i}$ *over* $i \in \mathrm{I}$.

Proof. Since restriction to the pairwise disjoint locus is fully faithful (*cf.* Proposition 1.3.13), this reduces to the case where I is a singleton. ☐

6.2.7. For any finite set I and étale X^{I}-scheme U, we apply the relative Tannakian reconstruction [FS24, Proposition VI.10.2] to the fiber functor (6.9), where the source is endowed with the filtration (6.17) on which (6.9) is co-representable (*cf.* Lemma 6.2.6).

This yields a bi-algebra object $\mathcal{A}_{\mathrm{U}}$ of $\mathrm{Ind}\,\mathsf{Lis}(\mathrm{U})$ and lifts (6.9) to a symmetric monoidal, $\mathsf{Lis}(\mathrm{U})$-linear equivalence natural in U:

$$ {}^+\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\mathrm{U}})_{-(\nu+\vartheta)^{\boxplus \mathrm{I}}} \simeq \mathcal{A}_{\mathrm{U}}\text{-}\mathsf{Comod}(\mathsf{Lis}(\mathrm{U})). \tag{6.18}$$

Denote by $\mathcal{A}_{\mathrm{X}}$ the result of this construction for $\mathrm{I} = \{1\}$ and $\mathrm{U} = \mathrm{X}$.

Proposition 6.2.8. *For any étale* X^{I}*-scheme* U*, the bi-algebra* $\mathcal{A}_{\mathrm{U}}$ *is canonically identified with the pullback along* $\mathrm{U} \to \mathrm{X}^{\mathrm{I}}$ *of the external tensor product* $\boxtimes_{i\in\mathrm{I}}\mathcal{A}_{\mathrm{X}}$.

Proof. By [FS24, Proposition VI.10.2] and Lemma 6.2.6, there is a canonical isomorphism of bi-algebras

$$\mathcal{A}_{\mathrm{U}} \simeq \operatorname*{colim}_{\{\mathrm{Z}_i\}} \omega^{\mathrm{I}}(\mathcal{P}_{\{\mathrm{Z}_i\}}|_{\mathrm{U}})^\vee,$$

where each $\mathcal{P}_{\{\mathrm{Z}_i\}}$ is the external fusion product of $\mathcal{P}_{\mathrm{Z}_i}$ over $i \in \mathrm{I}$. Since ω^{I} carries external fusion products to external tensor products, the result follows. ☐

6.2.9. Finally, we may define $\check{\mathrm{G}}$ to be the Tannaka dual of $\mathcal{A}_{\mathrm{X}}$, viewed as a locally constant étale sheaf of affine monoid e-schemes over X.

Corollary 6.2.10. *For any finite set* I*, there is a canonical equivalence of étale sheaves of symmetric monoidal* e*-linear categories over* X^{I}*:*

$$^{+}\mathcal{S}at_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\mathrm{I}})_{-(\nu+\vartheta)^{\boxplus \mathrm{I}}} \simeq \mathcal{R}ep_{\check{\mathrm{G}}^{\boxtimes \mathrm{I}}}. \tag{6.19}$$

Proof. This is a restatement of the equivalence (6.18), where U ranges over all étale X^{I}-schemes, and we use Proposition 6.2.8 to identify $\mathcal{A}_{\mathrm{U}}\text{-}\mathsf{Comod}(\mathsf{Lis}(\mathrm{U}))$ with $\mathcal{R}ep_{\check{\mathrm{G}}^{\boxtimes \mathrm{I}}}(\mathrm{U})$. ☐

6.3. Discrepancy A-torsors.

6.3.1. We assume that G is split and choose a Borel subgroup $\mathrm{B} \subset \mathrm{G}$ and a splitting of the quotient $\mathrm{B} \twoheadrightarrow \mathrm{T}$. We shall also specialize to $\mathrm{I} = \{1\}$ and omit $\{1\}$ from the notation (*i.e.* we write Gr_{G} instead of $\mathrm{Gr}_{\mathrm{G},\{1\}}$).

The goal of this subsection is to give an explicit description of the weight functor (6.11) applied to generators of the untwisted Satake category. For this purpose, we shall define certain A-torsors over Mirković–Vilonen cycles which measure the discrepancy between two trivializations of $\mathcal{G}_{\mathrm{Gr}_{\mathrm{G}}}$.

6.3.2. Fix $\lambda_1 \in \Lambda^{\sharp,+}$, $\lambda \in \Lambda^{\sharp}$ with equal image $\theta \in \hat{\mathrm{Z}}_{\mathrm{H}}$.

Denote by $\mathrm{S}^{-,\lambda}$ the LN^{-}-orbit of ϖ^{λ} in Gr_{G}, where $\mathrm{N}^{-} \subset \mathrm{B}^{-}$ is the unipotent radical of the Borel opposite to B. Consider the *Mirković–Vilonen cycle* $\mathrm{Gr}_{\mathrm{G}}^{\lambda_1} \cap \mathrm{S}^{-,\lambda}$ in Gr_{G}. It comes equipped with structural morphisms

$$\begin{tikzcd}
\mathrm{Gr}_{\mathrm{G}}^{\lambda_1} \arrow[d] & \supset & \mathrm{Gr}_{\mathrm{G}}^{\lambda_1} \cap \mathrm{S}^{-,\lambda} & \subset & \mathrm{S}^{-,\lambda} \arrow[d] \\
\mathrm{X} & & \simeq & & \mathrm{X}
\end{tikzcd}$$

The restriction of $\mathcal{G}_{\mathrm{Gr}_{\mathrm{G}}}$ to $\mathrm{Gr}_{\mathrm{G}}^{\lambda_1}$ is identified with the pullback of $\mathcal{G}_{\varpi^{\lambda_1}}$ (*cf.* Corollary 3.1.12). The restriction of $\mathcal{G}_{\mathrm{Gr}_{\mathrm{G}}}$ to $\mathrm{S}^{-,\lambda}$ is identified with the pullback of $\mathcal{G}_{\varpi^{\lambda}}$ by descent along $\mathrm{Gr}_{\mathrm{B}^{-}} \to \mathrm{Gr}_{\mathrm{T}}$. Meanwhile, $\mathcal{G}_{\varpi^{\lambda}}$, $\mathcal{G}_{\varpi^{\lambda_1}}$ may be identified with the values of $\nu + \vartheta$ at λ, respectively λ_1 by Proposition 4.2.6, which coincide because λ, λ_1 map to the same element $\theta \in \hat{\mathrm{Z}}_{\mathrm{H}}$ and $\nu + \vartheta$ factors through $\hat{\mathrm{Z}}_{\mathrm{H}}$ (*cf.* §2.1.18).

The composition of these identifications

$$\begin{aligned}(\nu+\vartheta)(\theta)|_{\mathrm{Gr}_{\mathrm{G}}^{\lambda_1} \cap \mathrm{S}^{-,\lambda}} \simeq \mathcal{G}_{\varpi^{\lambda_1}}|_{\mathrm{Gr}_{\mathrm{G}}^{\lambda_1} \cap \mathrm{S}^{-,\lambda}} &\simeq \mathcal{G}_{\mathrm{Gr}_{\mathrm{G}}}|_{\mathrm{Gr}_{\mathrm{G}}^{\lambda_1} \cap \mathrm{S}^{-,\lambda}} \\ &\simeq \mathcal{G}_{\varpi^{\lambda}}|_{\mathrm{Gr}_{\mathrm{G}}^{\lambda_1} \cap \mathrm{S}^{-,\lambda}} \simeq (\nu+\vartheta)(\theta)|_{\mathrm{Gr}_{\mathrm{G}}^{\lambda_1} \cap \mathrm{S}^{-,\lambda}}\end{aligned} \tag{6.20}$$

thus defines an A-torsor $\tau^{\lambda_1,\lambda}$ over $\mathrm{Gr}_{\mathrm{G}}^{\lambda_1} \cap \mathrm{S}^{-,\lambda}$. We refer to $\tau^{\lambda_1,\lambda}$ as the *discrepancy* A*-torsor* over the Mirković–Vilonen cycle $\mathrm{Gr}_{\mathrm{G}}^{\lambda_1} \cap \mathrm{S}^{-,\lambda}$.

The following lemma gives us some qualitative control of $\tau^{\lambda_1,\lambda}$ in special cases. It is a variant of the calculation performed in [FL10, §4.4].

Lemma 6.3.3. *Let* $\lambda_1 \in \Lambda^{\sharp,+}$ *and* $\lambda \in \Lambda^{\sharp}$ *be two elements with equal image in* $\hat{\mathrm{Z}}_{\mathrm{H}}$*. Then* $\tau^{\lambda_1,\lambda}$ *descends to* X *under either of the following assumptions:*

1. *λ belongs to the Weyl-orbit of λ_1;*
2. *G has semisimple rank 1.*

Proof. Under (1), the morphism $\mathrm{Gr}_{\mathrm{G}}^{\lambda_1} \cap \mathrm{S}^{-,\lambda} \to \mathrm{X}$ is a fibration in affine spaces. Indeed, it is identified with the $\mathrm{L}^+\mathrm{N}^-$-orbit of ϖ^λ (*cf.* [MV07, proof of Theorem 3.2]). This implies that any A-torsor descends to X.

Let us now assume (2). For $\mathrm{Gr}_{\mathrm{G}}^{\lambda_1} \cap \mathrm{S}^{-,\lambda}$ to be nonempty and λ_1, λ to have the same image in $\hat{\mathrm{Z}}_{\mathrm{H}}$, the cocharacter λ must belong to the string

$$\{\lambda_1, \lambda_1 - \alpha^\sharp, \cdots, s_{\check{\alpha}}(\lambda_1)\},$$

where α is the unique simple coroot of G. We shall further assume $\lambda \neq \lambda_1, s_{\check{\alpha}}(\lambda_1)$, or else we fall back to case (1).

Choose an isomorphism between $\mathbb{G}_a$ and the root subgroup $\mathrm{N}_{-\check{\alpha}}$ of G. We filter $\mathrm{L}\mathbb{G}_a$ by the group subschemes $\mathrm{L}^{\geq n}\mathbb{G}_a := \varpi^n \mathrm{L}^+\mathbb{G}_a$ for $n \in \mathbf{Z}$. Then the $\mathrm{L}\mathbb{G}_a$-action on ϖ^λ has stabilizer $\mathrm{L}^{\geq -\langle\check{\alpha},\lambda\rangle}\mathbb{G}_a$. Furthermore, $u \in \mathrm{L}\mathbb{G}_a$ satisfies $u\varpi^\lambda \in \mathrm{Gr}_{\mathrm{G}}^{\lambda_1}$ if and only if

$$u \in \mathrm{L}^{\geq -\langle\check{\alpha},\lambda\rangle - d}\mathbb{G}_a \smallsetminus \mathrm{L}^{\geq -\langle\check{\alpha},\lambda\rangle - d + 1}\mathbb{G}_a, \tag{6.21}$$

where $0 < d < \langle\check{\alpha}, \lambda_1\rangle$ is the integer defined by $\lambda_1 - \lambda = d \cdot \alpha$. (Thus, d coincides with the dimension of the Mirković–Vilonen cycle $\mathrm{Gr}_{\mathrm{G}}^{\lambda_1} \cap \mathrm{S}^{-,\lambda}$.)

Projecting $u\varpi^\lambda$ onto the leading coefficient of u defines a morphism

$$\mathrm{Gr}_{\mathrm{G}}^{\lambda_1} \cap \mathrm{S}^{-,\lambda} \to \Omega_{\mathrm{X}}^{-\langle\check{\alpha},\lambda\rangle - d} \smallsetminus 0, \tag{6.22}$$

where $\Omega_{\mathrm{X}}^{-\langle\check{\alpha},\lambda\rangle - d}$ denotes the total space of the corresponding power of the canonical line bundle Ω_{X}, and 0 denotes its zero section. The morphism (6.22) intertwines the $\mathrm{L}_+\mathrm{T}_{\mathrm{ad}}$-action on the source with the scaling $\mathbb{G}_m$-action on the target, with respect to

$$\mathrm{L}^+\mathrm{T}_{\mathrm{ad}} \simeq \mathrm{L}^+\mathbb{G}_m \twoheadrightarrow \mathbb{G}_m, \tag{6.23}$$

where the isomorphism is supplied by $\check{\alpha}$ and the second morphism is the projection.

We shall now endow $\mathcal{G}_{\mathrm{Gr}_{\mathrm{G}}^{[\theta]}}$ with the $\mathrm{L}^+\mathrm{G}_{\mathrm{ad}}$-equivariance determined by the image $\theta \in \hat{\mathrm{Z}}_{\mathrm{H}}$ of both λ_1 and λ (*cf.* Proposition 5.1.3, Remark 5.1.4). This in particular restricts to an $\mathrm{L}^+\mathrm{T}_{\mathrm{ad}}$-equivariance on $\mathcal{G}_{\mathrm{Gr}_{\mathrm{G}}^{[\theta]}}$ such that all isomorphisms in (6.20) are $\mathrm{L}^+\mathrm{T}_{\mathrm{ad}}$-equivariant (*cf.* Lemma 5.2.5, Lemma 5.2.7). Therefore, $\tau^{\lambda_1,\lambda}$ is an $\mathrm{L}^+\mathrm{T}_{\mathrm{ad}}$-equivariant A-torsor over $\mathrm{Gr}_{\mathrm{G}}^{\lambda_1} \cap \mathrm{S}^{-,\lambda}$. Since (6.22) is a fibration in affine spaces and (6.23) has a pro-unipotent kernel, all such A-torsors canonically descend to

$$(\Omega_{\mathrm{X}}^{-\langle\check{\alpha},\lambda\rangle - d} \smallsetminus 0)/\mathbb{G}_m \simeq \mathrm{X}.$$

This concludes the proof. □

Remark 6.3.4. Lemma 6.3.3 may suggest that $\tau^{\lambda_1,\lambda}$ descends to X in general. However, this is *not* true.

More precisely, when G has semisimple rank ≥ 2, it can happen that the restrictions of $\tau^{\lambda_1,\lambda}$ to *certain* irreducible components of $\mathrm{Gr}_{\mathrm{G}}^{\lambda_1} \cap \mathrm{S}^{-,\lambda}$ descend to X, while its restrictions to other irreducible components do not. We refer the reader to [Gai21, §7], where a similar phenomenon for line bundles over Zastava spaces is studied.

6.3.5 *Trivialization of* $\tau^{\alpha^\sharp,0}$**.** Let us now assume that G has a unique simple root $\check{\alpha}$.

According to Lemma 6.3.3, the discrepancy A-torsor $\tau^{\alpha^\sharp,0}$ canonically descends to X. We shall strengthen this result by constructing a *canonical* trivialization of $\tau^{\alpha^\sharp,0}$. It makes essential use of the trivalization of $\nu(\alpha^\sharp)$ defined in §2.1.18.

Lemma 6.3.6. *If* G *has a unique simple root* $\check{\alpha}$*, then* $\tau^{\alpha^\sharp,0}$ *is canonically trivial.*

Proof. It suffices to show that (6.20) is the identity automorphism for $\lambda_1 = \alpha^\sharp$ and $\lambda = 0$. Equivalently, we need to prove that the following diagram commutes

$$\begin{tikzcd}
\mathcal{G}_{\varpi^{\alpha^\sharp}}|_{\mathrm{Gr}_{\mathrm{G}}^{\alpha^\sharp}\cap \mathrm{S}^{-,0}} \arrow[r, phantom, "\simeq"] \arrow[d, "\simeq"] & \mathcal{G}_{\mathrm{Gr}_{\mathrm{G}}}|_{\mathrm{Gr}_{\mathrm{G}}^{\alpha^\sharp}\cap \mathrm{S}^{-,0}} \arrow[r, phantom, "\simeq"] & \mathcal{G}_{\varpi^0}|_{\mathrm{Gr}_{\mathrm{G}}^{\alpha^\sharp}\cap \mathrm{S}^{-,0}} \arrow[d, "\simeq"] \\
(\nu + \vartheta)(\alpha^\sharp)|_{\mathrm{Gr}_{\mathrm{G}}^{\alpha^\sharp}\cap \mathrm{S}^{-,0}} \arrow[rr, phantom, "\simeq"] & & (\nu + \vartheta)(0)|_{\mathrm{Gr}_{\mathrm{G}}^{\alpha^\sharp}\cap \mathrm{S}^{-,0}}
\end{tikzcd} \tag{6.24}$$

where the vertical isomorphisms are given by Proposition 4.2.6, and the lower horizontal isomorphism is induced from the canonical trivialization of $\nu(\alpha^\sharp)$ (*cf.* §2.1.18). The statement immediately reduces to the case where G is simply connected.

By the criterion (6.21) for the containment $u\varpi^0 \in \mathrm{Gr}_{\mathrm{G}}^{\alpha^\sharp}$, we have

$$\mathrm{Gr}_{\mathrm{G}}^{\alpha^\sharp} \cap \mathrm{S}^{-,0} \subset \mathrm{K}_{d\omega} \cdot \varpi^0$$

where for any $\lambda \in \Lambda_{\mathrm{ad}}$, we write K_λ for the subgroup scheme $\varpi^\lambda \mathrm{L}^+\mathrm{G}\varpi^{-\lambda}$ of LG, and here ω is the fundamental coweight of Λ_{ad} and $d := \mathrm{ord}\,\mathrm{Q}(\alpha)$. Furthermore, the orbit $\mathrm{K}_{d\omega} \cdot \varpi^0$ contains both ϖ^0 and $\varpi^{\alpha^\sharp}$. It thus suffices to prove the following statements

(1) the restriction of $\mathcal{G}_{\mathrm{Gr}_{\mathrm{G}}}$ to $\mathrm{K}_{d\omega} \cdot \varpi^0$ descends to X;

(2) the isomorphism $\mathcal{G}_{\varpi^{\alpha^\sharp}} \simeq \mathcal{G}_{\varpi^0}$ arising from (1) fits into the commutative square

$$\begin{tikzcd}
\mathcal{G}_{\varpi^{\alpha^\sharp}} \arrow[r, phantom, "\simeq"] \arrow[d, "\simeq"] & \mathcal{G}_{\varpi^0} \arrow[d, "\simeq"] \\
(\nu + \vartheta)(\alpha^\sharp) \arrow[r, phantom, "\simeq"] & (\nu + \vartheta)(0)
\end{tikzcd} \tag{6.25}$$

where the vertical isomorphisms are again given by Proposition 4.2.6 and the lower horizontal isomorphism is induced from the canonical trivialization of $\nu(\alpha^\sharp)$.

We shall prove both statements using the embedding $\mathrm{G} \subset \widetilde{\mathrm{G}} := \mathrm{G} \rtimes \mathrm{T}_{\mathrm{ad}}$ and the canonical extension of μ to an étale level $\widetilde{\mu}$ of $\widetilde{\mathrm{G}}$ (*cf.* the proof of Lemma 3.5.3). The latter allows us to extend $\mathcal{G}_{\mathrm{Gr}_{\mathrm{G}}}$ to an A-gerbe $\mathcal{G}_{\mathrm{Gr}_{\widetilde{\mathrm{G}}}}$. Identifying the maximal torus of $\widetilde{\mathrm{G}}$ with $\mathrm{T} \times \mathrm{T}_{\mathrm{ad}}$ and its adjoint group with G_{ad}, we may also consider the subgroup schemes

$$\widetilde{\mathrm{K}}_\lambda := \varpi^\lambda \mathrm{L}^+\widetilde{\mathrm{G}}\varpi^{-\lambda} \simeq \varpi^{(0,\lambda)}\mathrm{L}^+\widetilde{\mathrm{G}}\varpi^{-(0,\lambda)}$$

of $\mathrm{L}\widetilde{\mathrm{G}}$ for each $\lambda \in \Lambda_{\mathrm{ad}}$. The inclusion $\mathrm{Gr}_{\mathrm{G}} \to \mathrm{Gr}_{\widetilde{\mathrm{G}}}$ carries $\mathrm{K}_{d\omega} \cdot \varpi^0$ into $\widetilde{\mathrm{K}}_{d\omega} \cdot \varpi^0$, which is in turn the $\varpi^{(0,d\omega)}$-translate of the Schubert cell in $\mathrm{Gr}_{\widetilde{\mathrm{G}}}$ containing $\varpi^{(0,-d\omega)}$. Since the symmetric form (2.19) of $\widetilde{\mu}$ evaluates to the trivial character at $(0, -d\omega)$, Corollary 3.1.12 (applied to $\widetilde{\mathrm{G}}$) implies that the restriction of $\mathcal{G}_{\mathrm{Gr}_{\widetilde{\mathrm{G}}}}$ to this Schubert cell descends to X. Furthermore, the induced isomorphism

$$\mathcal{G}_{\varpi^{s_{\tilde{\alpha}}(0,-d\omega)}} \simeq \mathcal{G}_{\varpi^{(0,-d\omega)}}, \tag{6.26}$$

for $s_{\tilde{\alpha}}$ the simple reflection, coincides with the isomorphism arising from Weyl-equivariance (*cf.* Proposition 3.1.14).

Therefore, the restriction of $\mathcal{G}_{\mathrm{Gr}_{\mathrm{G}}}$ to $\mathrm{K}_{d\omega}\cdot\varpi^0$ descends to X, and the induced isomorphism $\mathcal{G}_{\varpi^{\alpha^\sharp}} \simeq \mathcal{G}_{\varpi^0}$ is the difference between (6.26) and the identity automorphism of $\mathcal{G}_{\varpi^{(0,-d\omega)}}$:

$$\mathcal{G}_{\varpi^{s_{\tilde{\alpha}}(0,-d\omega)}} - \mathcal{G}_{\varpi^{(0,-d\omega)}} \simeq \mathcal{G}_{\varpi^{(0,-d\omega)}} - \mathcal{G}_{\varpi^{(0,-d\omega)}}.$$

This isomorphism renders (6.25) commutative, by comparing with the construction of the trivialization of $\nu(\alpha^\sharp)$ (*cf.* §2.1.18). □

6.3.7 *Application to weight functors.* We shall use discrepancy A-torsors to give an explicit description of the image of standard objects under weight functors (*cf.* §6.2.2).

More precisely, consider the standard functor Δ^{λ_1} associated to $\lambda_1 \in \Lambda^{\sharp,+}$ (*cf.* §3.3.2). Upon untwisting the Satake category, we may regard it as a functor

$$\Delta^{\lambda_1} : \mathcal{L}is(\mathrm{X}) \to {}^{+}\mathcal{S}at^{\theta}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G}})_{-(\nu+\vartheta)(\lambda_1)}, \tag{6.27}$$

where $\theta \in \hat{\mathrm{Z}}_{\mathrm{H}}$ is the image of λ_1. Here, we invoked the fact that the image of Δ^{λ_1} is contained in the θ-graded component of the Satake category (*cf.* the proof of Proposition 5.3.3).

Consider now the composition of (6.27) and (6.11) for any $\lambda \in \Lambda^{\sharp}$ with image θ in $\hat{\mathrm{Z}}_{\mathrm{H}}$:

$$\omega^{\lambda} \circ \Delta^{\lambda_1} : \mathcal{L}is(\mathrm{X}) \to \mathcal{L}is(\mathrm{X}), \tag{6.28}$$

where we identify $(\nu+\vartheta)(\lambda_1)$ with $(\nu+\vartheta)(\lambda)$. On the other hand, we have the discrepancy A-torsor $\tau^{\lambda_1,\lambda}$ over the Mirković–Vilonen cycle $\mathrm{Gr}^{\lambda_1}_{\mathrm{G}} \cap \mathrm{S}^{-,\lambda}$. Along the character $\zeta : \mathrm{A} \subset \mathsf{e}^{\times}$, it defines a rank-1 e-local system $\tau^{\lambda_1,\lambda}_{\zeta}$.

Lemma 6.3.8. *The functor* (6.28) *is identified with tensor product with*

$$\mathrm{H}^{2d}(p^{-})_{!}\tau^{\lambda_1,\lambda}_{\zeta}(d), \tag{6.29}$$

where $p^{-} : \mathrm{Gr}^{\lambda_1}_{\mathrm{G}} \cap \mathrm{S}^{-,\lambda} \to \mathrm{X}$ *is the structural map and* d *is its relative dimension.*

Proof. We define the weight functor ω^{λ} using the constant term functor $\mathrm{CT}_{\mathrm{B}^-}$ associated to $\mathrm{B}^- \subset \mathrm{G}$ (*cf.* Lemma 6.1.10). Since $\mathrm{CT}_{\mathrm{B}^-}$ is perverse t-exact (*cf.* Proposition 1.5.6), we have an isomorphism for any $(\mathcal{G}_{\varpi^{\lambda_1}}, \zeta)$-twisted local system $\mathcal{E}$:

$$\mathrm{CT}_{\mathrm{B}^-}(\Delta^{\lambda_1}(\mathcal{E})) \simeq {}^{p}\mathrm{H}^0(\mathrm{CT}_{\mathrm{B}^-}((j^{\lambda_1})_{!}\widetilde{\mathcal{E}})), \tag{6.30}$$

where the notation j^{λ_1}, $\widetilde{\mathcal{E}}$ are as in §3.3.2. The right-hand-side coincides with tensor product with (6.29), by base change and the projection formula. □

Proposition 6.3.9. *Let* $\lambda_1 \in \Lambda^{\sharp,+}$ *and* $\lambda \in \Lambda^{\sharp}$ *be two elements with equal image in* $\hat{\mathrm{Z}}_{\mathrm{H}}$. *The functor* $\omega^{\lambda} \circ \Delta^{\lambda_1}$ *is an equivalence under either of the following assumptions:*

(1) λ *belongs to the Weyl-orbit of* λ_1*;*
(2) G *has semisimple rank* 1.

Proof. This follows from Lemma 6.3.3, Lemma 6.3.8, together with the fact that the structural map $p^{-} : \mathrm{Gr}^{\lambda_1}_{\mathrm{G}} \cap \mathrm{S}^{-,\lambda} \to \mathrm{X}$ has irreducible fibers under assumption (1) or (2). □

Proposition 6.3.10. *If* G *has a unique simple root* $\check{\alpha}$*, there is a canonical isomorphism*

$$\omega^{0} \circ \Delta^{\alpha^{\sharp}} \simeq \mathrm{id}_{\mathcal{L}is(\mathrm{X})}. \tag{6.31}$$

Proof. This follows from Lemma 6.3.6 and Lemma 6.3.8. □

Remark 6.3.11. Proposition 6.3.10 is stated with a given Borel subgroup $\mathrm{B} \subset \mathrm{G}$ and a section of $\mathrm{B} \twoheadrightarrow \mathrm{T}$. It is, however, indepedent of these choices.

First, let us argue that the left-hand-side of (6.31) is independent of these choices. We have seen that this is the case for the functor ω^0 (*cf.* §6.1.9). As for $\Delta^{\alpha^{\sharp}}$, note that the Schubert cell $\mathrm{Gr}^{\alpha^{\sharp}}_{\mathrm{G}} \subset \mathrm{Gr}_{\mathrm{G}}$ is independent of $\mathrm{T} \subset \mathrm{B} \subset \mathrm{G}$, as different choices are related by G-conjugation. It remains to show that the identification of A-gerbes

$$\mathcal{G}_{\mathrm{Gr}^{\alpha^{\sharp}}_{\mathrm{G}}} \simeq (\nu+\vartheta)(\alpha^{\sharp})|_{\mathrm{Gr}^{\alpha^{\sharp}}_{\mathrm{G}}}$$

is independent of $\mathrm{T} \subset \mathrm{B} \subset \mathrm{G}$. This follows from comparing the $\mathrm{L}^+\mathrm{G}$-equivariance on $\mathcal{G}_{\mathrm{Gr}_{\mathrm{G}}^{\alpha^\sharp}}$ with the canonical G-equivariance on μ (*cf.* the proof of Proposition 3.1.14).

Next, let us show that the isomorphism (6.31) is independent of $\mathrm{T} \subset \mathrm{B} \subset \mathrm{G}$. Unwinding the definitions, this amounts to showing that the trivialization of the descent of $\tau^{\alpha^\sharp,0}$ to X, constructed in Lemma 6.3.6, is independent of these choices. For this, we may perform the same construction using the universal Borel and maximal torus over Torel and verify that the resulting trivialization of $\tau^{\alpha^\sharp,0}$ descends along Torel → X (*cf.* Remark 2.1.15). However, since the geometric fibers of Torel are simply connected, pullback along Torel → X induces an equivalence on the categories of A-torsors.

6.4. The isomorphism $\check{\mathrm{G}} \simeq \mathrm{H}$.

6.4.1. We remain in the setting where $\mathrm{I} = \{1\}$ and omit it from the notation. Recall the étale sheaf $\check{\mathrm{G}}$ of affine monoid e-schemes over X.

The goal of this subsection is to construct an isomorphism

$$\check{\mathrm{G}} \simeq \mathrm{H}. \tag{6.32}$$

This isomorphism, combined with Corollary 6.2.10, will allow us to complete the construction of the Satake equivalence (*cf.* §6.4.11).

6.4.2. Let us first note a "pointwise" version of (6.19). Indeed, given a geometric point $\bar{x}$ of X, the stalk of $\mathcal{S}at_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G}})$ at $\bar{x}$ is canonically identified with $\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\bar{x}})$ via the pullback functor (*cf.* Proposition 3.2.5).

From this fact and (6.19), we deduce the equivalence

$${}^+\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\bar{x}})_{-(\nu+\vartheta)} \simeq \mathsf{Rep}_{\check{\mathrm{G}}_{\bar{x}}}, \tag{6.33}$$

where $\check{\mathrm{G}}_{\bar{x}}$ is the fiber of $\check{\mathrm{G}}$ at $\bar{x}$ and the symmetric monoidal structure on the left-hand-side is induced from that of ${}^+\mathcal{S}at_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G}})_{-(\nu+\vartheta)}$.

We use (6.33) to translate properties of the pointwise Satake category to those of $\check{\mathrm{G}}$.

Lemma 6.4.3. *Sections of* $\check{\mathrm{G}}$ *are reductive group* e*-schemes.*

Proof. It suffices to prove that the geometric fibers of $\check{\mathrm{G}}$ are reductive group e-schemes.

Let $\bar{x}$ be a geometric point of X. The Satake category ${}^+\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\bar{x}})$ is rigid (*cf.* Proposition 1.6.4), showing that $\check{\mathrm{G}}_{\bar{x}}$ is grouplike.

To prove that $\check{\mathrm{G}}_{\bar{x}}$ is of finite type, we need to find a finite set of objects of $\mathsf{Rep}_{\check{\mathrm{G}}_{\bar{x}}}$ which generate it under sums, monoidal products, and subquotients (*cf.* [DM82, Proposition 2.20(b)]). For this, we may choose a set $\lambda_1, \cdots, \lambda_n$ of generators of $\Lambda^{\sharp,+}$ as a monoid. The images of $\mathrm{IC}^{\lambda_1}, \cdots, \mathrm{IC}^{\lambda_n}$ generate ${}^+\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\bar{x}})$ and are homogeneous under the $\hat{\mathrm{Z}}_{\mathrm{H}}$-grading (*cf.* the proof of Proposition 5.3.3). They thus define generators of $\mathsf{Rep}_{\check{\mathrm{G}}_{\bar{x}}}$ upon untwisting.

Finally, ${}^+\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\bar{x}})$ is semisimple (*cf.* Corollary 3.3.7), showing that $\check{\mathrm{G}}_{\bar{x}}$ is linearly reductive. To prove that $\check{\mathrm{G}}_{\bar{x}}$ is reductive, it remains to show that it is connected, but this holds by the criterion of [DM82, Corollary 2.22]. ☐

6.4.4 *Maximal torus***.** By construction of (6.19), the forgetful functor $\mathsf{Rep}_{\check{\mathrm{G}}} \to \mathsf{Lis}(\mathrm{X})$ factors through the forgetful functor $\mathsf{Rep}_{\mathrm{T}_{\mathrm{H}}}$ (*cf.* §6.1.11).

Taking Tannaka duals, we obtain a morphism

$$\mathrm{T}_{\mathrm{H}} \to \check{\mathrm{G}} \tag{6.34}$$

of locally constant étale sheaves of group e-schemes.

Lemma 6.4.5. *Sections of* (6.34) *are maximal tori.*

Proof. It again suffices to prove this assertion for geometric fibers.

Fix a geometric point $\bar{x}$ of X. By Proposition 6.3.9(1), to each $\lambda \in \Lambda^\sharp$, viewed as a character of $\mathrm{T}_{\mathrm{H},\bar{x}}$, there exists an object $\mathrm{V} \in \mathsf{Rep}_{\check{\mathrm{G}}_{\bar{x}}}$ whose image in $\mathsf{Rep}_{\mathrm{T}_{\mathrm{H},\bar{x}}}$ has a 1-dimensional λ-weight space. Namely, this V may be chosen as the image of e under Δ^{λ_1}, where we take λ_1 to be the dominant representative of λ. It follows that $\mathrm{T}_{\mathrm{H},\bar{x}} \to \check{\mathrm{G}}_{\bar{x}}$ is a closed immersion.

To prove that $\mathrm{T}_{\mathrm{H},\bar{x}}$ is a maximal torus of $\mathrm{G}_{\bar{x}}$, one may now proceed as in the classical argument (*cf.* [BR18, §9.1]). □

6.4.6 *Borel subgroup.* Recall that the dual group H of (G, μ), being pinned, has a canonical Borel subgroup B_{H} containing the maximal torus T_{H} (*cf.* §2.1).

Let $\check{\mathrm{B}} \subset \check{\mathrm{G}}$ be the subsheaf of Borel subgroups containing T_{H} which renders the character $2\check{\rho}$ dominant. It is uniquely characterized by the property that T_{H} has the same set of dominant weights with respect to $\check{\mathrm{B}} \subset \check{\mathrm{G}}$ and $\mathrm{B}_{\mathrm{H}} \subset \mathrm{H}$ (*cf.* [BR18, Lemma 9.5]).

Proposition 6.4.7. *The root data of* $\mathrm{T}_{\mathrm{H}} \subset \check{\mathrm{B}} \subset \check{\mathrm{G}}$ *and* $\mathrm{T}_{\mathrm{H}} \subset \mathrm{B}_{\mathrm{H}} \subset \mathrm{H}$ *coincide.*

Proof. We again pass to fibers at a geometric point $\bar{x}$ of X and view $\Lambda^\sharp$ as the character lattice of $\mathrm{T}_{\mathrm{H},\bar{x}}$. Furthermore, we fix a maximal torus and a Borel subgroup $\mathrm{T} \subset \mathrm{B} \subset \mathrm{G}$.

Let us first assume that G has a unique simple root $\check{\alpha}$. In this case, H admits a unique simple root $\alpha^\sharp = \operatorname{ord} \mathrm{Q}(\alpha)\cdot\alpha$. By Corollary 5.3.6 and Proposition 6.3.9(2) (and the agreement of standard and intersection cohomology functors, *cf.* Corollary 3.3.8), the weights occurring in a simple $\check{\mathrm{G}}_{\bar{x}}$-representation of highest weight λ_1 are precisely elements of the set

$$\{\lambda_1, \lambda_1 - \alpha^\sharp, \cdots, s_{\check{\alpha}}(\lambda_1)\}.$$

This shows that $\alpha^\sharp$ is the unique simple root of $\check{\mathrm{G}}_{\bar{x}}$. Since $\check{\mathrm{B}}_{\bar{x}}$ and $\mathrm{B}_{\mathrm{H},\bar{x}}$ determine the same subset of dominant elements of $\Lambda^\sharp$, the coroot of $\check{\mathrm{G}}_{\bar{x}}$ associated to $\alpha^\sharp$ must be proportional to $\check{\alpha}$, as they annihilate the same hyperplane in $\Lambda^\sharp \otimes \mathbf{Q}$. The proportion is $\operatorname{ord}(\mathrm{Q}(\alpha))^{-1}$, because $\langle \check{\alpha}^\sharp, \alpha^\sharp \rangle = 2$.

The general case reduces to the above case by compatibility with constant term functors. More precisely, each simple root $\check{\alpha}$ of G determines a subminimal parabolic P with Levi quotient $\mathrm{P} \twoheadrightarrow \mathrm{M}$. By construction of (6.19) and the composition property of constant term functors (*cf.* Remark 1.5.3), the forgetful functor $\mathsf{Rep}_{\check{\mathrm{G}}} \to \mathsf{Lis}(\mathrm{X})$ factors through $\mathsf{Rep}_{\check{\mathrm{M}}}$, where $\check{\mathrm{M}}$ is the Tannaka dual of the untwisted Satake category for M, equipped with the induced étale level. This yields a morphism

$$\check{\mathrm{M}} \to \check{\mathrm{G}} \tag{6.35}$$

of locally constant étale sheaves of group e-schemes.

Note that (6.35) is a closed immersion: This follows from the same argument as in the proof of Lemma 6.4.5 and boils down to the fact that the $\mathrm{L}^+_{\bar{x}}\mathrm{N}_{\mathrm{P}}$-orbit of ϖ^λ in $\mathrm{Gr}_{\mathrm{G},\bar{x}}$ (for N_{P} the nilradical of P) is an affine space.

Thus, $\alpha^\sharp$ is a root of $\check{\mathrm{G}}$, so the set of roots of H is contained in the set of roots of $\check{\mathrm{G}}$, preserving positivity. This containment is an equality because the weights of $\Delta^{\lambda_1}(\mathsf{e}) \in \mathsf{Rep}_{\check{\mathrm{G}}_{\bar{x}}}$, for any $\lambda_1 \in \Lambda^{\sharp,+}$, belong to $\lambda_1 - \mathbf{Z}^{\geq 0}\Delta^\sharp$ (*cf.* Corollary 5.3.6). The identification of coroots follows, as (6.35) carries root subgroups of $\check{\mathrm{M}}_{\bar{x}}$ to those of $\check{\mathrm{G}}_{\bar{x}}$. □

6.4.8 *Pinning***.** Next, we shall construct a pinning on $\check{G}$ under the assumption that G is split. Namely, we shall construct a canonical isomorphism

$$\mathbb{G}_a \simeq N_{\check{\alpha}^\sharp}, \tag{6.36}$$

where $N_{\check{\alpha}^\sharp}$ is the root subgroup of $\check{G}$ corresponding to $\check{\alpha}^\sharp$.

Indeed, consider the object $\Delta^{\alpha^\sharp}(\mathsf{e}) \in \mathsf{Rep}_{\check{G}}$. By Proposition 6.3.10 and Remark 6.3.11, the weight space $\omega^0 \circ \Delta^{\alpha^\sharp}(\mathsf{e})$ is canonically isomorphic to the constant local system e. Since $\omega^{\alpha^\sharp} \circ \Delta^{\alpha^\sharp}(\mathsf{e})$ is tautologically isomorphic to e, we have distinguished sections

$$v_{\alpha^\sharp} : \mathsf{e} \simeq \omega^{\alpha^\sharp} \circ \Delta^{\alpha^\sharp}(\mathsf{e}),$$
$$v_0 : \mathsf{e} \simeq \omega^0 \circ \Delta^{\alpha^\sharp}(\mathsf{e}).$$

Under the $\check{G}$-action on the underlying local system of $\Delta^{\alpha^\sharp}(\mathsf{e})$, the subgroup $N_{\check{\alpha}^\sharp}$ acts on the subspace spanned by $\{v_{\alpha^\sharp}, v_0\}$ in a strictly upper triangular manner. We obtain (6.36) as the unique isomorphism under which $a \in \mathbb{G}_a$ acts by $a \cdot v_0 = v_0 - 2av_\alpha$.

Remark 6.4.9. Our definition of the pinning on $\check{G}$ ensures that when G has semisimple rank 1, under the identification $(\check{G})_{\mathrm{ad}} \simeq \mathrm{PGL}_2$ induced from this pinning, the object $\Delta^{\alpha^\sharp}(\mathsf{e}) \in \mathsf{Rep}_{\check{G}}$ is canonically the pullback of the adjoint representation of PGL_2 along $\check{G} \to (\check{G})_{\mathrm{ad}}$.

6.4.10 *Construction of* (6.32)**.** When G is split, we have endowed $\check{G}$ with the canonical structure of a constant sheaf of pinned split reductive group e-schemes whose root data coincide with those of H (*cf.* Proposition 6.4.7). We obtain (6.32) as the unique pinning-preserving isomorphism.

For general G, we choose an étale cover of X splitting G, over which we have a pinning-preserving isomorphism $\check{G} \simeq H$. To see that this isomorphism intertwines the descent data of $\check{G}$ and H, we may pass to the corresponding isomorphism of their root data, which is induced from the identity automorphism of T_H (as a maximal torus of both $\check{G}$ and H). We thus obtain the desired isomorphism (6.32) by étale descent.

6.4.11. Finally, we complete the construction of the geometric Satake equivalence.

Proof of Theorem 2.3.2. Combining (6.19) and (6.32), we obtain an equivalence

$${}^{+}\mathcal{S}at_{\mathcal{G},\zeta}(\mathrm{Hec}_{G,I})_{-(\nu+\vartheta)^{\boxplus I}} \simeq \mathcal{R}ep_{H^{\boxtimes I}} \tag{6.37}$$

of étale sheaves of $(\hat{Z}_H)^{\oplus I}$-graded symmetric monoidal e-linear categories over X^I for any finite set I. Twisting both sides of (6.37) by $(\nu+\vartheta)^{\boxplus I}$ (*cf.* §A.2.7), we obtain an equivalence of étale sheaves of symmetric monoidal e-linear categories

$${}^{+}\mathcal{S}at_{\mathcal{G},\zeta}(\mathrm{Hec}_{G,I}) \simeq \mathcal{R}ep_{H^{\boxtimes I},(\nu+\vartheta)^{\boxplus I}}. \tag{6.38}$$

The desired equivalence (2.36) arises by taking sections of (6.38) over X^I. □

Part 2. Global function fields

7. Preparation

In this section, we begin our treatment of global function fields. We let k be a field and X be a smooth, proper, geometrically connected curve over k. Let $\mathrm{D} \subset \mathrm{X}$ be a k-finite closed subscheme (the "ramification divisor") and write $\mathring{\mathrm{X}} := \mathrm{X} \setminus \mathrm{D}$.

Let A be a finite abelian group whose order is invertible in k. Let G be a smooth affine group X-scheme equipped with an A-valued étale level *over* $\mathring{\mathrm{X}}$, *i.e.* a morphism of pointed étale stacks $\mu : \mathsf{B}_{\mathring{\mathrm{X}}}\mathrm{G} \to \mathsf{B}^4_{\mathring{\mathrm{X}}}\mathrm{A}(1)$.

We begin by defining the "global A-gerbe" in §7.1. Then we explain a trace-of-Frobenius construction for gerbes (*cf.* §7.2) and use it to define ζ-genuine automorphic forms (*cf.* §7.3). These constructions are mild generalizations of standard ones in the literature (*cf.* [Lys06]), where we treat ramification and general A-gerbes. In §7.4, we prove a novel result, Theorem 7.4.6, which gives a geometric interpretation of Weissman's meta-Galois group in the function field setting.

7.1. The global A-gerbe.

7.1.1. Denote by $\mathrm{Bun}_{\mathrm{G},\mathrm{D}}$ the moduli stack of G-bundles over X rigidified along D, *i.e.* the prestack assigning to a k-algebra R the groupoid of pairs (P, ϕ), where P is a G-bundles over $\mathrm{X}_{\mathrm{R}} := \operatorname{Spec}\mathrm{R} \times \mathrm{X}$ and ϕ is a trivialization of the restriction of P to D_{R}.

By [Hei10, Proposition 1], $\mathrm{Bun}_{\mathrm{G},\varnothing}$ is representable by a smooth algebraic k-stack. The same then holds for $\mathrm{Bun}_{\mathrm{G},\mathrm{D}}$, since it is a torsor over $\mathrm{Bun}_{\mathrm{G},\varnothing}$ with structure group the Weil restriction of G along $\mathrm{D} \to \operatorname{Spec}\mathsf{k}$.

7.1.2. By convention, we shall write Ran for the Ran space *of* $\mathring{\mathrm{X}}$ (*cf.* §1.1.2).

Denote by $\mathrm{Hec}(\mathrm{Bun}_{\mathrm{G},\mathrm{D}})$ the global Hecke stack, *i.e.* an R-point of $\mathrm{Hec}(\mathrm{Bun}_{\mathrm{G},\mathrm{D}})$ consists of an R-point $\underline{x}$ of Ran, a pair of R-points (P^0, ϕ^0), (P^1, ϕ^1) of $\mathrm{Bun}_{\mathrm{G},\mathrm{D}}$ together with an isomorphism over $\mathrm{X}_{\mathrm{R}} \setminus \Gamma_{\underline{x}}$. This makes sense because D_{R} is contained in $\mathrm{X}_{\mathrm{R}} \setminus \Gamma_{\underline{x}}$.

We shall represent an R-point of $\mathrm{Hec}(\mathrm{Bun}_{\mathrm{G},\mathrm{D}})$ as:

$$(\mathrm{P}^0, \phi^0) \overset{\underline{x}}{\sim} (\mathrm{P}^1, \phi^1)$$

and refer to it as a *modification*, in parallel with §1.1.3.

7.1.3. More generally, we may define an "outer convolution diagram" version of the global Hecke stack as follows.

Let $n \geq 0$ be an integer. Denote by $\mathrm{Hec}^{[n]}(\mathrm{Bun}_{\mathrm{G},\mathrm{D}})$ the moduli stack parametrizing n points $\underline{x}_1, \cdots, \underline{x}_n$ of Ran, together with a chain of modifications

$$(\mathrm{P}^0, \phi^0) \overset{\underline{x}_1}{\sim} (\mathrm{P}^1, \phi^1) \overset{\underline{x}_2}{\sim} \cdots \overset{\underline{x}_n}{\sim} (\mathrm{P}^n, \phi^n). \tag{7.1}$$

By construction, we have structural morphisms

$$\begin{tikzcd}
\mathrm{Hec}^{[n]}(\mathrm{Bun}_{\mathrm{G},\mathrm{D}}) \arrow[r, "\prod r_j"] \arrow[d, "(p_0, p_n)"] & \prod_{j=1}^n \mathrm{Hec}_{\mathrm{G}} \\
\mathrm{Bun}_{\mathrm{G},\mathrm{D}} \times \mathrm{Bun}_{\mathrm{G},\mathrm{D}} &
\end{tikzcd} \tag{7.2}$$

where p_0, p_n send (7.1) to (P^0, ϕ^0), respectively (P^n, ϕ^n), and each r_j ($j = 1, \cdots, n$) sends it to the restriction of the modification $\mathrm{P}^{j-1} \overset{\underline{x}_j}{\sim} \mathrm{P}^j$ to $\mathrm{D}_{\underline{x}_j}$.

7.1.4. In parallel with §1.1.7, we shall construct a morphism of spaces

$$\int_{\mathrm{X}} : \mathrm{Maps}_*(\mathsf{B}_{\mathring{\mathrm{X}}}\mathrm{G}, \mathsf{B}^4_{\mathring{\mathrm{X}}}\mathrm{A}(1)) \to \mathrm{Maps}(\mathrm{Bun}_{\mathrm{G,D}}, \mathsf{B}^2\mathrm{A}). \tag{7.3}$$

Then we will set $\mathcal{G}_{\mathrm{Bun}_{\mathrm{G,D}}}$ to be the image of μ under (7.3). We shall refer to $\mathcal{G}_{\mathrm{Bun}_{\mathrm{G,D}}}$ as the *global* A*-gerbe* associated to the étale level μ.

Remark 7.1.5. Note that A-gerbes over $\mathrm{Bun}_{\mathrm{G,D}}$ are canonically equivalent to those over $\mathrm{Bun}_{\mathrm{G,D_{red}}}$, where $\mathrm{D_{red}} \subset \mathrm{D}$ is the reduced subscheme.

One may view this fact as saying that the our covers are "tamely ramified". We shall see another manifestation of this fact in §7.3.6.

7.1.6 *The global trace map***.** Given a k-algebra R, we shall construct a morphism in the pro-category of H**Z**-module spectra

$$\mathrm{tr}_{\mathrm{X}} : \Gamma(\mathrm{X_R}, \hat{\mathbf{Z}}(1)[2]) \to \Gamma(\mathrm{Spec\,R}, \hat{\mathbf{Z}}). \tag{7.4}$$

Indeed, write $\pi : \mathrm{X_R} \to \mathrm{Spec\,R}$ for the projection morphism. Since π is smooth of relative dimension 1, we may identify $\hat{\mathbf{Z}}(1)[2]$ with $\pi^!\hat{\mathbf{Z}}$. Since π is also proper, we may use the co-unit of the adjunction to form the composition

$$\begin{aligned}\Gamma(\mathrm{X_R}, \hat{\mathbf{Z}}(1)[2]) &\simeq \Gamma(\mathrm{X_R}, \pi^!\hat{\mathbf{Z}}) \\ &\simeq \Gamma(\mathrm{Spec\,R}, \pi_*\pi^!\hat{\mathbf{Z}}) \to \Gamma(\mathrm{Spec\,R}, \hat{\mathbf{Z}}),\end{aligned}$$

which is our definition of (7.4).

Remark 7.1.7. In parallel with Remark 1.1.9, we comment on the behavior of (7.4) with respect to change of coefficients: Given any complex $\mathcal{A}$ of torsion étale sheaves of invertible order, the same construction yields a morphism

$$\mathrm{tr}_{\mathrm{X}} : \Gamma(\mathrm{X_R}, \mathcal{A}(1)[2]) \to \Gamma(\mathrm{Spec\,R}, \mathcal{A}), \tag{7.5}$$

which is related to (7.4) by the commutative diagram

$$\begin{tikzcd}
\Gamma(\mathrm{Spec\,R}, \mathcal{A}) \otimes \Gamma(\mathrm{X_R}, \hat{\mathbf{Z}}(1)[2]) \arrow[r, "\otimes"] \arrow[d, "\mathrm{id}\otimes\mathrm{tr}_{\mathrm{X}}"] & \Gamma(\mathrm{X_R}, \mathcal{A}(1)[2]) \arrow[d, "\mathrm{tr}_{\mathrm{X}}"] \\
\Gamma(\mathrm{Spec\,R}, \mathcal{A}) \otimes \Gamma(\mathrm{Spec\,R}, \hat{\mathbf{Z}}) \arrow[r, "\otimes"] & \Gamma(\mathrm{Spec\,R}, \mathcal{A})
\end{tikzcd}$$

7.1.8 *Construction of* (7.3)**.** Using the cohomological interpretation of $\mathsf{B}^n\mathrm{A}$ (*cf.* §0.4.3), it suffices to construct a morphism of H**Z**-module spectra

$$\Gamma(\mathsf{B}_{\mathring{\mathrm{X}}}\mathrm{G} \bmod \mathring{\mathrm{X}}, \mathrm{A}(1)[4]) \to \Gamma(\mathrm{Bun}_{\mathrm{G,D}}, \mathrm{A}[2]). \tag{7.6}$$

Since the closed immersion $\mathrm{D} \subset \mathrm{X}$ is cohomologically pure, the associated Cousin sequence yields a fiber sequence of H**Z**-module spectra

$$\Gamma(\mathsf{B}_{\mathrm{D}}\mathrm{G} \bmod \mathrm{D}, \mathrm{A}[2]) \to \Gamma(\mathsf{B}_{\mathrm{X}}\mathrm{G} \bmod \mathrm{X}, \mathrm{A}(1)[4]) \to \Gamma(\mathsf{B}_{\mathring{\mathrm{X}}}\mathrm{G} \bmod \mathring{\mathrm{X}}, \mathrm{A}(1)[4]). \tag{7.7}$$

To construct (7.6), it thus suffices to construct a morphism

$$\Gamma(\mathsf{B}_{\mathrm{X}}\mathrm{G}, \mathrm{A}(1)[4]) \to \Gamma(\mathrm{Bun}_{\mathrm{G,D}}, \mathrm{A}[2]) \tag{7.8}$$

and trivialize its restriction to $\Gamma(\mathsf{B}_{\mathrm{D}}\mathrm{G} \bmod \mathrm{D}, \mathrm{A}[2])$.

Let P denote the universal G-bundle over $\mathrm{X} \times \mathrm{Bun}_{\mathrm{G,D}}$, which we view as a morphism $\mathrm{X} \times \mathrm{Bun}_{\mathrm{G,D}} \to \mathsf{B}_{\mathrm{X}}\mathrm{G}$. We define (7.8) as the composition $\mathrm{tr}_{\mathrm{X}} \circ \mathrm{P}^*$, where tr_{X} is the global trace map (7.5) (for $\mathcal{A} = \mathrm{A}[2]$). To trivialize $\mathrm{tr}_{\mathrm{X}} \circ \mathrm{P}^*$ over $\Gamma(\mathsf{B}_{\mathrm{D}}\mathrm{G} \text{ mod } \mathrm{D}, \mathrm{A}[2])$, we use the commutative square of $\mathrm{H}\mathbf{Z}$-module spectra

$$\begin{tikzcd}
\Gamma(\mathsf{B}_{\mathrm{D}}\mathrm{G}, \mathrm{A}[2]) \arrow[r] \arrow[d, "(\mathrm{P}|_{\mathrm{D}})^*"] & \Gamma(\mathsf{B}_{\mathrm{X}}\mathrm{G}, \mathrm{A}(1)[4]) \arrow[d, "\mathrm{P}^*"] \\
\Gamma(\mathrm{D} \times \mathrm{Bun}_{\mathrm{G},\mathrm{D}}, \mathrm{A}[2]) \arrow[r] & \Gamma(\mathrm{X} \times \mathrm{Bun}_{\mathrm{G},\mathrm{D}}, \mathrm{A}(1)[4])
\end{tikzcd}$$

where the horizontal arrows are given by cohomological purity. Note that $(\mathrm{P}|_{\mathrm{D}})^*$ is trivial over $\Gamma(\mathsf{B}_{\mathrm{D}}\mathrm{G} \text{ mod } \mathrm{D}, \mathrm{A}[2])$ since $\mathrm{P}|_{\mathrm{D}}$ factors through D via ϕ, as desired.

This concludes the construction of (7.3), hence the global A-gerbe $\mathcal{G}_{\mathrm{Bun}_{\mathrm{G},\mathrm{D}}}$.

Remark 7.1.9. For $\mathrm{D} = \varnothing$, the morphism (7.3) is simply the morphism of spaces induced from (7.8). In particular, it factors through $\mathrm{Maps}(\mathsf{B}_{\mathrm{X}}\mathrm{G}, \mathsf{B}^4_{\mathrm{X}}\mathrm{A}(1))$, *i.e.* the rigidification of μ at the neutral point is irrelevant for the construction.

On the other hand, the construction of $\mathcal{G}_{\mathrm{Bun}_{\mathrm{G},\mathrm{D}}}$ shows that it is canonically trivial along the neutral k-point of $\mathrm{Bun}_{\mathrm{G},\mathrm{D}}$, *i.e.* the pair (P, ϕ) where P is the trivial G-bundle over X and ϕ is the identity.

7.1.10. Next, we relate $\mathcal{G}_{\mathrm{Bun}_{\mathrm{G},\mathrm{D}}}$ to the local A-gerbe $\mathcal{G}_{\mathrm{Hec}_{\mathrm{G}}}$ (*cf.* §1.1.7) via the global Hecke action diagram (7.2) for a fixed integer $n \geq 0$.

Lemma 7.1.11. *There is a canonical isomorphism of* A*-gerbes over* $\mathrm{Hec}^{[n]}(\mathrm{Bun}_{\mathrm{G},\mathrm{D}})$*:*

$$(p_n)^*\mathcal{G}_{\mathrm{Bun}_{\mathrm{G},\mathrm{D}}} - (p_0)^*\mathcal{G}_{\mathrm{Bun}_{\mathrm{G},\mathrm{D}}} \simeq \sum_{j=1}^{n} (r_j)^*\mathcal{G}_{\mathrm{Hec}_{\mathrm{G}}}. \tag{7.9}$$

Proof. For $n = 0$, (7.9) is the tautological trivialization of $(p_0)^*\mathcal{G}_{\mathrm{Bun}_{\mathrm{G},\mathrm{D}}} - (p_0)^*\mathcal{G}_{\mathrm{Bun}_{\mathrm{G},\mathrm{D}}}$.

For $n \geq 1$, we may reduce to the case $n = 1$ by rewriting the left-hand-side of (7.9) as a telescopic sum. Then we need to prove the following: Given G-bundles P^0, P^1 over X_{R} with an isomorphism over $\mathrm{X}_{\mathrm{R}} \smallsetminus \Gamma_{\underline{x}}$, where $\underline{x}$ is an R-point of Ran, the image of

$$\begin{aligned}
\mu(\mathrm{P}^1) - \mu(\mathrm{P}^0) &\in \Gamma(\mathring{\mathrm{X}}_{\mathrm{R}} \text{ mod } \mathring{\mathrm{X}}_{\mathrm{R}} \smallsetminus \Gamma_{\underline{x}}, \mathsf{B}^4\mathrm{A}(1)) \\
&\simeq \Gamma(\mathrm{X}_{\mathrm{R}} \text{ mod } \mathrm{X}_{\mathrm{R}} \smallsetminus \Gamma_{\underline{x}}, \mathsf{B}^4\mathrm{A}(1))
\end{aligned}$$

under the global trace map (7.5) coincides with the image of

$$\mu(\mathrm{P}^1|_{\mathrm{D}_{\underline{x}}}) - \mu(\mathrm{P}^0|_{\mathrm{D}_{\underline{x}}}) \in \Gamma(\mathrm{D}_{\underline{x}} \text{ mod } \mathring{\mathrm{D}}_{\underline{x}}, \mathsf{B}^4\mathrm{A}(1))$$

under the local trace map (1.10).

This follows by identifying the morphism of $\mathrm{H}\mathbf{Z}$-module spectra

$$\begin{aligned}
\Gamma(\mathrm{X}_{\mathrm{R}} \text{ mod } \mathrm{X}_{\mathrm{R}} \smallsetminus \Gamma_{\underline{x}}, \mathrm{A}(1)[4]) &\xrightarrow{\simeq} \Gamma(\mathrm{D}_{\underline{x}} \text{ mod } \mathring{\mathrm{D}}_{\underline{x}}, \mathrm{A}(1)[4]) \\
&\xrightarrow{\mathrm{tr}_{\underline{x}}} \Gamma(\mathrm{Spec}\,\mathrm{R}, \mathrm{A}[2])
\end{aligned} \tag{7.10}$$

with the restriction of the global trace map, where the isomorphism is given by formal base change (*cf.* §1.1.8). Note, however, that the composition (7.10) is defined by adjunction for the proper morphism $x_{\mathrm{R}} \to \mathrm{Spec}\,\mathrm{R}$, so the desired identification follows from naturality with respect to the morphism $x_{\mathrm{R}} \to \mathrm{X}_{\mathrm{R}}$ of proper R-schemes. □

7.1.12 *Variant for* **∞D.** Let us note a variant of the construction of $\mathcal{G}_{\mathrm{Bun}_{\mathrm{G},\mathrm{D}}}$ at "infinite level", where it is enough to start with a group scheme over $\mathring{\mathrm{X}}$.

More precisely, we let G be a smooth affine group $\mathring{\mathrm{X}}$-scheme. Denote by $\mathrm{Bun}_{\mathrm{G},\infty\mathrm{D}}$ the moduli stack of pairs (P,ϕ), where P is a G-bundle over $\mathring{\mathrm{X}}_{\mathrm{R}} := \mathring{\mathrm{X}} \times \operatorname{Spec}\mathrm{R}$ and ϕ is a trivialization of P over $\hat{\mathrm{D}}_{\mathrm{R}} \smallsetminus \mathrm{D}_{\mathrm{R}}$, for $\hat{\mathrm{D}}_{\mathrm{R}}$ the formal completion of X_{R} along D_{R}. (Here, R is a test k-algebra.)

There is a morphism of spaces

$$\int_{\mathrm{X}} : \mathrm{Maps}(\mathsf{B}_{\mathring{\mathrm{X}}}\mathrm{G}, \mathsf{B}^4_{\mathring{\mathrm{X}}}\mathrm{A}(1)) \to \mathrm{Maps}(\mathrm{Bun}_{\mathrm{G},\infty\mathrm{D}}, \mathsf{B}^2\mathrm{A}), \tag{7.11}$$

constructed in analogy with (7.3): We use the fact that any section of $\mathsf{B}^4\mathrm{A}(1)$ over $\mathring{\mathrm{X}}_{\mathrm{R}}$ trivialized over $\hat{\mathrm{D}}_{\mathrm{R}} \smallsetminus \mathrm{D}_{\mathrm{R}}$ admits a canonical extension to X_{R} and then apply the global trace map (7.5) (for $\mathcal{A} := \mathrm{A}[2]$). Given an étale level μ of G, we thus obtain an A-gerbe $\mathcal{G}_{\mathrm{Bun}_{\mathrm{G},\infty\mathrm{D}}}$ over $\mathrm{Bun}_{\mathrm{G},\infty\mathrm{D}}$ as its image under (7.11).

Remark 7.1.13. Let G be a smooth affine group X-scheme. By the Beauville–Laszlo lemma (*cf.* [BL95]), an R-point of $\mathrm{Bun}_{\mathrm{G},\infty\mathrm{D}}$ is equivalent to a G-bundle over X_{R} trivialized over $\hat{\mathrm{D}}_{\mathrm{R}}$. In particular, we have a canonical morphism

$$\mathrm{Bun}_{\mathrm{G},\infty\mathrm{D}} \to \mathrm{Bun}_{\mathrm{G},\mathrm{D}}$$

under which $\mathcal{G}_{\mathrm{Bun}_{\mathrm{G},\mathrm{D}}}$ pulls back to $\mathcal{G}_{\mathrm{Bun}_{\mathrm{G},\infty\mathrm{D}}}$.

7.2. Interlude: trace of Frobenius.

7.2.1. We now suppose that k is a finite field of cardinality q. For a k-prestack $\mathcal{X}$, we write $\mathrm{Fr}_{\mathcal{X}} : \mathcal{X} \to \mathcal{X}$ for the qth power absolute Frobenius endomorphism, *i.e.* it sends $x \in \mathcal{X}(\mathrm{R})$ to $x \circ \mathrm{Fr}_{\mathrm{R}}$, where Fr_{R} is induced from the ring map $\mathrm{R} \to \mathrm{R}$, $a \mapsto a^q$.

We shall describe a "trace-of-Frobenius" construction for A-gerbes, which takes as input a k-prestack $\mathcal{X}$ equipped with an A-gerbe $\mathcal{G}$, and gives as output an A-torsor $\mathrm{Tr}(\mathrm{Fr} \mid \mathcal{G})$ over the fixed-point prestack $\mathcal{X}^{\mathrm{Fr}}$, *i.e.* the fiber product

$$\begin{tikzcd}
\mathcal{X}^{\mathrm{Fr}} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "\Delta"] \\
\mathcal{X} \arrow[r, "(\mathrm{id}, \mathrm{Fr}_{\mathcal{X}})"] & \mathcal{X} \times \mathcal{X}
\end{tikzcd} \tag{7.12}$$

Remark 7.2.2. Let $\mathcal{X}$ be an algebraic k-stack locally of finite type. Regarding the groupoid $\mathcal{X}(\mathsf{k})$ as a discrete k-stack, we obtain a natural morphism of k-stacks

$$\mathcal{X}(\mathsf{k}) \to \mathcal{X}^{\mathrm{Fr}}. \tag{7.13}$$

By [Var04, Lemma 3.3], $\mathcal{X}^{\mathrm{Fr}}$ is an étale Deligne–Mumford k-stack and the morphism (7.13) is an open and closed immersion; it is an isomorphism if $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ has connected geometric fibers. Furthermore, by the proof of [Var04, Proposition 2.16(c)] (*cf.* [Laf18, §12.3.2]), (7.13) is an isomorphism for $\mathcal{X} := \mathrm{Bun}_{\mathrm{G},\mathrm{D}}$, where G is a smooth group X-scheme with connected geometric fibers, which is reductive over $\mathring{\mathrm{X}}$.

7.2.3. Recall that for any k-scheme S, the endofunctor $\mathrm{Fr}^*_{\mathrm{S}}$ on the étale site of S is naturally isomorphic to the identity (*cf.* [Sta18, 03SN]).

Let us explicitly describe the value of this natural isomorphism at an A-gerbe $\mathcal{G}$:

$$\mathrm{Fr}^*_{\mathrm{S}}\,\mathcal{G} \simeq \mathcal{G}. \tag{7.14}$$

For any étale morphism $f : \mathrm{S}_1 \to \mathrm{S}$, the groupoid $\mathrm{Fr}^*_{\mathrm{S}}(\mathcal{G})(\mathrm{S}_1)$ is the filtered colimit of $\mathcal{G}(\mathrm{U})$ over étale morphisms $u : \mathrm{U} \to \mathrm{S}$ through which $\mathrm{Fr}_{\mathrm{S}} \circ f$ factors. This index category has an initial object, namely $(\mathrm{U}, u) = (\mathrm{S}_1, f)$ with the factorization $\mathrm{Fr}_{\mathrm{S}} \circ f = f \circ \mathrm{Fr}_{\mathrm{S}_1}$. The colimit is thus identified with $\mathcal{G}(\mathrm{S}_1)$.

Since the isomorphism (7.14) is natural in S, we obtain a natural isomorphism

$$\mathrm{Fr}_{\mathcal{X}}^* \mathcal{G} \simeq \mathcal{G} \tag{7.15}$$

for any k-prestack $\mathcal{X}$.

7.2.4 *Construction of* $\mathrm{Tr}(\mathrm{Fr} \mid \mathcal{G})$**.** Let $\mathcal{X}$ be a k-prestack equipped with an A-gerbe $\mathcal{G}$. Since $\mathrm{Fr}_{\mathcal{X}}$ restricts to the identity map on $\mathcal{X}^{\mathrm{Fr}}$, we obtain an isomorphism

$$\mathrm{Fr}_{\mathcal{X}}^*(\mathcal{G}|_{\mathcal{X}^{\mathrm{Fr}}}) \simeq \mathcal{G}|_{\mathcal{X}^{\mathrm{Fr}}}. \tag{7.16}$$

The A-gerbe $\mathrm{Tr}(\mathrm{Fr} \mid \mathcal{G})$ is defined as the difference between the restriction of (7.15) to $\mathcal{X}^{\mathrm{Fr}}$ and (7.16), *i.e.* it corresponds to the composition

$$\mathcal{G}|_{\mathcal{X}^{\mathrm{Fr}}} \xrightarrow{(7.16)} \mathrm{Fr}_{\mathcal{X}}^*(\mathcal{G}|_{\mathcal{X}^{\mathrm{Fr}}}) \simeq (\mathrm{Fr}_{\mathcal{X}}^* \mathcal{G})|_{\mathcal{X}^{\mathrm{Fr}}} \xrightarrow{(7.14)} \mathcal{G}|_{\mathcal{X}^{\mathrm{Fr}}}. \tag{7.17}$$

7.2.5. We now fix an algebraic k-stack $\mathcal{X}$ locally of finite type such that $\mathcal{X}(\mathsf{k}) \to \mathcal{X}^{\mathrm{Fr}}$ is an isomorphism (*cf.* Remark 7.2.2). Let $\mathcal{G}$ be an A-gerbe over $\mathcal{X}$.

Taking k-points of $\mathrm{Tr}(\mathrm{Fr} \mid \mathcal{G})$, we obtain a morphism of groupoids

$$\widetilde{\mathcal{X}} := \mathrm{Tr}(\mathrm{Fr} \mid \mathcal{G})(\mathsf{k}) \to \mathcal{X}^{\mathrm{Fr}}(\mathsf{k}) \simeq \mathcal{X}(\mathsf{k}). \tag{7.18}$$

Note that (7.18) is surjective: By functoriality of the construction, it suffices to show that for any k-point x of $\mathcal{X}$, the A-torsor $\mathrm{Tr}(\mathrm{Fr} \mid \mathcal{G}|_x)$ over x is trivial, but this holds because $\mathcal{G}|_x$ is trivial, as $\mathrm{H}^2_{\text{ét}}(\mathrm{Spec}\,\mathsf{k}, \mathrm{A}) \simeq 0$. It follows that (7.18) is a set-theoretic A-torsor.

Remark 7.2.6. Let us give a more concrete description of (7.18). Given $x \in \mathcal{X}(\mathsf{k})$, we choose an algebraic closure $\bar{\mathsf{k}}$ of k and a trivialization $\bar{g}$ of $\mathcal{G}$ over $\bar{x} := \mathrm{Spec}\,\bar{\mathsf{k}}$. The resulting fundamental group of $\mathcal{G}|_x$ fits into a short exact sequence (*cf.* [Wei18, Theorem 19.6])

$$1 \to \mathrm{A} \to \pi_1^{\text{ét}}(\mathcal{G}|_x, \bar{g}) \to \pi_1^{\text{ét}}(x, \bar{x}) \to 1. \tag{7.19}$$

The fiber of (7.18) at x, as an A-torsor, is identified with the preimage of the *geometric* Frobenius element under the surjection of (7.19). To see this, it suffices to observe that the composition (7.17) (for $\mathcal{X} := \mathrm{Spec}\,\mathsf{k}$) is *inverse* to the Frobenius-pullback on $\bar{\mathsf{k}}$-points.

Remark 7.2.7. Let k_1 be a finite extension of k of cardinality q_1 and $\mathcal{X}_1$ be a k_1-prestack. Write $\mathcal{X}$ for the Weil restriction of $\mathcal{X}_1$ along $\mathrm{Spec}\,\mathsf{k}_1 \to \mathrm{Spec}\,\mathsf{k}$. There is a *norm* map:

$$\mathrm{Nm} : \mathrm{Maps}(\mathcal{X}_1, \mathsf{B}^2\mathrm{A}) \to \mathrm{Maps}(\mathcal{X}, \mathsf{B}^2\mathrm{A}). \tag{7.20}$$

Indeed, by adjunction, it suffices to construct a natural map of $\mathrm{H}\mathbf{Z}$-module spectra

$$\Gamma(\mathrm{Spec}\,\mathrm{R} \otimes_{\mathsf{k}} \mathsf{k}_1, \mathrm{A}) \to \Gamma(\mathrm{Spec}\,\mathrm{R}, \mathrm{A}) \tag{7.21}$$

for any k-algebra R. This is given by the trace map on étale cochains, using the fact that $\mathrm{Spec}\,\mathsf{k}_1 \to \mathrm{Spec}\,\mathsf{k}$ is finite étale.

The norm map (7.20) is compatible with the trace-of-Frobenius construction in the following sense: For $(\mathcal{X}_1, \mathcal{G}_1)$ as in §7.2.5, the A-torsor $\mathrm{Tr}(\mathrm{Fr} \mid \mathcal{G})(\mathsf{k}_1)$ over $\mathcal{X}_1(\mathsf{k}_1)$, formed with respect to the q_1th power Frobenius, coincides with $\mathrm{Tr}(\mathrm{Fr} \mid \mathrm{Nm}\,\mathcal{G})$ over $\mathcal{X}(\mathsf{k})$, formed with respect to the qth power Frobenius, under the natural identification $\mathcal{X}(\mathsf{k}_1) \simeq \mathcal{X}(\mathsf{k})$.

7.2.8. Given the coefficient data of §1.3.1, we write $\mathrm{Fun}_\zeta(\widetilde{\mathcal{X}}, \mathsf{e})$ for the e-vector space of ζ-*genuine* functions on $\widetilde{\mathcal{X}}$, *i.e.* functions $f : \widetilde{\mathcal{X}} \to \mathsf{e}$ satisfying

$$f(x \cdot a) = f(x) \cdot \zeta(a)$$

for each $x \in \widetilde{\mathcal{X}}$ and $a \in \mathrm{A}$. We denote by $\mathrm{Fun}_{c,\zeta}(\widetilde{\mathcal{X}}, \mathsf{e})$ the subspace of compactly supported ζ-genuine functions on $\widetilde{\mathcal{X}}$.

Denote by $\mathcal{L}_{\mathrm{Tr}(\mathrm{Fr}|\mathcal{G}),\zeta}$ the 1-dimensional e-local system over $\mathcal{X}^{\mathrm{Fr}}$ induced from $\mathrm{Tr}(\mathrm{Fr} \mid \mathcal{G})$ along ζ. Since $\mathcal{X}^{\mathrm{Fr}}$ is identified with $\mathcal{X}(\mathsf{k})$, we have a canonical isomorphism

$$\mathrm{Fun}_\zeta(\widetilde{\mathcal{X}}, \mathsf{e}) \simeq \mathrm{H}^0(\mathcal{X}^{\mathrm{Fr}}, (\mathcal{L}_{\mathrm{Tr}(\mathrm{Fr}|\mathcal{G}),\zeta})^{\otimes -1}). \tag{7.22}$$

Here, the inverse appears because sections of $(\mathcal{L}_{\tau,\zeta})^{\otimes -1}$, for a trivial A-torsor τ over $\mathrm{Spec}\, \mathsf{k}$, are canonically identified with ζ-genuine functions $\tau(\mathsf{k}) \to \mathsf{e}$.

Moreover, any $\mathcal{A} \in \mathsf{D}_{\mathcal{G},\zeta}(\mathcal{X})$ (*cf.* §B.1.3) induces a ζ-genuine function $\mathrm{Tr}(\mathrm{Fr} \mid \mathcal{A})$ on $\widetilde{\mathcal{X}}$ as follows: For each $x \in \mathcal{X}(\mathsf{k})$, a choice of an algebraic closure $\bar{\mathsf{k}}$ of k and a trivialization $\bar{g}$ of $\mathcal{G}$ over $\bar{x} \coloneqq \mathrm{Spec}\, \bar{\mathsf{k}}$ defines a ζ-genuine representation of $\pi_1^{\text{ét}}(\mathcal{G}|_x, \bar{g})$ on the constructible complex of e-vector spaces $\mathcal{A}_{\bar{x}}$, whose trace defines $\mathrm{Tr}(\mathrm{Fr} \mid \mathcal{A})$ over x (*cf.* Remark 7.2.6).

7.3. Automorphic forms.

7.3.1. We continue to assume k to be a finite field of cardinality q and invoke the coefficient data of §1.3.1. Instead of e, we shall work with an algebraic closure $\overline{\mathbf{Q}}_\ell$ of $\mathbf{Q}_\ell$ and define "ζ-genuine automorphic forms" using the trace-of-Frobenius construction of A-gerbes.

Assume that $\mathrm{G} \to \mathrm{X}$ has connected geometric fibers and is reductive over $\mathring{\mathrm{X}}$. Then $(\mathrm{Bun}_{\mathrm{G},\mathrm{D}})^{\mathrm{Fr}}$ coincides with the discrete stack $\mathrm{Bun}_{\mathrm{G},\mathrm{D}}(\mathsf{k})$ (*cf.* Remark 7.2.2). Applying the trace-of-Frobenius construction to the global A-gerbe $\mathcal{G}_{\mathrm{Bun}_{\mathrm{G},\mathrm{D}}}$, we obtain an A-torsor

$$\widetilde{\mathrm{Bun}}_{\mathrm{G},\mathrm{D}} \coloneqq \mathrm{Tr}(\mathrm{Fr} \mid \mathcal{G}_{\mathrm{Bun}_{\mathrm{G},\mathrm{D}}})(\mathsf{k}) \tag{7.23}$$

over $\mathrm{Bun}_{\mathrm{G},\mathrm{D}}(\mathsf{k})$. We shall refer to ζ-genuine functions over $\widetilde{\mathrm{Bun}}_{\mathrm{G},\mathrm{D}}$ as ζ-*genuine automorphic forms* (with ramification divisor D). They form the $\overline{\mathbf{Q}}_\ell$-vector space

$$\mathrm{Fun}_\zeta(\widetilde{\mathrm{Bun}}_{\mathrm{G},\mathrm{D}}, \overline{\mathbf{Q}}_\ell). \tag{7.24}$$

Remark 7.3.2. Let us note a variant of $\widetilde{\mathrm{Bun}}_{\mathrm{G},\mathrm{D}}$ at "infinite level", for any smooth affine group $\mathring{\mathrm{X}}$-scheme G (*cf.* §7.1.12).

Namely, restricting $\mathrm{Tr}(\mathrm{Fr} \mid \mathcal{G}_{\mathrm{Bun}_{\mathrm{G},\infty\mathrm{D}}})(\mathsf{k})$ along $\mathrm{Bun}_{\mathrm{G},\infty\mathrm{D}}(\mathsf{k}) \subset (\mathrm{Bun}_{\mathrm{G},\infty\mathrm{D}})^{\mathrm{Fr}}(\mathsf{k})$, we obtain a set-theoretic k-torsor $\widetilde{\mathrm{Bun}}_{\mathrm{G},\infty\mathrm{D}}$ over $\mathrm{Bun}_{\mathrm{G},\infty\mathrm{D}}(\mathsf{k})$.

7.3.3. The A-torsor (7.23) may be expressed in an adèlic form, as follows.

Denote by F the field of fractions of X. For each closed point $x \in \mathrm{X}$, we write F_x for the local field and $\mathcal{O}_x$ for its ring of integers. Denote by $\mathbb{A}_{\mathrm{F}}$ (respectively, $\mathbb{O}_{\mathrm{F}}$) the topological ring of (respectively, integral) adèles of F. By Weil uniformization, we have a fully faithful embedding of groupoids

$$\mathrm{G}(\mathrm{F})\backslash\mathrm{G}(\mathbb{A}_{\mathrm{F}})/\mathrm{K}_{\mathrm{D}} \subset \mathrm{Bun}_{\mathrm{G},\mathrm{D}}(\mathsf{k}), \tag{7.25}$$

where $\mathrm{K}_{\mathrm{D}} \subset \mathrm{G}(\mathbb{O}_{\mathrm{F}})$ is the kernel of $\mathrm{G}(\mathbb{O}_{\mathrm{F}}) \to \mathrm{G}(\mathcal{O}_{\mathrm{D}})$.

Let us describe the pullback of $\widetilde{\mathrm{Bun}}_{\mathrm{G},\mathrm{D}}$ along (7.25) in terms of a cover of $\mathrm{G}(\mathbb{A}_{\mathrm{F}})$ equipped with canonical sections over $\mathrm{G}(\mathrm{F})$ and over K_{D}. To do so, we need a mild generalization of the construction of $\mathcal{G}_{\mathrm{Hec}_{\mathrm{G}}}$ (*cf.* §1.1.7). Indeed, we shall construct a morphism

$$\int_{\mathring{\mathrm{D}}_x} : \mathrm{Maps}_*(\mathsf{B}_{\mathrm{F}_x}\mathrm{G}, \mathsf{B}^4_{\mathrm{F}_x}\mathrm{A}(1)) \to \mathrm{Maps}_{\mathbb{E}_1}(\mathrm{L}_x\mathrm{G}, \mathsf{B}^2\mathrm{A}), \tag{7.26}$$

where $\mathsf{B}_{\mathrm{F}_x}$ denotes the deloop functor on the big étale site of $\operatorname{Spec}\mathrm{F}_x$, and $\mathrm{L}_x\mathrm{G}$ is the loop group associated to the k_x-point $x : \operatorname{Spec}\mathsf{k}_x \to \mathrm{X}$ (*cf.* §1.1.3).

7.3.4 *Construction of* (7.26)**.** For any k_x-algebra R, we shall construct a morphism

$$\operatorname{Maps}_*(\mathsf{B}_{\mathrm{F}_x}\mathrm{G}, \mathsf{B}^4_{\mathrm{F}_x}\mathrm{A}(1)) \to \operatorname{Maps}_{\mathbb{E}_1}(\mathrm{G}(\mathring{\mathrm{D}}_{x_{\mathrm{R}}}), \Gamma(\operatorname{Spec}\mathrm{R}, \mathsf{B}^2\mathrm{A})) \tag{7.27}$$

natural in R. Here, x_{R} denotes the R-point of X defined by x.

By rewriting the left-hand-side of (7.27) as $\operatorname{Maps}_{\mathbb{E}_1}(\mathrm{G}_{\mathrm{F}_x}, \mathsf{B}^3_{\mathrm{F}_x}\mathrm{A}(1))$, we shall obtain (7.27) as the composition of the evaluation over $\mathring{\mathrm{D}}_{x_{\mathrm{R}}}$, followed by the morphism of $\mathbb{E}_1$-monoids induced from some morphism of $\mathbf{H}\mathbf{Z}$-module spectra

$$\Gamma(\mathring{\mathrm{D}}_{x_{\mathrm{R}}}, \mathrm{A}(1)[3]) \to \Gamma(\operatorname{Spec}\mathrm{R}, \mathrm{A}[2]). \tag{7.28}$$

The morphism (7.28) is the composition of the second map in the fiber sequence

$$\Gamma(\mathrm{D}_{x_{\mathrm{R}}}, \mathrm{A}(1)[3]) \to \Gamma(\mathring{\mathrm{D}}_{x_{\mathrm{R}}}, \mathrm{A}(1)[3]) \to \Gamma(\mathrm{D}_{x_{\mathrm{R}}} \bmod \mathring{\mathrm{D}}_{x_{\mathrm{R}}}, \mathrm{A}(1)[4]) \tag{7.29}$$

with the local trace map (1.10) (for $\mathcal{A} := \mathrm{A}[2]$).

Remark 7.3.5. Given a pointed morphism $\mathsf{B}_{\mathrm{F}_x}\mathrm{G} \to \mathsf{B}^4_{\mathrm{F}_x}\mathrm{A}(1)$ equipped with an extension to $\operatorname{Spec}\mathcal{O}_x$, its image under (7.26) admits a canonical trivialization over $\mathrm{L}^+_x\mathrm{G}$. Indeed, this follows from the fiber sequence (7.29).

In this situation, we obtain an A-gerbe over $\operatorname{Hec}_{\mathrm{G},x} \simeq \mathrm{L}^+_x\mathrm{G}\backslash\mathrm{L}_x\mathrm{G}/\mathrm{L}^+_x\mathrm{G}$. This coincides with the (pullback of the) A-gerbe $\mathcal{G}_{\operatorname{Hec}_{\mathrm{G}}}$ constructed in §1.1.7.

7.3.6. For each closed point $x \in \mathrm{X}$, we may restrict the étale level μ to $\operatorname{Spec}\mathrm{F}_x$ and apply (7.26) to obtain a monoidal A-gerbe $\mathcal{G}_{\mathrm{L}_x\mathrm{G}}$ over $\mathrm{L}_x\mathrm{G}$.

Taking trace-of-Frobenius of $\mathcal{G}_{\mathrm{L}_x\mathrm{G}}$ (with respect to the q_xth power Frobenius, for q_x the cardinality of k_x), we obtain a central extension

$$1 \to \mathrm{A} \to \widetilde{\mathrm{G}}_x \to \mathrm{G}(\mathrm{F}_x) \to 1 \tag{7.30}$$

canonically split over $\mathrm{G}(\mathrm{X} \smallsetminus x)$ by the sum-of-residue formula.

Since the principal congruence subgroup $\mathrm{L}^{\geq 1}_x\mathrm{G} \subset \mathrm{L}^+_x\mathrm{G}$ is pro-unipotent, any monoidal A-gerbe over $\mathrm{L}^{\geq 1}_x\mathrm{G}$ is canonically trivial. This implies that (7.30) admits a canonical section over $\ker(\mathrm{G}(\mathcal{O}_x) \to \mathrm{G}(\mathsf{k}_x))$. On the other hand, if $x \in \mathring{\mathrm{X}}$, then this section extends to $\mathrm{G}(\mathcal{O}_x)$ by Remark 7.3.5.

Repeating the construction of [Zha22, §2.2], we obtain from (7.30) a central extension of the adèlic group

$$1 \to \mathrm{A} \to \widetilde{\mathrm{G}} \to \mathrm{G}(\mathbb{A}_{\mathrm{F}}) \to 1 \tag{7.31}$$

equipped with canonical sections over $\mathrm{G}(\mathrm{F})$ and over K_{D}. In particular, it gives rise to an A-torsor $\mathrm{G}(\mathrm{F})\backslash\widetilde{\mathrm{G}}/\mathrm{K}_{\mathrm{D}}$ over $\mathrm{G}(\mathrm{F})\backslash\mathrm{G}(\mathbb{A}_{\mathrm{F}})/\mathrm{K}_{\mathrm{D}}$.

Remark 7.3.7. Note that [Zha22, §2] supplies an *a priori* different construction of the local covering group (7.30), and consequently another global covering group (7.31). However, these constructions coincide for an appropriate normalization of the Tate duality isomorphism $\mathrm{H}^2_{\text{ét}}(\operatorname{Spec}\mathrm{F}_x, \mathrm{A}(1)) \simeq \mathrm{A}$. Namely, we exhibit it as the composition

$$\begin{aligned}\mathrm{H}^2_{\text{ét}}(\operatorname{Spec}\mathrm{F}_x, \mathrm{A}(1)) &\simeq \mathrm{H}^1_{\text{ét}}(\operatorname{Spec}\mathsf{k}_x, \mathrm{A}) \\ &\simeq \operatorname{Hom}(\operatorname{Gal}(\bar{\mathsf{k}}_x/\mathsf{k}_x), \mathrm{A}) \simeq \mathrm{A},\end{aligned}$$

where the last map is evaluation on the *geometric* Frobenius.

Lemma 7.3.8. *The* A*-torsor* $\mathrm{G(F)}\backslash\widetilde{\mathrm{G}}/\mathrm{K_D}$ *over* $\mathrm{G(F)}\backslash\mathrm{G}(\mathbb{A}_\mathrm{F})/\mathrm{K_D}$ *is canonically identified with the restriction of* $\widetilde{\mathrm{Bun}}_{\mathrm{G,D}}$ *along* (7.25)*.*

Proof. For each $x \in \mathrm{X}$, consider the Weil restriction $\mathrm{res}_{\mathsf{k}}(\mathrm{L}_x\mathrm{G})$ of the loop group $\mathrm{L}_x\mathrm{G}$ along $\mathrm{Spec}\,\mathsf{k}_x \to \mathrm{Spec}\,\mathsf{k}$. Beauville–Laszlo gluing yields a map

$$\mathrm{res}_{\mathsf{k}}(\mathrm{L}_x\mathrm{G}) \to \mathrm{Bun}_{\mathrm{G,D}}, \tag{7.32}$$

which recovers the map $\mathrm{G}(\mathrm{F}_x) \to \mathrm{Bun}_{\mathrm{G,D}}(\mathsf{k})$ in Weil uniformization upon taking k-points.

By Remark 7.2.7, it suffices to identify the pullback of $\mathcal{G}_{\mathrm{Bun}_{\mathrm{G,D}}}$ along (7.32) with the norm of the A-gerbe $\mathcal{G}_{\mathrm{L}_x\mathrm{G}}$, compatibly with the canonical sections. Unwinding the definitions, this amounts to identifying the composition

$$\begin{aligned}\Gamma(\mathrm{X_R} \bmod \mathrm{X_R} \smallsetminus x_\mathrm{R}, \mathrm{A}(1)[4]) &\xrightarrow{\cong} \Gamma(\mathrm{D}_{x_\mathrm{R}} \bmod \mathring{\mathrm{D}}_{x_\mathrm{R}}, \mathrm{A}(1)[4]) \\ &\xrightarrow{\mathrm{tr}_{x_\mathrm{R}}} \Gamma(\mathrm{Spec}\,\mathrm{R}\otimes_{\mathsf{k}}\mathsf{k}_x, \mathrm{A}[2]) \xrightarrow{(7.21)} \Gamma(\mathrm{Spec}\,\mathrm{R}, \mathrm{A}[2])\end{aligned}$$

with the restriction of the global trace map (7.5) (for $\mathcal{A} = \mathrm{A}[2]$), where x_R denotes the $(\mathrm{R}\otimes_{\mathsf{k}}\mathsf{k}_x)$-point of X defined by x. As in the proof of Lemma 7.1.11, this follows from naturality with respect to the morphism of proper R-schemes $x_\mathrm{R} \to \mathrm{X_R}$. We omit verifying the compatibilities with canonical sections. ☐

Remark 7.3.9. The essential image of the Weil uniformization map (7.25) consists of pairs (P, ϕ), where P is trivial over the generic point $\mathrm{Spec}\,\mathrm{F}$ of X. To describe the remaining points of $\mathrm{Bun}_{\mathrm{G,D}}(\mathsf{k})$, we need to consider inner forms of $\mathrm{G_F}$.

Define the set

$$\ker^1(\mathrm{F},\mathrm{G}) := \ker(\mathrm{H}^1_{\text{ét}}(\mathrm{Spec}\,\mathrm{F},\mathrm{G}) \to \prod_{x\in\mathrm{X}} \mathrm{H}^1_{\text{ét}}(\mathrm{Spec}\,\mathrm{F}_x,\mathrm{G})),$$

where the product is taken over closed points of X. By [Laf18, Remark 12.2], $\ker^1(\mathrm{F},\mathrm{G})$ is finite and, after choosing a G-bundle P_α over $\mathrm{Spec}\,\mathrm{F}$ trivialized over $\mathrm{Spec}\,\mathrm{F}_x$ ($x \in \mathrm{X}$) representing each $\alpha \in \ker^1(\mathrm{F},\mathrm{G})$, the collection of Weil uniformization maps for all $\mathrm{G}_\alpha := \mathrm{Aut}(\mathrm{P}_\alpha)$ defines an equivalence of groupoids

$$\bigsqcup_{\alpha\in\ker^1(\mathrm{F},\mathrm{G})} \mathrm{G}_\alpha(\mathrm{F})\backslash\mathrm{G}_\alpha(\mathbb{A}_\mathrm{F})/\mathrm{K_D} \simeq \mathrm{Bun}_{\mathrm{G,D}}(\mathsf{k}). \tag{7.33}$$

Here, we implicitly extended P_α (hence G_α) to X using its trivializations over $\mathrm{Spec}\,\mathrm{F}_x$.

On the other hand, the étale level μ of G induces an étale level μ_α of G_α, thanks to its G-equivariance structure (*cf.* §2.1.11). Repeating the construction of (7.31) with $(\mathrm{G}_\alpha, \mu_\alpha)$ instead of (G, μ), we obtain a central extension $\widetilde{\mathrm{G}}_\alpha$ of $\mathrm{G}_\alpha(\mathbb{A}_\mathrm{F})$ by A, and by Lemma 7.3.8, an identification of A-torsors over (7.33):

$$\bigsqcup_{\alpha\in\ker^1(\mathrm{F},\mathrm{G})} \mathrm{G}_\alpha(\mathrm{F})\backslash\widetilde{\mathrm{G}}_\alpha/\mathrm{K_D} \simeq \widetilde{\mathrm{Bun}}_{\mathrm{G,D}}. \tag{7.34}$$

7.3.10 *The "sharp center".* In order to obtain some finiteness, we need to restrict the action of the connected component Z° of the center Z of G over $\mathring{\mathrm{X}}$. In the twisted setting, the role of the connected center is played by a torus isogenic to Z°.

To wit, denote by $\Lambda_{\mathrm{Z}^\circ}$ the sheaf of cocharacters of Z° and set

$$\Lambda_{\mathrm{Z}^\sharp} := \Lambda_{\mathrm{Z}^\circ} \cap \Lambda^\sharp \subset \Lambda,$$

where $\Lambda^\sharp$ is the kernel of the symmetric form b (*cf.* §2.1.6). Then the inclusion $\Lambda_{\mathrm{Z}^\sharp} \subset \Lambda_{\mathrm{Z}^\circ}$ corresponds to an isogeny of tori $\mathrm{Z}^\sharp \to \mathrm{Z}^\circ$. Since the étale level μ is $\mathbb{E}_\infty$-monoidal over $\mathrm{Z}^\sharp$

(*cf.* [Zha22, Proposition 4.6.2]), the associated cover $\widetilde{\mathrm{Z}}^\sharp$ of $\mathrm{Z}^\sharp(\mathbb{A}_\mathrm{F})$ is commutative. Moreover, the "infinite level" construction (*cf.* Remark 7.3.2) applies to $\mathrm{Z}^\sharp$ and yields a symmetric monoidal extension

$$\mathrm{A} \to \widetilde{\mathrm{Bun}}_{\mathrm{Z}^\sharp,\infty\mathrm{D}} \to \mathrm{Bun}_{\mathrm{Z}^\sharp,\infty\mathrm{D}}(\mathsf{k}).$$

There is a natural $\mathrm{Bun}_{\mathrm{Z}^\sharp,\infty\mathrm{D}}$-action on $\mathrm{Bun}_{\mathrm{G},\mathrm{D}}$ defined by the natural $\mathsf{B}\mathrm{Z}$-action and Beauville–Laszlo gluing. More precisely, an R-point $(\mathrm{P}_{\mathrm{Z}^\sharp}, \phi_{\mathrm{Z}^\sharp})$ of $\mathrm{Bun}_{\mathrm{Z}^\sharp,\infty\mathrm{D}}$ carries $(\mathrm{P}, \phi) \in \mathrm{Bun}_{\mathrm{G},\mathrm{D}}$ to the pair (P', ϕ'), where P' is the gluing of $\mathrm{P} \otimes \mathrm{P}_{\mathrm{Z}^\sharp}$ (defined over $\mathrm{X}_\mathrm{R} \smallsetminus \mathrm{D}_\mathrm{R}$) with $\mathrm{P}|_{\hat{\mathrm{D}}_\mathrm{R}}$ along the isomorphism

$$\mathrm{id}_\mathrm{P} \otimes \phi_{\mathrm{Z}^\sharp} : \mathrm{P} \otimes \mathrm{P}_{\mathrm{Z}^\sharp}|_{\hat{\mathrm{D}}_\mathrm{R} \smallsetminus \mathrm{D}_\mathrm{R}} \simeq \mathrm{P}|_{\hat{\mathrm{D}}_\mathrm{R} \smallsetminus \mathrm{D}_\mathrm{R}},$$

and ϕ' is the trivialization of $\mathrm{P}'|_{\mathrm{D}_\mathrm{R}}$ corresponding to ϕ.

Since μ is $\mathsf{B}\mathrm{Z}^\sharp$-equivariant (*cf.* §3.1.6), the $\mathrm{Bun}_{\mathrm{Z}^\sharp,\infty\mathrm{D}}(\mathsf{k})$-action on $\mathrm{Bun}_{\mathrm{G},\mathrm{D}}$ lifts to an action of $\widetilde{\mathrm{Bun}}_{\mathrm{Z}^\sharp,\infty\mathrm{D}}$-action on $\widetilde{\mathrm{Bun}}_{\mathrm{G},\mathrm{D}}$, compatibly with the natural A-action.

7.3.11. We shall fix a cocompact lattice $\Xi \subset \mathrm{Z}^\sharp(\mathrm{F})\backslash\widetilde{\mathrm{Z}}^\sharp$ which maps isomorphically onto its image in $\mathrm{Bun}_{\mathrm{Z}^\sharp,\infty\mathrm{D}}(\mathsf{k})$.

Using the $\widetilde{\mathrm{Bun}}_{\mathrm{Z}^\sharp,\infty\mathrm{D}}$-action on $\widetilde{\mathrm{Bun}}_{\mathrm{G},\mathrm{D}}$, we may consider Ξ-invariant ζ-genuine automorphic forms, *i.e.* the $\overline{\mathbf{Q}}_\ell$-vector space

$$\mathrm{Fun}_\zeta(\widetilde{\mathrm{Bun}}_{\mathrm{G},\mathrm{D}}/\Xi, \overline{\mathbf{Q}}_\ell). \tag{7.35}$$

Next, we shall define the subspace of (7.35) consisting of "cusp forms".

7.3.12 *Cusp forms***.** Note that the central extension (7.31) splits canonically over $\mathrm{N}(\mathbb{A}_\mathrm{F}) \subset \mathrm{G}(\mathbb{A}_\mathrm{F})$, where $\mathrm{N} \subset \mathrm{G}$ is any unipotent subgroup.

We shall call a ζ-genuine function $f : \mathrm{G}(\mathrm{F})\backslash\widetilde{\mathrm{G}} \to \overline{\mathbf{Q}}_\ell$ *cuspidal* if for any $\tilde{g} \in \widetilde{\mathrm{G}}$ and the unipotent radical N of any proper parabolic subgroup of G_F, there holds

$$\int_{\mathrm{N}(\mathrm{F})\backslash\mathrm{N}(\mathbb{A}_\mathrm{F})} f(n\tilde{g})dn = 0,$$

with respect to some Haar measure dn over $\mathrm{N}(\mathbb{A}_\mathrm{F})$.

We likewise have the notion of cuspidality of ζ-genuine functions over $\mathrm{G}_\alpha(\mathrm{F})\backslash\widetilde{\mathrm{G}}_\alpha$ for each $\alpha \in \ker^1(\mathrm{F}, \mathrm{G})$ (*cf.* Remark 7.3.9). An element f of (7.35) is called a *cusp form* if its restriction to each $\mathrm{G}_\alpha(\mathrm{F})\backslash\widetilde{\mathrm{G}}_\alpha$ satisfies the cuspidality condition.

In other words, the space of ζ-genuine cusp forms is given by

$$\mathrm{Fun}_{\mathrm{cusp},\zeta}(\widetilde{\mathrm{Bun}}_{\mathrm{G},\mathrm{D}}/\Xi, \overline{\mathbf{Q}}_\ell) \simeq \bigoplus_{\alpha\in\ker^1(\mathrm{F},\mathrm{G})} \mathrm{Fun}_{\mathrm{cusp},\zeta}(\mathrm{G}_\alpha(\mathrm{F})\backslash\widetilde{\mathrm{G}}_\alpha/\mathrm{K}_\mathrm{D}\Xi, \overline{\mathbf{Q}}_\ell). \tag{7.36}$$

Lemma 7.3.13. *The* $\overline{\mathbf{Q}}_\ell$*-vector space* (7.36) *is finite-dimensional.*

Proof. Since $\ker^1(\mathrm{F}, \mathrm{G})$ is finite, it suffices to show that each summand in (7.36) is finite-dimensional. However, by the proof of [Har74, Corollary 1.2.3], cuspidal ζ-genuine functions over $\mathrm{G}_\alpha(\mathrm{F})\backslash\widetilde{\mathrm{G}}_\alpha$ are uniformly supported on a compact subset. □

Remark 7.3.14. Instead of choosing Ξ in the definition of ζ-genuine cusp forms (7.36), we may fix a ζ-genuine character $\omega : \widetilde{\mathrm{Bun}}_{\mathrm{Z}^\sharp,\infty\mathrm{D}} \to \overline{\mathbf{Q}}_\ell^\times$ and consider the $\overline{\mathbf{Q}}_\ell$-vector space

$$\mathrm{Fun}^\omega_{\mathrm{cusp},\zeta}(\widetilde{\mathrm{Bun}}_{\mathrm{G},\mathrm{D}}, \overline{\mathbf{Q}}_\ell) \tag{7.37}$$

of ζ-genuine cusp forms which are $\widetilde{\mathrm{Bun}}_{\mathrm{Z}^\sharp,\infty\mathrm{D}}$-equivariant against ω.

An analogue of Lemma 7.3.13 asserts that (7.37) is finite-dimensional. In fact, all our considerations concerning (7.36) apply to (7.37) with minor modifications.

7.3.15 *The spherical Hecke algebra.* As the final topic of this subsection, we define the spherical Hecke algebra and its action on the space of ζ-genuine automorphic forms.

Let x be a closed point of $\mathring{\mathrm{X}}$. Recall that the étale level μ defines a multiplicative A-gerbe $\mathcal{G}_{\mathrm{Hec}_{\mathrm{G},x}}$ over the local Hecke stack $\mathrm{Hec}_{\mathrm{G},x}$ (*cf.* §1.1.7). Its trace-of-Frobenius yields the groupoid $\mathrm{G}(\mathcal{O}_x)\backslash\widetilde{\mathrm{G}}_x/\mathrm{G}(\mathcal{O}_x)$ over $\mathrm{Hec}_{\mathrm{G},x}(\mathsf{k})$, where $\widetilde{\mathrm{G}}_x$ is the cover of $\mathrm{G}(\mathrm{F}_x)$ associated to μ (*cf.* Remark 7.3.5).[17] The induced multiplicative structure on $\mathrm{G}(\mathcal{O}_x)\backslash\widetilde{\mathrm{G}}_x/\mathrm{G}(\mathcal{O}_x)$ coincides with the one defined by the group structure on $\widetilde{\mathrm{G}}_x$ its splitting over $\mathrm{G}(\mathcal{O}_x)$.

In particular, the $\overline{\mathbf{Q}}_\ell$-vector space

$$\mathrm{Fun}_{c,\zeta}(\mathrm{G}(\mathcal{O}_x)\backslash\widetilde{\mathrm{G}}_x/\mathrm{G}(\mathcal{O}_x), \overline{\mathbf{Q}}_\ell) \tag{7.38}$$

of compactly supported ζ-genuine functions on $\mathrm{G}(\mathcal{O}_x)\backslash\widetilde{\mathrm{G}}_x/\mathrm{G}(\mathcal{O}_x)$ admits an algebra structure by convolution. We shall refer to (7.38) as the *spherical Hecke algebra* at x.

7.3.16. We shall obtain distinguished elements of (7.38) using the geometric Satake equivalence (2.39) (for $\mathrm{I} = \{1\}$). We fix a square root $q^{1/2} \in \overline{\mathbf{Q}}_\ell$ which determines a half-integral Tate twist $\overline{\mathbf{Q}}_\ell(\frac{1}{2})$.

Indeed, denote by ${}^{\mathrm{L}}\mathrm{H}_x$ the *unramified local* L*-group* at x, *i.e.* the base change of the L-group (2.33) along $\pi_1^{\text{ét}}(x, \bar{x}) \to \pi_1^{\text{ét}}(\mathrm{X}, \bar{x})$ (where we assume that $\bar{x}$ lies over x). The stalk of (2.39) at x (*cf.* Proposition 3.2.5) yields a *monoidal* equivalence

$$\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},x}) \simeq \mathsf{Rep}({}^{\mathrm{L}}\mathrm{H}_x). \tag{7.39}$$

Thus, to each $\mathrm{V} \in \mathsf{Rep}({}^{\mathrm{L}}\mathrm{H}_x)$ corresponds an object $\mathcal{S}_{\mathrm{V}}$ of $\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},x})$.

Taking Grothendieck groups and applying trace-of-Frobenius of twisted constructible complexes (*cf.* §7.2.8), we obtain a map of $\overline{\mathbf{Q}}_\ell$-algebras

$$\begin{aligned} \mathrm{K}_0(\mathsf{Rep}({}^{\mathrm{L}}\mathrm{H}_x)) &\simeq \mathrm{K}_0(\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},x})) \\ &\to \mathrm{Fun}_{c,\zeta}(\mathrm{G}(\mathcal{O}_x)\backslash\widetilde{\mathrm{G}}_x/\mathrm{G}(\mathcal{O}_x), \overline{\mathbf{Q}}_\ell) \\ \mathrm{V} &\mapsto h_{\mathrm{V},x} := \mathrm{Tr}(\mathrm{Fr} \mid \mathcal{S}_{\mathrm{V}}). \end{aligned} \tag{7.40}$$

Remark 7.3.17. Denote by ${}^{\mathrm{L}}\mathrm{H}_x^1$ the fiber of ${}^{\mathrm{L}}\mathrm{H}_x \to \pi_1^{\text{ét}}(x, \bar{x})$ at the geometric Frobenius element Fr_x. It admits the natual structure of an affine variety over $\overline{\mathbf{Q}}_\ell$ and an H-action by conjugation. Consider the $\overline{\mathbf{Q}}_\ell$-algebra of H-invariant algebraic functions on it:

$$\Gamma({}^{\mathrm{L}}\mathrm{H}_x^1/\!/\mathrm{H}, \mathcal{O}) := \Gamma({}^{\mathrm{L}}\mathrm{H}_x^1, \mathcal{O})^{\mathrm{H}}.$$

It receives a map from $\mathrm{K}_0(\mathsf{Rep}({}^{\mathrm{L}}\mathrm{H}_x))$, mapping V to the character of its restriction to ${}^{\mathrm{L}}\mathrm{H}_x^1$.

By [Zhu17, §5.6], (7.40) factors through an isomorphism of $\overline{\mathbf{Q}}_\ell$-algebras

$$\Gamma({}^{\mathrm{L}}\mathrm{H}_x^1/\!/\mathrm{H}, \mathcal{O}) \simeq \mathrm{Fun}_{c,\zeta}(\mathrm{G}(\mathcal{O}_x)\backslash\widetilde{\mathrm{G}}_x/\mathrm{G}(\mathcal{O}_x), \overline{\mathbf{Q}}_\ell), \tag{7.41}$$

which is our version of the *classical* Satake isomorphism.

We note that a version of the classical Satake isomorphism for covering groups (over any nonarchimedean local field) has been obtained by McNamara (*cf.* [McN12]). In the remainder of this article, we will not use the isomorphism (7.41).

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[17] By Remark 1.1.5, $\mathrm{Hec}_{\mathrm{G},x}$ admits an ind-presentation by limits of algebraic stacks $\mathcal{X}$ such that $\mathcal{X}(\mathsf{k})$ is canonically isomorphic to $\mathcal{X}^{\mathrm{Fr}}$ (*cf.* Remark 7.2.2).

7.3.18 *Hecke action.* The spherical Hecke algebra (7.38) acts on the $\overline{\mathbf{Q}}_\ell$-vector space of ζ-genuine automorphic forms (7.24).

Indeed, by Lemma 7.1.11 (for $n = 1$), the trace-of-Frobenius of $(p_1)^*\mathcal{G}_{\mathrm{Bun}_{\mathrm{G},\mathrm{D}}}$ over the global Hecke stack at x may be identified with the groupoid

$$\widetilde{\mathrm{Bun}}_{\mathrm{G},\mathrm{D}+\infty x}\times^{\mathrm{G}(\mathcal{O}_x)\times\mathrm{A}}(\mathrm{G}(\mathcal{O}_x)\backslash\widetilde{\mathrm{G}}_x/\mathrm{G}(\mathcal{O}_x)),$$

where $\widetilde{\mathrm{Bun}}_{\mathrm{G},\mathrm{D}+\infty x}$ is the pullback of $\widetilde{\mathrm{Bun}}_{\mathrm{G},\mathrm{D}}$, and the superscript $\mathrm{G}(\mathcal{O}_x)\times\mathrm{A}$ means quotient by the anti-diagonal action. It admits structural morphisms p_0, p_1, r:

$$\begin{tikzcd}
\widetilde{\mathrm{Bun}}_{\mathrm{G},\mathrm{D}+\infty x}\times^{\mathrm{G}(\mathcal{O}_x)\times\mathrm{A}}(\mathrm{G}(\mathcal{O}_x)\backslash\widetilde{\mathrm{G}}_x/\mathrm{G}(\mathcal{O}_x)) \arrow[r, "r"] \arrow[d, "{(p_0,p_1)}"] & \mathrm{G}(\mathcal{O}_x)\backslash\widetilde{\mathrm{G}}_x/\mathrm{G}(\mathcal{O}_x) \\
\widetilde{\mathrm{Bun}}_{\mathrm{G},\mathrm{D}}\times\widetilde{\mathrm{Bun}}_{\mathrm{G},\mathrm{D}} &
\end{tikzcd} \tag{7.42}$$

where p_0, r are projections onto the first, respectively second factor, and p_1 is the multiplication map. Integral transform along (7.42) thus defines an action

$$\mathrm{Fun}_\zeta(\widetilde{\mathrm{Bun}}_{\mathrm{G},\mathrm{D}},\overline{\mathbf{Q}}_\ell)\circlearrowleft\mathrm{Fun}_{c,\zeta}(\mathrm{G}(\mathcal{O}_x)\backslash\widetilde{\mathrm{G}}_x/\mathrm{G}(\mathcal{O}_x),\overline{\mathbf{Q}}_\ell) \tag{7.43}$$
$$f\star h := (p_1)_!(p_0^*f\cdot r^*h),$$

where upper-$*$ indicates pullback and lower-$*$ indicates summation along fibers.

The action (7.43) preserves the subspace of Ξ-invariant automorphic forms (7.35), as it commutes with the $\widetilde{\mathrm{Bun}}_{\mathrm{Z}^\sharp,\infty\mathrm{D}}$-action. It also preserves the subspace of cusp forms (7.36), by compatibility with the constant term functors (*cf.* §1.5).

7.4. Meta-Galois group *vs.* ϑ-characteristics.

7.4.1. As the final topic before we tackle the spectral decomposition of (7.36), we shall prove a result that "explains" the appearance of Weissman's meta-Galois twist in the formation of the L-group (*cf.* [Wei18, §4]).

This relies on results of §4.2, together with a categorical analogue of geometric class field theory, which we shall presently explain.

7.4.2. We fix a geometric point $\bar{\eta}$ lying over the generic point $\eta := \operatorname{Spec}\mathrm{F}$ of X. We also assume that D is nonempty, so $\mathrm{Bun}_{\mathbb{G}_m,\infty\mathrm{D}}$ is a k-scheme.

The Artin reciprocity map is an isomorphism of topological abelian groups:

$$\mathrm{Art}:\pi_1(\mathring{\mathrm{X}},\bar{\eta})^{\mathrm{ab}}\xrightarrow{\simeq}\mathrm{Bun}_{\mathbb{G}_m,\infty\mathrm{D}}(\mathsf{k})^{\mathrm{profin}}, \tag{7.44}$$

where the target denotes the profinite completion of $\mathrm{Bun}_{\mathbb{G}_m,\infty\mathrm{D}}(\mathsf{k})$. The isomorphism (7.44) is normalized so that the geometric Frobenius element $\varphi_x\in\mathrm{Gal}(\bar{\mathsf{k}}_x/\mathsf{k}_x)$, for each closed point $x\in\mathring{\mathrm{X}}$, maps to $\mathcal{O}(x)$.

On the other hand, consider the Abel–Jacobi morphism:

$$\mathrm{AJ}:\mathring{\mathrm{X}}\to\mathrm{Bun}_{\mathbb{G}_m,\infty\mathrm{D}} \tag{7.45}$$

sending an R-point x to $\mathcal{O}(x)$, equipped with its canonical trivialization over $\hat{\mathrm{D}}_{\mathrm{R}}$. The Abel–Jacobi map (7.45) recovers (7.44) via trace-of-Frobenius as follows: Given a multiplicative A-torsor τ over $\mathrm{Bun}_{\mathbb{G}_m,\infty\mathrm{D}}$, defining a character $\mathrm{Tr}(\mathrm{Fr}\mid\tau):\mathrm{Bun}_{\mathbb{G}_m,\infty\mathrm{D}}(\mathsf{k})\to\mathrm{A}$ by the classical trace-of-Frobenius construction, the character $\mathrm{Tr}(\mathrm{Fr}\mid\tau)\circ\mathrm{Art}$ coincides with the character of $\pi_1(\mathring{\mathrm{X}},\bar{\eta})$ associated to the A-torsor $\mathrm{AJ}^*(\tau)$ over $\mathring{\mathrm{X}}$.

7.4.3. We shall prove a categorical analogue of the aforementioned compatibility.

Denote by $\mathrm{CExt}(\pi_1^{\text{ét}}(\mathring{\mathrm{X}},\bar\eta),\mathrm{A})$ the groupoid of central extensions of $\pi_1^{\text{ét}}(\mathring{\mathrm{X}},\bar\eta)$ by A, so the map sending an A-gerbe $\mathcal{G}$ over $\mathring{\mathrm{X}}$ to the étale fundamental group of $\mathcal{G} - (\mathcal{G}|_{\bar\eta})$ (*cf.* [Wei18, Theorem 19.6]), where $\mathcal{G}|_{\bar\eta}$ is viewed as a constant A-gerbe over $\mathring{\mathrm{X}}$, defines a map

$$\mathrm{Maps}(\mathring{\mathrm{X}},\mathsf{B}^2\mathrm{A}) \to \mathrm{CExt}(\pi_1(\mathring{\mathrm{X}},\bar\eta),\mathrm{A}). \tag{7.46}$$

Proposition 7.4.4. *The following diagram canonically commutes:*

$$\begin{tikzcd}
\mathrm{Maps}_{\mathbb{E}_\infty}(\mathrm{Bun}_{\mathbb{G}_m,\infty\mathrm{D}},\mathsf{B}^2\mathrm{A}) \arrow[r, "\mathrm{AJ}^*"] \arrow[d, "\mathrm{Tr}(\mathrm{Fr}|\cdot)(\mathsf{k})"] & \mathrm{Maps}(\mathring{\mathrm{X}},\mathsf{B}^2\mathrm{A}) \arrow[d, "(7.46)"] \\
\mathrm{Maps}_{\mathbb{E}_\infty}(\mathrm{Bun}_{\mathbb{G}_m,\infty\mathrm{D}}(\mathsf{k}),\mathsf{B}\mathrm{A}) \arrow[r, "\mathrm{Art}^*"] & \mathrm{CExt}(\pi_1(\mathring{\mathrm{X}},\bar\eta),\mathrm{A})
\end{tikzcd} \tag{7.47}$$

Proof. Recall that the containment

$$\mathrm{Maps}_{\mathbf{Z}}(\mathrm{Bun}_{\mathbb{G}_m,\infty\mathrm{D}},\mathsf{B}^2\mathrm{A}) \subset \mathrm{Maps}_{\mathbb{E}_\infty}(\mathrm{Bun}_{\mathbb{G}_m,\infty\mathrm{D}},\mathsf{B}^2\mathrm{A})$$

admits a retract, sending an $\mathbb{E}_\infty$-monoidal morphism to a $\mathbf{Z}$-linear morphism with the same underlying $\mathbb{E}_1$-monoidal morphism (*cf.* §2.2.5).

Note that both $\mathrm{Tr}(\mathrm{Fr}\mid\cdot)(\mathsf{k})$ and AJ^* factor through this retract: This is obvious for AJ^*, while for $\mathrm{Tr}(\mathrm{Fr}\mid\cdot)(\mathsf{k})$, this holds as any symmetric monoidal morphism $\mathrm{Bun}_{\mathbb{G}_m,\infty\mathrm{D}}(\mathsf{k}) \to \mathsf{B}\mathrm{A}$ is uniquely determined by its underlying monoidal morphism. We thus reduce to proving the commutativity of (7.47) over the subspace $\mathrm{Maps}_{\mathbf{Z}}(\mathrm{Bun}_{\mathbb{G}_m,\infty\mathrm{D}},\mathsf{B}^2\mathrm{A})$.

Over the neutral connected component of $\mathrm{Maps}_{\mathbf{Z}}(\mathrm{Bun}_{\mathbb{G}_m,\infty\mathrm{D}},\mathsf{B}^2\mathrm{A})$, we may establish the commutativity of (7.47) after taking loop spaces, where it reduces to the classical compatibility between Art and AJ (*cf.* §7.4.2).

By étale descent, it remains to prove that any $\mathbf{Z}$-linear morphism $\mathrm{Bun}_{\mathbb{G}_m,\infty\mathrm{D}} \to \mathsf{B}^2\mathrm{A}$ is trivial over a finite extension of k, *i.e.*

$$\mathcal{E}xt^2(\mathrm{Bun}_{\mathbb{G}_m,\infty\mathrm{D}},\mathrm{A}) \simeq 0,$$

where $\mathcal{E}xt$ denotes the internal Ext-group for étale sheaves over $\mathrm{Spec}\,\mathsf{k}$.

Replacing k by a finite extension if necessary, we may assume that the reduced subscheme of D is a finite (nonempty) collection of k-points $x^{\mathrm{I}} = (x^i)_{i\in\mathrm{I}}$ of X. We choose an element $i_0 \in \mathrm{I}$ and fit $\mathrm{Bun}_{\mathbb{G}_m,\infty\mathrm{D}}$ into a system of three short exact sequences:

$$\begin{tikzcd}
 & \prod_{i\in\mathrm{I}} \ker(\mathrm{L}^+_{x^i}\mathbb{G}_m \to \mathbb{G}_m) \arrow[d, hook] & \\
 & \mathrm{Bun}_{\mathbb{G}_m,\infty\mathrm{D}} \arrow[d, two heads] & \mathrm{Pic}^0 \arrow[d, hook] \\
\prod_{i\neq i_0}\mathbb{G}_m \arrow[r, hook] & \mathrm{Bun}_{\mathbb{G}_m,x^{\mathrm{I}}} \arrow[r, two heads] & \mathrm{Pic} \arrow[d, two heads] \\
 & & \mathbf{Z}
\end{tikzcd}$$

where $\mathrm{Pic} \simeq \mathrm{Bun}_{\mathbb{G}_m,x^{i_0}}$ is the Picard scheme of X and Pic^0 is its neutral component. It thus suffices to prove $\mathcal{E}xt^2(\mathrm{M},\mathrm{A}) \simeq 0$, for M a pro-unipotent group scheme, $\mathbb{G}_m$, an abelian variety, or the constant sheaf $\mathbf{Z}$.

The pro-unipotent case and the case $\mathrm{M} = \mathbf{Z}$ are clear. For $\mathrm{M} = \mathbb{G}_m$ or an abelian variety, we may replace A by μ_n, for an integer $n \geq 1$ invertible in k. Morphisms $\mathrm{M} \to \mu_n[2]$ of complexes are equivalent to morphisms $\mathrm{M}_{n\text{-tors}} \to \mathbb{G}_m[1]$, where $\mathrm{M}_{n\text{-tors}} \subset \mathrm{M}$ is the subgroup k-scheme of n-torsion elements. However, $\mathcal{E}xt^1(\mathrm{M}_{n\text{-tors}}, \mathbb{G}_m) \simeq 0$ because $\mathrm{M}_{n\text{-tors}}$ is finite (locally) free (*cf.* [GRR72, Exposé VIII, Proposition 3.3.1]). ☐

7.4.5. Let us now assume that $\operatorname{char} \mathsf{k} \neq 2$ and A is the subgroup $\{\pm 1\}$ of $\mathsf{e}^\times$.

Consider the étale level $\mu : \mathsf{B}\mathbb{G}_m \to \mathsf{B}^4\{\pm 1\}^{\otimes 2}$ defined by self-tensor product of Ψ mod 2, *i.e.* the morphism $\mathsf{B}\mathbb{G}_m \to \mathsf{B}^2\{\pm 1\}$ given by the double cover of $\mathbb{G}_m$. The topological cover of $\mathbb{A}_\mathrm{F}^\times$ induced from μ is the central extension

$$1 \to \{\pm 1\} \to \widetilde{\mathbb{A}}_\mathrm{F}^\times \to \mathbb{A}_\mathrm{F}^\times \to 1 \tag{7.48}$$

defined by the cocycle

$$\begin{aligned} \mathbb{A}_\mathrm{F}^\times \otimes \mathbb{A}_\mathrm{F}^\times &\to \{\pm 1\} \\ a \otimes b &\mapsto \prod_{x \in \mathrm{X}} \{a, b\}_x, \end{aligned}$$

where $\{\cdot, \cdot\}_x$ is the quadratic Hilbert symbol at $x \in \mathrm{X}$. Indeed, by Remark 7.3.7, the central extension (7.48) may be obtained from the construction of [Zha22, §2.2], so this identification follows from [Zha22, Proposition 2.3.12].

The central extension (7.48) is equipped with canonical splittings over $\mathrm{F}^\times$ and $\mathbb{O}_\mathrm{F}^\times$. It induces, in particular, a central extension of $\mathrm{Bun}_{\mathbb{G}_m, \infty \mathrm{D}}(\mathsf{k})$, whose pullback along Art is the *meta-Galois group* of $\mathring{\mathrm{X}}$ (*cf.* [Wei18, §4]):

$$1 \to \{\pm 1\} \to \widetilde{\pi}_1^{\text{ét}}(\mathring{\mathrm{X}}, \bar{\eta}) \to \pi_1^{\text{ét}}(\mathring{\mathrm{X}}, \bar{\eta}) \to 1. \tag{7.49}$$

We shall identify (7.49) as the étale fundamental group of the $\{\pm 1\}$-gerbe $\Psi(\Omega_{\mathring{\mathrm{X}}}^{-1})$ mod 2 of ϑ-characteristics of $\mathring{\mathrm{X}}$ (*cf.* §2.1.21).

Theorem 7.4.6. *The topological central extension* (7.49) *is canonically identified with the image of* $\Psi(\Omega_{\mathring{\mathrm{X}}}^{-1})$ *mod* 2 *under* (7.46).

Proof. First, we observe that the following diagram commutes

$$\begin{tikzcd} \mathrm{Maps}_{\mathbb{E}_\infty}(\mathsf{B}_{\mathring{\mathrm{X}}}\mathbb{G}_m, \mathsf{B}^4_{\mathring{\mathrm{X}}}\{\pm 1\}^{\otimes 2}) \arrow[r, "\int_\mathrm{X}"] \arrow[d] & \mathrm{Maps}_{\mathbb{E}_\infty}(\mathrm{Bun}_{\mathbb{G}_m, \infty \mathrm{D}}, \mathsf{B}^2\{\pm 1\}) \arrow[d, "\mathrm{Tr}(\mathrm{Fr}|\cdot)(\mathsf{k})"] \\ \mathrm{Maps}_{\mathbb{E}_\infty}(\mathrm{F}^\times \backslash \mathbb{A}_\mathrm{F}^\times / \mathrm{K}_{\infty \mathrm{D}}, \mathsf{B}\{\pm 1\}) \arrow[r, phantom, "\simeq"] & \mathrm{Maps}_{\mathbb{E}_\infty}(\mathrm{Bun}_{\mathbb{G}_m, \infty \mathrm{D}}(\mathsf{k}), \mathsf{B}\{\pm 1\}) \end{tikzcd} \tag{7.50}$$

where the top horizontal arrow is given by the $\mathbb{E}_\infty$-monoidal version of (7.11) and the left vertical arrow is the construction of adèlic covers from étale levels. The commutativity of (7.50) is an $\mathbb{E}_\infty$-monoidal version of Lemma 7.3.8 and follows from its proof.

By construction, (7.49) is the image of μ under the lower circuit of (7.50), followed by pullback along Artin reciprocity 7.44. By the commutativity of (7.50) and (7.47), we may also express it as the image of μ under the composition

$$\begin{aligned} \mathrm{Maps}_{\mathbb{E}_\infty}(\mathsf{B}_{\mathring{\mathrm{X}}}\mathbb{G}_m, \mathsf{B}^4_{\mathring{\mathrm{X}}}\{\pm 1\}^{\otimes 2}) &\xrightarrow{\int_\mathrm{X}} \mathrm{Maps}_{\mathbb{E}_\infty}(\mathrm{Bun}_{\mathbb{G}_m, \infty \mathrm{D}}, \mathsf{B}^2\{\pm 1\}) \\ &\xrightarrow{\mathrm{AJ}^*} \mathrm{Maps}(\mathring{\mathrm{X}}, \mathsf{B}^2\{\pm 1\}) \xrightarrow{(7.46)} \mathrm{CExt}(\pi_1^{\text{ét}}(\mathring{\mathrm{X}}, \bar{\eta}), \{\pm 1\}). \end{aligned}$$

It therefore remains to identify the image of μ under $\mathrm{AJ}^* \circ \int_{\mathrm{X}}$ with $\Psi(\Omega^{-1}_{\mathring{\mathrm{X}}})$ mod 2. However, since AJ factors through the map $\varpi^1 : \mathring{\mathrm{X}} \to \mathrm{Gr}_{\mathbb{G}_m,\{1\}}$ (*cf.* §4.1.4 for $\Lambda \simeq \mathbf{Z}$), this identification is a special case of Proposition 4.2.6. □

Remark 7.4.7. It follows from Theorem 7.4.6 that the meta-Galois group for function fields (global, local, and local integral) is *non-canonically* split, and is functorial with respect to finite separable extensions. These facts have been established by Weissman by different means (*cf.* [Wei18, §4.2, §4.4]).

Moreover, for k of characteristic 2, Weissman stipulates the meta-Galois group to be the split extension of $\pi_1^{\text{ét}}(\mathring{\mathrm{X}}, \bar{\eta})$ by $\{\pm 1\}$ (*cf.* [Wei18, §4.1]). This appears to align well with the fact that $\mathring{\mathrm{X}}$ admits a canonical ϑ-characteristic in characteristic 2 (*cf.* [Ati71]).

8. Spectral decomposition

The goal of this section is to construct the spectral decomposition of ζ-genuine cusp forms, following V. Lafforgue's method (*cf.* [Laf18]). All arguments of this section are minor variants of those of *op.cit.* and are already sketched in [Laf18, §14]. Thus, one may view this section as explaining how to put our results in contact with Lafforgue's work.

Let k be a finite field of cardinality q. Let X be a smooth, proper, geometrically connected k-curve with a nonempty k-finite closed subscheme $\mathrm{D} \subset \mathrm{X}$. Write $\mathring{\mathrm{X}} \coloneqq \mathrm{X} \smallsetminus \mathrm{D}$ and Ran for the Ran space of $\mathring{\mathrm{X}}$ (*cf.* §1.1.2). Let ℓ be a prime not dividing q and $\overline{\mathbf{Q}}_\ell$ be an algebraic closure of $\mathbf{Q}_\ell$ with a chosen square root $q^{1/2}$. Let A be a finite subgroup of $\overline{\mathbf{Q}}_\ell$ and write $\zeta : \mathrm{A} \hookrightarrow \overline{\mathbf{Q}}_\ell$ for the inclusion. Let G be a smooth affine group X-scheme which is reductive over $\mathring{\mathrm{X}}$. Let μ be an A-valued étale level of the base change of G to $\mathring{\mathrm{X}}$. Let Ξ be as in §7.3.11.

8.1. Statement.

8.1.1 L-*parameters***.** We fix an algebraic closure $\bar{\mathrm{F}}$ of the field of fractions F of X and write $\bar{\eta} \coloneqq \operatorname{Spec} \bar{\mathrm{F}}$ for the corresponding geometric point of X. The L-group (2.33), applied to $\mathring{\mathrm{X}}$, yields a short exact sequence of topological groups

$$1 \to \mathrm{H}_{\bar{\eta}}(\overline{\mathbf{Q}}_\ell) \to {}^{\mathrm{L}}\mathrm{H}_{\mathring{\mathrm{X}}} \to \pi_1^{\text{ét}}(\mathring{\mathrm{X}}, \bar{\eta}) \to 1. \tag{8.1}$$

By an L-*parameter*, we mean an $\mathrm{H}_{\bar{\eta}}(\overline{\mathbf{Q}}_\ell)$-conjugacy class of continuous sections

$$\sigma : \pi_1^{\text{ét}}(\mathring{\mathrm{X}}, \bar{\eta}) \to {}^{\mathrm{L}}\mathrm{H}_{\mathring{\mathrm{X}}} \tag{8.2}$$

of the surjection in (8.1).

8.1.2. By construction, (8.1) is the pullback of an extension ${}^{\mathrm{L}}\mathrm{H}_{\Gamma}$ of some finite quotient $\pi_1^{\text{ét}}(\mathring{\mathrm{X}}, \bar{\eta}) \twoheadrightarrow \Gamma$ by $\mathrm{H}_{\bar{\eta}}(\overline{\mathbf{Q}}_\ell)$. Thus, we may view ${}^{\mathrm{L}}\mathrm{H}_{\Gamma}$ as (the $\overline{\mathbf{Q}}_\ell$-points of) an algebraic group over $\overline{\mathbf{Q}}_\ell$ whose neutral component is reductive.

We may use this observation to define "semisimplicity" of an L-parameter. Namely, a continuous section (8.2) is *semisimple* if the Zariski closure of its image in some ${}^{\mathrm{L}}\mathrm{H}_{\Gamma}$ has reductive neutral component. An L-parameter $[\sigma]$ is *semisimple* if some (equivalently, any) representative of $[\sigma]$ is semisimple.

Remark 8.1.3. Since (8.1) is induced from an extension ${}^{\mathrm{L}}\mathrm{H}_{\mathring{\mathrm{X}},\mathsf{e}}$ of $\pi_1^{\text{ét}}(\mathring{\mathrm{X}}, \bar{\eta})$ by $\mathrm{H}_{\bar{\eta}}(\mathsf{e})$ for some finite extension e of $\mathbf{Q}_\ell$, it also makes sense to say that an L-parameter is "defined" over an intermediate field between e and $\overline{\mathbf{Q}}_\ell$.

To be more precise, an L-parameter is *defined* over e_1, for an intermediate field $\mathsf{e} \subset \mathsf{e}_1 \subset \overline{\mathbf{Q}}_\ell$, if a representative of it factors through a continuous homomorphism $\pi_1^{\text{ét}}(\mathring{\mathrm{X}}, \bar{\eta}) \to {}^{\mathrm{L}}\mathrm{H}_{\mathring{\mathrm{X}},\mathsf{e}_1}$, where ${}^{\mathrm{L}}\mathrm{H}_{\mathring{\mathrm{X}},\mathsf{e}_1}$ is the extension of $\pi_1^{\text{ét}}(\mathring{\mathrm{X}}, \bar{\eta})$ by $\mathrm{H}_{\bar{\eta}}(\mathsf{e}_1)$ induced from ${}^{\mathrm{L}}\mathrm{H}_{\mathring{\mathrm{X}},\mathsf{e}}$ along the inclusion $\mathrm{H}_{\bar{\eta}}(\mathsf{e}) \subset \mathrm{H}_{\bar{\eta}}(\mathsf{e}_1)$.

8.1.4. For each closed point $x \in \mathring{\mathrm{X}}$, we choose a $\bar{\mathsf{k}}$-point $\bar{x}$ lying over x to form $\pi_1^{\text{ét}}(x, \bar{x})$. A choice of an étale path $\bar{x} \simeq \bar{\eta}$ in $\mathring{\mathrm{X}}$ yields a continuous homomorphism $\pi_1^{\text{ét}}(x, \bar{x}) \to \pi_1^{\text{ét}}(\mathring{\mathrm{X}}, \bar{\eta})$, along which we may pull back (8.1) to obtain the unramified local L-group (*cf.* §7.3.16)

$$1 \to \mathrm{H}_{\bar{\eta}}(\overline{\mathbf{Q}}_\ell) \to {}^{\mathrm{L}}\mathrm{H}_x \to \pi_1^{\text{ét}}(x, \bar{x}) \to 1. \tag{8.3}$$

Thus, given an L-parameter $[\sigma]$, we obtain a $\mathrm{H}_{\bar{\eta}}(\overline{\mathbf{Q}}_\ell)$-conjugacy class $[\sigma_x]$ of continuous sections $\sigma_x : \pi_1^{\text{ét}}(x, \bar{x}) \to {}^{\mathrm{L}}\mathrm{H}_x$ of the surjection in (8.3).

Writing $\mathrm{Fr}_x \in \pi_1^{\text{ét}}(x, \bar{x})$ for the geometric Frobenius element, we obtain an $\mathrm{H}_{\bar{\eta}}(\overline{\mathbf{Q}}_\ell)$-conjugacy class $[\sigma_x](\mathrm{Fr}_x)$ in ${}^{\mathrm{L}}\mathrm{H}_x$, which we may regard as the (unramified) local L-parameter associated to σ at x.

8.1.5. In §7.3, we have defined the $\overline{\mathbf{Q}}_\ell$-vector space of ζ-genuine cusp forms (7.36) equipped with the action of the spherical Hecke algebra at each $x \in \mathring{\mathrm{X}}$ (*cf.* §7.3.15).

Our main result is a deomposition of (7.36) indexed by semisimple L-parameters, where each summand is a simultaneous eigenspace for all the Hecke operators.

Theorem 8.1.6. *There is a canonical isomorphism*

$$\mathrm{Fun}_{\mathrm{cusp},\zeta}(\widetilde{\mathrm{Bun}}_{\mathrm{G},\mathrm{D}}/\Xi, \overline{\mathbf{Q}}_\ell) \simeq \bigoplus_{[\sigma]} \mathbf{H}_{\mathrm{D},[\sigma]} \tag{8.4}$$

indexed by a set of L*-parameters* $[\sigma]$*, such that:*

1. *each* $[\sigma]$ *is semisimple (*cf. *§8.1.2);*
2. *each* $[\sigma]$ *is defined over some finite extension of* $\mathbf{Q}_\ell$ *(*cf. *Remark 8.1.3);*
3. *each* $\mathbf{H}_{\mathrm{D},[\sigma]}$ *is a Hecke eigenspace: For any* $x \in \mathring{\mathrm{X}}$ *and* $\mathrm{V} \in \mathsf{Rep}({}^{\mathrm{L}}\mathrm{H}_x)$*, the operator* $h_{\mathrm{V},x}$ *acts on* $\mathbf{H}_{\mathrm{D},[\sigma]}$ *by the scalar* $\mathrm{Tr}([\sigma_x](\mathrm{Fr}_x) \mid \mathrm{V})$ *(*cf. *§8.1.4).*

8.1.7. The spectral decomposition (8.4) is compatible with inclusions of nonempty k-finite closed subschemes $\mathrm{D} \subset \mathrm{D}_1$ of X, *i.e.* the restriction map

$$\mathrm{Fun}_{\mathrm{cusp},\zeta}(\widetilde{\mathrm{Bun}}_{\mathrm{G},\mathrm{D}}/\Xi, \overline{\mathbf{Q}}_\ell) \subset \mathrm{Fun}_{\mathrm{cusp},\zeta}(\widetilde{\mathrm{Bun}}_{\mathrm{G},\mathrm{D}_1}/\Xi, \overline{\mathbf{Q}}_\ell) \tag{8.5}$$

carries $\mathbf{H}_{\mathrm{D},[\sigma]}$ into $\mathbf{H}_{\mathrm{D}_1,[\sigma]}$, where we view $[\sigma]$ as an L-parameter over $\mathrm{X} \smallsetminus \mathrm{D}_1$ by restriction along $\mathrm{X} \smallsetminus \mathrm{D}_1 \subset \mathrm{X} \smallsetminus \mathrm{D} = \mathring{\mathrm{X}}$.

Assuming that G splits, we may take the union of (8.4) over the poset of nonempty k-finite closed subschemes D to obtain the "generic form" of spectral decomposition

$$\mathrm{Fun}_{\mathrm{cusp},\zeta}(\mathrm{G}(\mathrm{F})\backslash\widetilde{\mathrm{G}}/\Xi, \overline{\mathbf{Q}}_\ell) \simeq \bigoplus_{[\sigma]} \mathbf{H}_{[\sigma]}, \tag{8.6}$$

where $[\sigma]$ ranges over $\mathrm{H}_{\bar{\eta}}(\overline{\mathbf{Q}}_\ell)$-conjugacy classes of continuous sections $\mathrm{Gal}(\bar{\mathrm{F}}/\mathrm{F}) \to {}^{\mathrm{L}}\mathrm{H}_\eta$. Here, ${}^{\mathrm{L}}\mathrm{H}_\eta$ denotes the pullback of (8.2) along $\mathrm{Gal}(\bar{\mathrm{F}}/\mathrm{F}) \simeq \pi_1^{\text{ét}}(\eta, \bar{\eta}) \to \pi_1^{\text{ét}}(\mathring{\mathrm{X}}, \bar{\eta})$.

Remark 8.1.8. If one is only interested in the generic form (8.6) of the spectral decomposition, it suffices to replace our input (G, μ) by a reductive group F-scheme G equipped with an A-valued étale level μ. Indeed, one extends G to a smooth group X-scheme as in [Laf18, §12.1] and μ likewise extends to some open subscheme of X by [Zha22, Lemma 2.2.5].

However, we note that an extension of μ across a closed point x of X is *additional data*. Thus, in order to state the compatibility between (8.6) and the Satake isomorphism at a point $x \in \mathrm{X}$ (*i.e.* the analogue of statement (3) of Theorem 8.1.6), we must assume that G is reductive over $\mathcal{O}_x$ and *choose* an extension of μ along $\operatorname{Spec} \mathrm{F}_x \subset \operatorname{Spec} \mathcal{O}_x$.

Remark 8.1.9. In the statement of Theorem 8.1.6, we assumed $\mathrm{D} \neq \varnothing$. However, it also applies when $\mathrm{D} = \varnothing$ and X is *not* isomorphic to $\mathbb{P}^1$. Indeed, this is because we only use the property that $\mathring{\mathrm{X}}$ is an algebraic $\mathrm{K}(\pi, 1)$-space (*cf.* §2.2.8), which holds as long as it is not isomorphic to $\mathbb{P}^1$.

8.2. Cohomology of Shtukas.

8.2.1. The remainder of this article is devoted to the proof of Theorem 8.1.6. The main construction involved is the moduli space of Shtukas, which dates back to Drinfeld's pioneering works [Dri87a, Dri87b, Dri88].

In this subsection, we shall define the cohomology of Shtukas in the twisted setting. This relies on the twisted geometric Satake equivalence (*cf.* Theorem 2.3.2) and the compatibility between the local and global A-gerbes (*cf.* Lemma 7.1.11).

8.2.2 *Moduli of Shtukas*. For any integer $n \geq 0$, we define the moduli stack of Shtukas to be the fiber product of k-stacks

$$\begin{tikzcd} \mathrm{Sht}^{[n]}_{\mathrm{G,D}} \arrow[r] \arrow[d] & \mathrm{Hec}^{[n]}(\mathrm{Bun}_{\mathrm{G,D}}) \arrow[d, "(p_0, p_n)"] \\ \mathrm{Bun}_{\mathrm{G,D}} \arrow[r, "(\mathrm{id}, \mathrm{Fr})"] & \mathrm{Bun}_{\mathrm{G,D}} \times \mathrm{Bun}_{\mathrm{G,D}} \end{tikzcd} \tag{8.7}$$

where the structural maps p_0, p_n are as in §7.1.3.

In other words, an R-point of $\mathrm{Sht}^{[n]}_{\mathrm{G,D}}$ consists of R-points $\underline{x}_1, \cdots, \underline{x}_n$ of Ran, a chain of n modifications along with an isomorphism

$$(\mathrm{P}^0, \phi^0) \overset{\underline{x}_1}{\sim} (\mathrm{P}^1, \phi^1) \overset{\underline{x}_2}{\sim} \cdots \overset{\underline{x}_n}{\sim} (\mathrm{P}^n, \phi^n) \simeq ({}^{\tau}\mathrm{P}^0, {}^{\tau}\phi^n), \tag{8.8}$$

where the superscript τ means pullback along the endomorphism $\mathrm{id}_{\mathrm{X}} \times \mathrm{Fr}_{\mathrm{R}}$ of X_{R}.

The R-points $\underline{x}_1, \cdots, \underline{x}_n$ of Ran^n are called the *paws* of the Shtuka (8.8). By [Var04, Proposition 2.16] (*cf.* [Laf18, Lemma 12.19] for nonsplit G), $\mathrm{Sht}^{[n]}_{\mathrm{G,D}}$ is a relative ind-Deligne–Mumford stack over Ran^n.

Remark 8.2.3. For $n = 0$, the Cartesian square (8.7) reduces to the square (7.12) defining the Frobenius-fix point stack of $\mathrm{Bun}_{\mathrm{G,D}}$. Thanks to Remark 7.2.2, the latter is identified with the discrete stack $\mathrm{Bun}_{\mathrm{G,D}}(\mathsf{k})$, so we have a canonical isomorphism

$$\mathrm{Sht}^{[0]}_{\mathrm{G,D}} \simeq \mathrm{Bun}_{\mathrm{G,D}}(\mathsf{k}). \tag{8.9}$$

8.2.4. For each $j = 1, \cdots, n$, we have a morphism

$$r_j : \mathrm{Sht}^{[n]}_{\mathrm{G,D}} \to \mathrm{Hec}_{\mathrm{G}} \tag{8.10}$$

sending (8.8) to the restriction of $\mathrm{P}^{n-1} \overset{\underline{x}_n}{\sim} \mathrm{P}^n$ to $\mathrm{D}_{\underline{x}_n}$.

Recall the local A-gerbe $\mathcal{G}_{\mathrm{Hec}_{\mathrm{G}}}$ over $\mathrm{Hec}_{\mathrm{G}}$ (*cf.* §1.1.7). We shall need a canonical trivialization of the sum of pullbacks:

$$\mathcal{G}_{\mathrm{Sht}^{[n]}_{\mathrm{G,D}}} := \sum_{j=1}^{n} (r_j)^* \mathcal{G}_{\mathrm{Hec}_{\mathrm{G}}}. \tag{8.11}$$

Indeed, by Lemma 7.1.11, the A-gerbe (8.11) is canonically identified with the pullback of $(p_n)^*\mathcal{G}_{\mathrm{Bun}_{\mathrm{G,D}}} - (p_0)^*\mathcal{G}_{\mathrm{Bun}_{\mathrm{G,D}}}$ along the top horizontal arrow of (8.7). Using the commutativity of (8.7), we may rewrite it as the pullback of

$$(\mathrm{Fr}_{\mathrm{Bun}_{\mathrm{G,D}}})^*\mathcal{G}_{\mathrm{Bun}_{\mathrm{G,D}}} - \mathcal{G}_{\mathrm{Bun}_{\mathrm{G,D}}} \tag{8.12}$$

along the left vertical arrow. However, the A-gerbe (8.12) admits a canonical trivialization by (7.15). This yields the desired trivialization of (8.11).

8.2.5. Given a family of finite sets $\mathrm{I}_1, \cdots, \mathrm{I}_n$, we write $\mathrm{Sht}^{\mathrm{I}_1,\cdots,\mathrm{I}_n}_{\mathrm{G,D}}$ for the base change of $\mathrm{Sht}^{[n]}_{\mathrm{G,D}}$ along the product of the maps $\mathring{\mathrm{X}}^{\mathrm{I}_j} \to \mathrm{Ran}$ over $j = 1, \cdots, n$, with respect to the paws.

We shall construct a functor

$$\mathsf{Rep}\Big(\prod_{j=1}^{n} {}^{\mathrm{L}}\mathrm{H}_{\mathring{\mathrm{X}}^{\mathrm{I}_j}}\Big) \to \mathsf{D}(\mathrm{Sht}^{\mathrm{I}_1,\cdots,\mathrm{I}_n}_{\mathrm{G,D}}), \tag{8.13}$$

where ${}^{\mathrm{L}}\mathrm{H}_{\mathring{\mathrm{X}}^{\mathrm{I}_j}}$ is the L-group defined in §2.2.10 (for $j = 1, \cdots, n$). Here, $\prod_{j=1}^{n} {}^{\mathrm{L}}\mathrm{H}_{\mathring{\mathrm{X}}^{\mathrm{I}_j}}$ is viewed as an extension of $\prod_{j=1}^{n} \pi_1^{\text{ét}}(\mathring{\mathrm{X}}^{\mathrm{I}_j}, \bar{\eta})$ by $\mathrm{H}^{\mathrm{I}}_{\bar{\eta}}(\overline{\mathbf{Q}}_\ell)$ (with $\mathrm{I} := \mathrm{I}_1 \sqcup \cdots \sqcup \mathrm{I}_n$) and the source of (8.13) is the category of its continuous representations on finite-dimensional $\overline{\mathbf{Q}}_\ell$-vector spaces which are algebraic over $\mathrm{H}^{\mathrm{I}}_{\bar{\eta}}(\overline{\mathbf{Q}}_\ell)$.

8.2.6 *Construction of* (8.13)**.** Recall the outer convolution diagram $\widetilde{\mathrm{Hec}}^{\mathrm{I}_1,\cdots,\mathrm{I}_n}_{\mathrm{G}}$ together with the structural morphisms $\tilde{p}_1, \cdots, \tilde{p}_n$ (*cf.* §1.4.2). Denote by $\mathsf{Sat}_{\mathcal{G},\zeta}(\widetilde{\mathrm{Hec}}^{\mathrm{I}_1,\cdots,\mathrm{I}_n}_{\mathrm{G}})$ the category of $(\sum_{j=1}^{n} \tilde{p}_j^*\mathcal{G}_{\mathrm{Hec}_{\mathrm{G}}}, \zeta)$-twisted perverse ULA sheaves over $\widetilde{\mathrm{Hec}}^{\mathrm{I}_1,\cdots,\mathrm{I}_n}_{\mathrm{G}}$ relative to $\mathring{\mathrm{X}}^{\mathrm{I}}$. (As in §1.3.7, perversity is defined with respect to the pullback to the corresponding version of the affine Grassmannian $\widetilde{\mathrm{Gr}}^{\mathrm{I}_1,\cdots,\mathrm{I}_n}_{\mathrm{G}}$, parametrizing modifications (1.39) where P^0 is trivial.) There is a morphism

$$r^{\mathrm{I}_1,\cdots,\mathrm{I}_n} : \mathrm{Sht}^{\mathrm{I}_1,\cdots,\mathrm{I}_n}_{\mathrm{G,D}} \to \widetilde{\mathrm{Hec}}^{\mathrm{I}_1,\cdots,\mathrm{I}_n}_{\mathrm{G}} \tag{8.14}$$

sending (8.8) to its restriction over $\mathrm{D}_{\underline{x}_1 \cup \cdots \cup \underline{x}_n}$ (and forgetting the last isomorphism).

Consider the functor

$$\prod_{j=1}^{n} \mathsf{Rep}({}^{\mathrm{L}}\mathrm{H}_{\mathring{\mathrm{X}}^{\mathrm{I}_j}}) \to \mathsf{Sat}_{\mathcal{G},\zeta}(\widetilde{\mathrm{Hec}}^{\mathrm{I}_1,\cdots,\mathrm{I}_n}_{\mathrm{G}}) \tag{8.15}$$

$$(\mathrm{V}_1, \cdots, \mathrm{V}_n) \mapsto \bigotimes_{j=1}^{n} \tilde{p}_j^* \mathcal{S}_{\mathrm{I}_j, \mathrm{V}_j},$$

where $\mathcal{S}_{\mathrm{I}_j,\mathrm{V}_j} \in \mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\mathrm{I}_j})$ denotes the image of V_j under the equivalence (2.39). Since (8.15) is $\overline{\mathbf{Q}}_\ell$-multilinear and right exact in each factor, it factors through the tensor product of the Tannakian categories $\mathsf{Rep}({}^{\mathrm{L}}\mathrm{H}_{\mathring{\mathrm{X}}^{\mathrm{I}_j}})$, which is identified with $\mathsf{Rep}(\prod_{j=1}^{n} {}^{\mathrm{L}}\mathrm{H}_{\mathring{\mathrm{X}}^{\mathrm{I}_j}})$ (*cf.* [Del90, §5.18]). This yields a functor

$$\mathsf{Rep}\Big(\prod_{j=1}^{n} {}^{\mathrm{L}}\mathrm{H}_{\mathring{\mathrm{X}}^{\mathrm{I}_j}}\Big) \to \mathsf{Sat}_{\mathcal{G},\zeta}(\widetilde{\mathrm{Hec}}^{\mathrm{I}_1,\cdots,\mathrm{I}_n}_{\mathrm{G}}). \tag{8.16}$$

The desired morphism (8.13) is the composition of (8.16) with the pullback along (8.14), which is an *untwisted* complex by the canonical trivialization of the A-gerbe (8.11).

Remark 8.2.7. Let us express $\widetilde{\mathrm{Hec}}^{\mathrm{I}_1,\cdots,\mathrm{I}_n}_{\mathrm{G}}$ as the quotient of $\widetilde{\mathrm{Gr}}^{\mathrm{I}_1,\cdots,\mathrm{I}_n}_{\mathrm{G}}$ by $\mathrm{L}^+_{\mathrm{I}}\mathrm{G}$, which acts by modifying the trivialization of P^0. It acts on any closed subscheme Z_α of $\widetilde{\mathrm{Gr}}^{\mathrm{I}_1,\cdots,\mathrm{I}_n}_{\mathrm{G}}$ through some finite type quotient $\mathrm{L}^+_{\mathrm{I}}\mathrm{G} \to \mathrm{H}_\beta$ (*cf.* Remark 1.1.5).

The pullback of $\mathrm{Sht}_{\mathrm{G,D}}^{\mathrm{I}_1,\cdots,\mathrm{I}_n}$ to the stack $\mathrm{L}_{\mathrm{I}}^{+}\mathrm{G}\backslash \mathrm{Z}_\alpha$ along (8.14) is smooth over $\mathrm{H}_\beta\backslash \mathrm{Z}_\alpha$ of the same relative dimension as H_β over $\mathring{\mathrm{X}}^{\mathrm{I}}$ (*cf.* [Laf18, Proposition 2.8]). This implies that the image of (8.13) consists of perverse ULA sheaves relative to $\mathring{\mathrm{X}}^{\mathrm{I}}$.

8.2.8 $\mathrm{Bun}_{\mathrm{Z}^\sharp,\infty\mathrm{D}}(\mathsf{k})$*-equivariance***.** Recall the $\mathrm{Bun}_{\mathrm{Z}^\sharp,\infty\mathrm{D}}$-action on $\mathrm{Bun}_{\mathrm{G,D}}$ (*cf.* §7.3.10). This induces a $\mathrm{Bun}_{\mathrm{Z}^\sharp,\infty\mathrm{D}}$-action on $\mathrm{Hec}^{[n]}(\mathrm{Bun}_{\mathrm{G,D}})$, by acting on each term in a chain of modifications (7.1). This action lifts to a $\mathrm{Bun}_{\mathrm{Z}^\sharp,\infty\mathrm{D}}(\mathsf{k})$-action on $\mathrm{Sht}_{\mathrm{G,D}}^{[n]}$.

We shall argue that (8.13) factors through the category of $\mathrm{Bun}_{\mathrm{Z}^\sharp,\infty\mathrm{D}}(\mathsf{k})$-equivariant objects in $\mathsf{D}(\mathrm{Sht}_{\mathrm{G,D}}^{\mathrm{I}_1,\cdots,\mathrm{I}_n})$. Indeed, by the equivariance structure on (8.14):

$$\begin{tikzcd}
\mathrm{Bun}_{\mathrm{Z}^\sharp,\infty\mathrm{D}}(\mathsf{k}) \arrow[d] \arrow[r, phantom, "\circlearrowright"] & \mathrm{Sht}_{\mathrm{G,D}}^{\mathrm{I}_1,\cdots,\mathrm{I}_n} \arrow[d, "(8.14)"] \\
\mathrm{L}_{\mathrm{I}}^{+}\mathsf{BZ}^\sharp \arrow[r, phantom, "\circlearrowright"] & \widetilde{\mathrm{Hec}}_{\mathrm{G}}^{\mathrm{I}_1,\cdots,\mathrm{I}_n}
\end{tikzcd}$$

where the left vertical arrow is the restriction to the appropriate formal disk, this follows from the observation below.

Lemma 8.2.9. *The functor forgetting* $\mathrm{L}_{\mathrm{I}}^{+}\mathsf{BZ}^\sharp$*-equivariance is an equivalence:*

$$\mathsf{Sat}_{\mathcal{G},\zeta}(\widetilde{\mathrm{Hec}}_{\mathrm{G}}^{\mathrm{I}_1,\cdots,\mathrm{I}_n})^{\mathrm{L}_{\mathrm{I}}^{+}\mathsf{BZ}^\sharp} \simeq \mathsf{Sat}_{\mathcal{G},\zeta}(\widetilde{\mathrm{Hec}}_{\mathrm{G}}^{\mathrm{I}_1,\cdots,\mathrm{I}_n}).$$

Proof. The forgetful functor is fully faithful, as $\mathrm{L}_{\mathrm{I}}^{+}\mathsf{BZ}^\sharp$ may be written as an inverse limit of connected smooth algebraic $\mathring{\mathrm{X}}^{\mathrm{I}}$-stacks. It remains to prove that it is essentially surjective.

Let $\mathcal{A}$ be an object of $\mathsf{Sat}_{\mathcal{G},\zeta}(\widetilde{\mathrm{Hec}}_{\mathrm{G}}^{\mathrm{I}_1,\cdots,\mathrm{I}_n})$. Denote by a and p the action, respectively projection map from $\mathrm{L}_{\mathrm{I}}^{+}\mathsf{BZ}^\sharp \times \widetilde{\mathrm{Hec}}_{\mathrm{G}}^{\mathrm{I}_1,\cdots,\mathrm{I}_n}$ to $\widetilde{\mathrm{Hec}}_{\mathrm{G}}^{\mathrm{I}_1,\cdots,\mathrm{I}_n}$. It suffices to find an isomorphism

$$a^*\mathcal{A} \simeq p^*\mathcal{A} \tag{8.17}$$

extending the natural isomorphism over the neutral section $e \times \widetilde{\mathrm{Hec}}_{\mathrm{G}}^{\mathrm{I}_1,\cdots,\mathrm{I}_n}$, as such an isomorphism is unique if it exists.

By universal local acyclicity, it suffices to construct (8.17) over the pointwise disjoint locus in $\mathring{\mathrm{X}}^{\mathrm{I}}$ (*cf.* [HS23, Theorem 6.8]). Using the factorization structure, we may further reduce to the case where I is a singleton, where (8.17) exists by Proposition 5.3.3. ☐

8.2.10. Let us now invoke the lattice Ξ. Consider the structural morphism

$$\nu_{\Xi}^{\mathrm{I}_1,\cdots,\mathrm{I}_n} : \mathrm{Sht}_{\mathrm{G,D}}^{\mathrm{I}_1,\cdots,\mathrm{I}_n}/\Xi \to \mathring{\mathrm{X}}^{\mathrm{I}}, \tag{8.18}$$

which is a relative ind-Deligne–Mumford stack (*cf.* §8.2.2).

On the other hand, the functor (8.13) factors through the category of Ξ-equivariant objects (*cf.* §8.2.8), defining a functor

$$\begin{aligned} \mathsf{Rep}\Big(\prod_{j=1}^{n} {}^{\mathrm{L}}\mathrm{H}_{\mathring{\mathrm{X}}^{\mathrm{I}_j}}\Big) &\to \mathsf{D}(\mathrm{Sht}_{\mathrm{G,D}}^{\mathrm{I}_1,\cdots,\mathrm{I}_n}/\Xi), \\ \mathrm{V} &\mapsto \mathcal{F}_{\mathrm{I}_1,\cdots,\mathrm{I}_n,\mathrm{V}}. \end{aligned} \tag{8.19}$$

For any object $\mathrm{V} \in \mathsf{Rep}(\prod_{j=1}^{n} {}^{\mathrm{L}}\mathrm{H}_{\mathring{\mathrm{X}}^{\mathrm{I}_j}})$, we shall write

$$\mathcal{H}_{\mathrm{I}_1,\cdots,\mathrm{I}_n,\mathrm{V}} := (\nu_{\Xi}^{\mathrm{I}_1,\cdots,\mathrm{I}_n})_!\mathcal{F}_{\mathrm{I}_1,\cdots,\mathrm{I}_n,\mathrm{V}} \in \mathrm{Ind}\,\mathsf{D}(\mathring{\mathrm{X}}^{\mathrm{I}}) \tag{8.20}$$

and refer to it as the *cohomology of Shtukas* with coefficients in (the perverse sheaf associated to) the representation V.

Remark 8.2.11. There is a canonical isomorphism in $\operatorname{Ind}\mathsf{D}(\mathring{\mathrm{X}}^{\mathrm{I}})$:

$$\mathcal{H}_{\mathrm{I}_1,\cdots,\mathrm{I}_n,\mathrm{V}} \simeq \mathcal{H}_{\mathrm{I},\mathrm{V}}, \tag{8.21}$$

where the right-hand-side is the cohomology of Shtukas with coefficients in the restriction of V along the natural map ${}^{\mathrm{L}}\mathrm{H}_{\mathring{\mathrm{X}}^{\mathrm{I}}} \to \prod_{j=1}^{n} {}^{\mathrm{L}}\mathrm{H}_{\mathring{\mathrm{X}}^{\mathrm{I}_j}}$.

Indeed, (8.21) follows from the compatibility between (8.16) and (a mild generalization of) the outer convolution product (*cf.* §1.4.4, for n possibly distinct finite sets $\mathrm{I}_1, \cdots, \mathrm{I}_n$.)

8.2.12 *The case* $n = 0$**.** For $n = 0$, the source of (8.19) is the category of finite-dimensional $\overline{\mathbf{Q}}_\ell$-vector spaces, while the right-hand-side is $\mathsf{D}(\mathrm{Bun}_{\mathrm{G},\mathrm{D}}(\mathsf{k})/\Xi)$ (*cf.* Remark 8.2.3). Let us identify the image $\mathcal{S}_{\mathbf{1}}$ of the 1-dimensional vector space $\mathbf{1} := \overline{\mathbf{Q}}_\ell$.

By construction, $\mathcal{S}_{\mathbf{1}}$ is obtained as follows: Consider the constant 1-dimensional local system $\overline{\mathbf{Q}}_\ell$ over $\mathrm{Bun}_{\mathrm{G},\mathrm{D}}$, regarded as twisted by the pullback of $(-\mathcal{G}_{\mathrm{Bun}_{\mathrm{G},\mathrm{D}}}) \boxplus \mathcal{G}_{\mathrm{Bun}_{\mathrm{G},\mathrm{D}}}$ along the diagonal of $\mathrm{Bun}_{\mathrm{G},\mathrm{D}}$. The pullback of $\overline{\mathbf{Q}}_\ell$ to $\mathrm{Bun}_{\mathrm{G},\mathrm{D}}(\mathsf{k})$ may be viewed as an *untwisted* complex by trivializing the pullback of $(-\mathcal{G}_{\mathrm{Bun}_{\mathrm{G},\mathrm{D}}}) \boxplus \mathcal{G}_{\mathrm{Bun}_{\mathrm{G},\mathrm{D}}}$ along

$$(\mathrm{id}, \mathrm{Fr}_{\mathrm{Bun}_{\mathrm{G},\mathrm{D}}}) : \mathrm{Bun}_{\mathrm{G},\mathrm{D}} \to \mathrm{Bun}_{\mathrm{G},\mathrm{D}} \times \mathrm{Bun}_{\mathrm{G},\mathrm{D}}$$

via (7.15). This complex is the 1-dimesional local system $(\mathcal{L}_{\mathrm{Tr}(\mathrm{Fr}|\mathcal{G}),\zeta})^{\otimes -1}$ (*cf.* §7.2.4).

Thus, (7.22) (applied to quasi-compact substacks of $\mathrm{Bun}_{\mathrm{G},\mathrm{D}}$) yields an isomorphism

$$\mathcal{H}_{\varnothing,\overline{\mathbf{Q}}_\ell} \simeq \mathrm{Fun}_{c,\zeta}(\widetilde{\mathrm{Bun}}_{\mathrm{G},\mathrm{D}}/\Xi, \overline{\mathbf{Q}}_\ell). \tag{8.22}$$

8.3. Hecke action.

8.3.1. In this subsection, we construct an action of the spherical Hecke algebra (*cf.* §7.3.15) on the cohomology of Shtukas (*cf.* §8.2.10)

More precisely, for each closed point $x \in \mathring{\mathrm{X}}$ and element $h \in \mathrm{Fun}_{c,\zeta}(\mathrm{G}(\mathcal{O}_x)\backslash\widetilde{\mathrm{G}}_x/\mathcal{O}_x)$, we shall construct an endomorphism $\mathrm{T}(h)$ of the *restriction* of $\mathcal{H}_{\mathrm{I}_1,\cdots,\mathrm{I}_n,\mathrm{V}}$ to $(\mathring{\mathrm{X}} \smallsetminus x)^{\mathrm{I}}$, where $\mathrm{I}_1, \cdots, \mathrm{I}_n$ is any family of finite sets with $\mathrm{I} := \mathrm{I}_1 \sqcup \cdots \sqcup \mathrm{I}_n$ and $\mathrm{V} \in \mathsf{Rep}(\prod_{j=1}^{n} {}^{\mathrm{L}}\mathrm{H}_{\mathring{\mathrm{X}}^{\mathrm{I}_j}})$.

8.3.2. Theis construction will makes use of the Hecke correspondence for $\mathrm{Sht}_{\mathrm{G},\mathrm{D}}^{[n]}$.

Namely, write $\mathrm{Hec}(\mathrm{Sht}_{\mathrm{G},\mathrm{D}}^{[n]})$ for the prestack parametrizing two R-points

$$(\mathrm{P}^0, \phi^0) \overset{\underline{x}_1}{\sim} \cdots \overset{\underline{x}_{n-1}}{\sim} (\mathrm{P}^{n-1}, \phi^{n-1}) \overset{\underline{x}_n}{\sim} ({}^{\tau}\mathrm{P}^0, {}^{\tau}\phi^0) \tag{8.23}$$

$$(\mathrm{Q}^0, \psi^0) \overset{\underline{x}_1}{\sim} \cdots \overset{\underline{x}_{n-1}}{\sim} (\mathrm{Q}^{n-1}, \psi^{n-1}) \overset{\underline{x}_n}{\sim} ({}^{\tau}\mathrm{Q}^0, {}^{\tau}\psi^0) \tag{8.24}$$

of $\mathrm{Sht}_{\mathrm{G},\mathrm{D}}^{[n]}$ over the same R-point $(\underline{x}_1, \cdots, \underline{x}_n)$ of Ran^n, equipped with an isomorphism off the graph of another R-point $\underline{x}$ of Ran disjoint from $\underline{x}_1, \cdots, \underline{x}_n$.

Sending an R-point of $\mathrm{Hec}(\mathrm{Sht}_{\mathrm{G},\mathrm{D}}^{[n]})$ to (8.23), respectively (8.24), as well as the restriction of $\mathrm{P}^0 \overset{\underline{x}}{\sim} \mathrm{Q}^0$ to $\mathrm{D}_{\underline{x}}$ define morphisms p, q, and r in the following diagram:

$$\begin{tikzcd}
\mathrm{Hec}(\mathrm{Sht}_{\mathrm{G},\mathrm{D}}^{[n]}) \arrow[r, "r"] \arrow[d, "{(p,q)}"] & \mathrm{Hec}_{\mathrm{G}}(\mathsf{k}) \\
\mathrm{Sht}_{\mathrm{G},\mathrm{D}}^{[n]} \times_{\mathrm{Ran}^n} \mathrm{Sht}_{\mathrm{G},\mathrm{D}}^{[n]} &
\end{tikzcd} \tag{8.25}$$

where we have identified $\mathrm{Hec}_{\mathrm{G}}(\mathsf{k})$ with the Frobenius-fixed point stack of $\mathrm{Hec}_{\mathrm{G}}$.

8.3.3. We shall formulate a compatibility statement for A-gerbes with respect to (8.25).

Recall the A-gerbe $\mathcal{G}_{\mathrm{Sht}^{[n]}_{\mathrm{G,D}}}$ over $\mathrm{Sht}^{[n]}_{\mathrm{G,D}}$ (*cf.* §8.2.4). Since the post-compositions of p, q with each r_j ($j = 1, \cdots, n$) coincide, we obtain an isomorphism of A-gerbes

$$p^* \mathcal{G}_{\mathrm{Sht}^{[n]}_{\mathrm{G,D}}} \simeq q^* \mathcal{G}_{\mathrm{Sht}^{[n]}_{\mathrm{G,D}}}. \tag{8.26}$$

On the other hand, *loc.cit.* supplies a canonical trivialization of $\mathcal{G}_{\mathrm{Sht}^{[n]}_{\mathrm{G,D}}}$, whose pullback along p and q do *not* coincide under the isomorphism (8.26).

Lemma 8.3.4. *With respect to the pullbacks of the canonical trivialization of* $\mathcal{G}_{\mathrm{Sht}^{[n]}_{\mathrm{G,D}}}$ *under* p *and* q*, the isomorphism* (8.26) *is given by the* A*-torsor* $r^*(-\mathrm{Tr}(\mathrm{Fr} \mid \mathcal{G}_{\mathrm{Hec}_{\mathrm{G}}}))$.

Proof. The value of (8.26) at an R-point of $\mathrm{Hec}(\mathrm{Sht}^{[n]}_{\mathrm{G,D}})$ is the sum of isomorphisms

$$\mathcal{G}_{\mathrm{Hec}_{\mathrm{G}}}(\mathrm{P}^{j-1} \overset{x_j}{\sim} \mathrm{P}^j) \simeq \mathcal{G}_{\mathrm{Hec}_{\mathrm{G}}}(\mathrm{Q}^{j-1} \overset{x_j}{\sim} \mathrm{Q}^j), \tag{8.27}$$

supplied by the identifications $(\mathrm{P}^{j-1} \overset{x_j}{\sim} \mathrm{P}^j) \simeq (\mathrm{Q}^{j-1} \overset{x_j}{\sim} \mathrm{Q}^j)$ over $\mathrm{D}_{\underline{x}_j}$, over $j = 1, \cdots, n$.

On the other hand, we may use Lemma 7.1.11 to rewrite (8.27) as an isomorphism

$$\mathcal{G}_{\mathrm{Bun}_{\mathrm{G,D}}}(\mathrm{P}^j, \phi^j) - \mathcal{G}_{\mathrm{Bun}_{\mathrm{G,D}}}(\mathrm{P}^{j-1}, \phi^{j-1}) \simeq \mathcal{G}_{\mathrm{Bun}_{\mathrm{G,D}}}(\mathrm{Q}^j, \psi^j) - \mathcal{G}_{\mathrm{Bun}_{\mathrm{G,D}}}(\mathrm{Q}^{j-1}, \psi^{j-1}). \tag{8.28}$$

Since (8.28) is obtained from an identification of Hecke modifications along $\underline{x}_j$, it admits the following description: Taking the difference of the isomorphisms from Lemma 7.1.11

$$\mathcal{G}_{\mathrm{Bun}_{\mathrm{G,D}}}(\mathrm{P}^j, \phi^j) + \mathcal{G}_{\mathrm{Hec}_{\mathrm{G}}}(\mathrm{P}^j \overset{x}{\sim} \mathrm{Q}^j) \simeq \mathcal{G}_{\mathrm{Bun}_{\mathrm{G,D}}}(\mathrm{Q}^j, \psi^j),$$
$$\mathcal{G}_{\mathrm{Bun}_{\mathrm{G,D}}}(\mathrm{P}^{j-1}, \phi^{j-1}) + \mathcal{G}_{\mathrm{Hec}_{\mathrm{G}}}(\mathrm{P}^{j-1} \overset{x}{\sim} \mathrm{Q}^{j-1}) \simeq \mathcal{G}_{\mathrm{Bun}_{\mathrm{G,D}}}(\mathrm{Q}^{j-1}, \psi^{j-1}),$$

and applying the identification in $\mathrm{Hec}_{\mathrm{G},x}$

$$(\mathrm{P}^{j-1} \overset{x}{\sim} \mathrm{Q}^{j-1}) \simeq (\mathrm{P}^j \overset{x}{\sim} \mathrm{Q}^j),$$

supplied by the Hecke modifications along $\underline{x}_j$, we again arrive at (8.28).

Using this description, we see that the sum of (8.27) over $j = 1, \cdots, n$ yields the difference of the two isomorphisms from Lemma 7.1.11

$$\mathcal{G}_{\mathrm{Bun}_{\mathrm{G,D}}}({}^{\tau}\mathrm{P}^0, {}^{\tau}\phi^0) + \mathcal{G}_{\mathrm{Hec}_{\mathrm{G}}}({}^{\tau}\mathrm{P}^0 \overset{x}{\sim} {}^{\tau}\mathrm{Q}^0) \simeq \mathcal{G}_{\mathrm{Bun}_{\mathrm{G,D}}}({}^{\tau}\mathrm{Q}^0, {}^{\tau}\psi^0), \tag{8.29}$$

$$\mathcal{G}_{\mathrm{Bun}_{\mathrm{G,D}}}(\mathrm{P}^0, \phi^0) + \mathcal{G}_{\mathrm{Hec}_{\mathrm{G}}}(\mathrm{P}^0 \overset{x}{\sim} \mathrm{Q}^0) \simeq \mathcal{G}_{\mathrm{Bun}_{\mathrm{G,D}}}(\mathrm{Q}^0, \psi^0) \tag{8.30}$$

under the composition of identifications in $\mathrm{Hec}_{\mathrm{G},x}$:

$$(\mathrm{P}^0 \overset{x}{\sim} \mathrm{Q}^0) \simeq (\mathrm{P}^1 \overset{x}{\sim} \mathrm{Q}^1) \simeq \cdots \simeq ({}^{\tau}\mathrm{P}^0 \overset{x}{\sim} {}^{\tau}\mathrm{Q}^0) \tag{8.31}$$

supplied by the chains of Hecke modifications along $\underline{x}_1, \cdots, \underline{x}_n$.

Identifying the values of $\mathcal{G}_{\mathrm{Bun}_{\mathrm{G,D}}}$ at $({}^{\tau}\mathrm{P}^0, {}^{\tau}\phi^0)$ and (P^0, ϕ^0), respectively $({}^{\tau}\mathrm{Q}^0, {}^{\tau}\psi^0)$ and (Q^0, ψ^0), using Frobenius-equivariance (7.15), we may write the difference of (8.29) from (8.30) as an automorphism of the trivial A-gerbe, *i.e.* an A-torsor.

The above discussion allows us to identify this A-torsor with the difference of the Frobenius-equivariance isomorphism

$$\mathcal{G}_{\mathrm{Hec}_{\mathrm{G}}}(\mathrm{P}^0 \overset{x}{\sim} \mathrm{Q}^0) \simeq \mathcal{G}_{\mathrm{Hec}_{\mathrm{G}}}({}^{\tau}\mathrm{P}^0 \overset{x}{\sim} {}^{\tau}\mathrm{Q}^0)$$

from the isomorphism induced from the identification (8.31). By construction, this is inverse to the A-torsor $\mathrm{Tr}(\mathrm{Fr} \mid \mathcal{G}_{\mathrm{Hec}_{\mathrm{G}}})$ over $\mathrm{Hec}_{\mathrm{G}}(\mathsf{k})$ (*cf.* §7.2.4). □

8.3.5 *Construction of Hecke action.* Let $\mathrm{I}_1, \cdots, \mathrm{I}_n$ be a family of finite sets, with $\mathrm{I} := \mathrm{I}_1 \sqcup \cdots \sqcup \mathrm{I}_n$. Fix a closed point x of $\mathring{\mathrm{X}}$. Consider the base change of (8.25), where the paws of the Shtukas are restricted to $(\mathring{\mathrm{X}} \smallsetminus x)^{\mathrm{I}}$ and the copy of local Hecke stack $\mathrm{Hec}_{\mathrm{G}}(\mathsf{k})$ is restricted to x:

$$\begin{tikzcd}
\mathrm{Hec}_x(\mathrm{Sht}^{\mathrm{I}_1,\cdots,\mathrm{I}_n}_{\mathrm{G},\mathrm{D}}) \arrow[r, "r"] \arrow[d, "{(p,q)}"] & \mathrm{Hec}_{\mathrm{G},x}(\mathsf{k}) \\
\mathrm{Sht}^{\mathrm{I}_1,\cdots,\mathrm{I}_n}_{\mathrm{G},\mathrm{D}} \times_{(\mathring{\mathrm{X}} \smallsetminus x)^{\mathrm{I}}} \mathrm{Sht}^{\mathrm{I}_1,\cdots,\mathrm{I}_n}_{\mathrm{G},\mathrm{D}} &
\end{tikzcd} \tag{8.32}$$

Given $\mathrm{V} \in \mathsf{Rep}(\prod_{j=1}^n {}^{\mathrm{L}}\mathrm{H}_{\mathring{\mathrm{X}}^{\mathrm{I}_j}})$, we have defined the Ξ-equivariant perverse sheaf $\mathcal{F}_{\mathrm{I}_1,\cdots,\mathrm{I}_n,\mathrm{V}}$ over $\mathrm{Sht}^{\mathrm{I}_1,\cdots,\mathrm{I}_n}_{\mathrm{G},\mathrm{D}}$ (*cf.* §8.2.10). Using Lemma 8.3.4, we obtain an isomorphism of Ξ-equivariant perverse sheaves

$$p^*\mathcal{F}_{\mathrm{I}_1,\cdots,\mathrm{I}_n,\mathrm{V}} \otimes r^*(\mathcal{L}_{\mathrm{Tr}(\mathrm{Fr}|\mathcal{G}_{\mathrm{Hec}_{\mathrm{G}}})})^{\otimes -1} \simeq q^*\mathcal{F}_{\mathrm{I}_1,\cdots,\mathrm{I}_n,\mathrm{V}}. \tag{8.33}$$

By expressing q as a union of étale morphisms, we obtain from (8.33) a morphism

$$q_!(p^*\mathcal{F}_{\mathrm{I}_1,\cdots,\mathrm{I}_n,\mathrm{V}} \otimes r^*(\mathcal{L}_{\mathrm{Tr}(\mathrm{Fr}|\mathcal{G}_{\mathrm{Hec}_{\mathrm{G}}})})^{\otimes -1}) \to \mathcal{F}_{\mathrm{I}_1,\cdots,\mathrm{I}_n,\mathrm{V}}$$

and thus by taking compactly supported cohomology along $\nu_{\Xi}^{\mathrm{I}_1,\cdots,\mathrm{I}_n}$ (*cf.* §8.2.10) and using the isomorphism (7.22), we obtain a morphism in $\mathrm{Ind}\,\mathsf{D}((\mathring{\mathrm{X}} \smallsetminus x)^{\mathrm{I}})$:

$$\mathcal{H}_{\mathrm{I}_1,\cdots,\mathrm{I}_n,\mathrm{V}}|_{(\mathring{\mathrm{X}} \smallsetminus x)^{\mathrm{I}}} \otimes \mathrm{Fun}_{c,\zeta}(\mathrm{G}(\mathcal{O}_x)\backslash \widetilde{\mathrm{G}}_x/\mathrm{G}(\mathcal{O}_x)) \to \mathcal{H}_{\mathrm{I}_1,\cdots,\mathrm{I}_n,\mathrm{V}}|_{(\mathring{\mathrm{X}} \smallsetminus x)^{\mathrm{I}}},$$

and thus a map

$$\mathrm{T} : \mathrm{Fun}_{c,\zeta}(\mathrm{G}(\mathcal{O}_x)\backslash \widetilde{\mathrm{G}}_x/\mathrm{G}(\mathcal{O}_x)) \to \mathrm{End}(\mathcal{H}_{\mathrm{I}_1,\cdots,\mathrm{I}_n,\mathrm{V}}|_{(\mathring{\mathrm{X}} \smallsetminus x)^{\mathrm{I}}}). \tag{8.34}$$

Remark 8.3.6. By working with the convolution local Hecke stack instead of $\mathrm{Hec}_{\mathrm{G},x}$, we also obtain an isomorphism of endomorphisms

$$\mathrm{T}(h_1 \star h_2) \simeq \mathrm{T}(h_1) \circ \mathrm{T}(h_2), \tag{8.35}$$

for any $h_1, h_2 \in \mathrm{Fun}_{c,\zeta}(\mathrm{G}(\mathcal{O}_x)\backslash \widetilde{\mathrm{G}}_x/\mathrm{G}(\mathcal{O}_x))$. Furthermore, T carries the unit for the convolution product to the identity endomorphism.

Note that we certainly expect (8.34) to lift to a morphism of $\mathbb{E}_1$-monoids, but we will not supply the construction. (We only need that T induces a morphism of associative algebras after taking H^0 of $\mathcal{H}_{\mathrm{I}_1,\cdots,\mathrm{I}_n,\mathrm{V}}|_{(\mathring{\mathrm{X}} \smallsetminus x)^{\mathrm{I}}}$, which is guaranteed by (8.35).)

8.4. Partial Frobenius-equivariance.

8.4.1. Given n finite sets $\mathrm{I}_1, \cdots, \mathrm{I}_n$ ($n \geq 0$) with $\mathrm{I} := \mathrm{I}_1 \sqcup \cdots \sqcup \mathrm{I}_n$, we may consider the *partial Frobenius* endomorphism $\mathrm{F}_{\mathrm{I}_j} := \mathrm{Fr}_{\mathrm{X}^{\mathrm{I}_j}} \times \mathrm{id}_{\mathrm{X}^{\mathrm{I} \smallsetminus \mathrm{I}_j}}$ of X^{I} (for $j = 1, \cdots, n$). Note that the composition $\mathrm{F}_{\mathrm{I}_1} \circ \cdots \circ \mathrm{F}_{\mathrm{I}_n}$ equals $\mathrm{Fr}_{\mathrm{X}^{\mathrm{I}}}$. Let V be an object of $\mathsf{Rep}(\prod_{j=1}^n {}^{\mathrm{L}}\mathrm{H}_{\mathring{\mathrm{X}}^{\mathrm{I}_j}})$.

In this subsection, we shall equip the corresponding cohomology of Shtukas (8.20) with an $\mathrm{F}_{\mathrm{I}_1}$-equivariance structure, in the sense of an isomorphism

$$(\mathrm{F}_{\mathrm{I}_1})^*\mathcal{H}_{\mathrm{I}_1,\cdots,\mathrm{I}_n,\mathrm{V}} \simeq \mathcal{H}_{\mathrm{I}_2,\cdots,\mathrm{I}_n,\mathrm{I}_1,\mathrm{V}}, \tag{8.36}$$

such that its n-fold composition

$$\begin{aligned}
(\mathrm{Fr}_{\mathring{\mathrm{X}}^{\mathrm{I}}})^*\mathcal{H}_{\mathrm{I}_1,\cdots,\mathrm{I}_n,\mathrm{V}} &\simeq (\mathrm{F}_{\mathrm{I}_n})^* \circ \cdots \circ (\mathrm{F}_{\mathrm{I}_1})^*\mathcal{H}_{\mathrm{I}_1,\cdots,\mathrm{I}_n,\mathrm{V}} \\
&\simeq (\mathrm{F}_{\mathrm{I}_n})^* \circ \cdots \circ (\mathrm{Fr}_{\mathrm{I}_2})^*\mathcal{H}_{\mathrm{I}_2,\cdots,\mathrm{I}_1,\mathrm{V}} \simeq \cdots \simeq \mathcal{H}_{\mathrm{I}_1,\cdots,\mathrm{I}_n,\mathrm{V}}
\end{aligned} \tag{8.37}$$

is induced from the isomorphism $(\mathrm{Fr}_{\mathring{\mathrm{X}}^{\mathrm{I}}})^* \simeq \mathrm{id}$ on the étale site of $\mathring{\mathrm{X}}^{\mathrm{I}}$.

8.4.2. To construct (8.36), we consider the morphism

$$\mathrm{F}_{\mathrm{I}_1} : \mathrm{Sht}_{\mathrm{G},\mathrm{D}}^{\mathrm{I}_1,\cdots,\mathrm{I}_n} \to \mathrm{Sht}_{\mathrm{G},\mathrm{D}}^{\mathrm{I}_2,\cdots,\mathrm{I}_1},$$

lifting the corresponding partial Frobenius endomorphism of $\mathring{\mathrm{X}}^{\mathrm{I}}$, sending an R-point

$$(\mathrm{P}^0,\phi^0) \overset{x_1}{\sim} (\mathrm{P}^1,\phi^1) \overset{x_2}{\sim} \cdots \overset{x_n}{\sim} ({}^{\tau}\mathrm{P}^0, {}^{\tau}\phi^0) \tag{8.38}$$

to the R-point

$$(\mathrm{P}^1,\phi^1) \overset{x_2}{\sim} \cdots \overset{x_n}{\sim} ({}^{\tau}\mathrm{P}^0, {}^{\tau}\phi^0) \overset{{}^{\tau}x_1}{\sim} ({}^{\tau}\mathrm{P}^1, {}^{\tau}\phi^1).$$

By definition, we have a commutative square

$$\begin{tikzcd}
\mathrm{Sht}_{\mathrm{G},\mathrm{D}}^{\mathrm{I}_1,\cdots,\mathrm{I}_n} \arrow[r, "\prod r_j"] \arrow[d, "\mathrm{F}_{\mathrm{I}_1}"] & \prod_{j=1}^n \mathrm{Hec}_{\mathrm{G},\mathrm{I}_j} \arrow[d, "\mathrm{Fr}_{\mathrm{Hec}_{\mathrm{G},\mathrm{I}_1}} \times \mathrm{id}"] \\
\mathrm{Sht}_{\mathrm{G},\mathrm{D}}^{\mathrm{I}_2,\cdots,\mathrm{I}_1} \arrow[r, "\prod r_j"] & \prod_{j=1}^n \mathrm{Hec}_{\mathrm{G},\mathrm{I}_j}
\end{tikzcd} \tag{8.39}$$

where the horizontal morphisms are restrictions to the formal disks.

It remains to construct, for an n-tuple $\mathrm{V}_j \in \mathsf{Rep}({}^{\mathrm{L}}\mathrm{H}_{\mathring{\mathrm{X}}^{\mathrm{I}_j}})$, an isomorphism

$$(\mathrm{F}_{\mathrm{I}_1})^* \bigotimes_{j=1}^{n} (r_j)^* \mathcal{S}_{\mathrm{I}_j,\mathrm{V}_j} \simeq \bigotimes_{j=1}^{n} (r_j)^* \mathcal{S}_{\mathrm{I}_j,\mathrm{V}_j} \tag{8.40}$$

of (untwisted) perverse sheaves over $\mathrm{Sht}_{\mathrm{G},\mathrm{D}}^{\mathrm{I}_1,\cdots,\mathrm{I}_n}$, compatibly with the Ξ-equivariance structure (*cf.* §8.2.6). Indeed, we then obtain (8.36) for $\mathrm{V} \coloneqq \boxtimes_{j=1}^n \mathrm{V}_j$ by taking compactly supported direct image of (8.40) along $\nu_{\Xi}^{\mathrm{I}_1,\cdots,\mathrm{I}_n}$ (using the remark on base change in [Laf18, §4.3]) and the case for general V by the universal property.

8.4.3 *Construction of* (8.40)**.** Since $(\mathrm{Fr}_{\mathrm{Hec}_{\mathrm{G},\mathrm{I}_1}})^*$ acts as the identity on the étale site of $\mathrm{Hec}_{\mathrm{G},\mathrm{I}_1}$, we have a canonical isomorphism

$$(\mathrm{Fr}_{\mathrm{Hec}_{\mathrm{G},\mathrm{I}_1}})^* \mathcal{S}_{\mathrm{I}_1,\mathrm{V}_1} \simeq \mathcal{S}_{\mathrm{I}_1,\mathrm{V}_1} \tag{8.41}$$

with respect to the canonical isomorphism (7.15) for the local A-gerbe over $\mathrm{Hec}_{\mathrm{G},\mathrm{I}_1}$.

The isomorphism (8.40) is obtained as the tensor product of the pullback of (8.41) along r_1 with the identity on $(r_j)^*\mathcal{S}_{\mathrm{I}_j,\mathrm{V}_j}$ for $j = 2, \cdots, n$, using the commutativity of (8.39).

For this procedure to yield an isomorphism of *untwisted* perverse sheaves, the following compatibility of the trivializations of A-gerbes is required: Along the isomorphisms

$$\begin{aligned}
(\mathrm{F}_{\mathrm{I}_1})^* \sum_{j=1}^{n} (r_j)^* \mathcal{G}_{\mathrm{Hec}_{\mathrm{G},\mathrm{I}_j}} &\simeq (r_1)^* (\mathrm{Fr}_{\mathrm{Hec}_{\mathrm{G},\mathrm{I}_1}})^* \mathcal{G}_{\mathrm{Hec}_{\mathrm{G},\mathrm{I}_1}} + \sum_{j=2}^{n} (r_j)^* \mathcal{G}_{\mathrm{Hec}_{\mathrm{G},\mathrm{I}_j}} \\
&\simeq \sum_{j=1}^{n} (r_j)^* \mathcal{G}_{\mathrm{Hec}_{\mathrm{G},\mathrm{I}_j}}
\end{aligned} \tag{8.42}$$

supplied by (8.39) and (7.15) (for $\mathcal{G}_{\mathrm{Hec}_{\mathrm{G},\mathrm{I}_1}}$), the trivialization of $\sum_{j=1}^n (r_j)^* \mathcal{G}_{\mathrm{Hec}_{\mathrm{G},\mathrm{I}_j}}$ constructed in §8.2.6 corresponds to its own pullback by $\mathrm{F}_{\mathrm{I}_1}$. However, the two sides of (8.42) evaluate at an R-point (8.38) of $\mathrm{Sht}_{\mathrm{G},\mathrm{D}}^{\mathrm{I}_1,\cdots,\mathrm{I}_n}$ to the same telescopic sum (*cf.* the proof of Lemma 7.1.11), where both trivializations are induced from (7.15) for $\mathcal{G}_{\mathrm{Bun}_{\mathrm{G},\mathrm{D}}}$.

The fact that the n-fold composition (8.37) is the canonical $\mathrm{Fr}_{\mathrm{X}^{\mathrm{I}}}$-equivariance structure on $\mathcal{H}_{\mathrm{I}_1,\cdots,\mathrm{I}_n,\mathrm{V}}$ follows immediately from the construction.

Remark 8.4.4. Using (8.21), we may write (8.36) as an isomorphism

$$\varphi_{\mathrm{I}_1} : (\mathrm{F}_{\mathrm{I}_1})^* \mathcal{H}_{\mathrm{I},\mathrm{V}} \simeq \mathcal{H}_{\mathrm{I},\mathrm{V}}. \tag{8.43}$$

By the compatibility between the construction of (8.36) with outer convolution product (along $\mathrm{I}_2, \cdots, \mathrm{I}_n$), the isomorphism (8.43) depends only on I_1 as a subset of I, and not on the partition of $\mathrm{I} \smallsetminus \mathrm{I}_1$ into $\mathrm{I}_2, \cdots, \mathrm{I}_n$.

Likewise, for two disjoint subsets I_1, I_2 of I, there is a commutative diagram

$$\begin{tikzcd}
(\mathrm{F}_{\mathrm{I}_2})^* (\mathrm{F}_{\mathrm{I}_1})^* \mathcal{H}_{\mathrm{I},\mathrm{V}} \arrow[r, "(\mathrm{F}_{\mathrm{I}_2})^* \varphi_{\mathrm{I}_1}"] \arrow[d, "\simeq"] & (\mathrm{F}_{\mathrm{I}_2})^* \mathcal{H}_{\mathrm{I},\mathrm{V}} \arrow[d, "\varphi_{\mathrm{I}_2}"] \\
(\mathrm{F}_{\mathrm{I}_1 \sqcup \mathrm{I}_2})^* \mathcal{H}_{\mathrm{I},\mathrm{V}} \arrow[r, "\varphi_{\mathrm{I}_1 \sqcup \mathrm{I}_2}"] & \mathcal{H}_{\mathrm{I},\mathrm{V}}
\end{tikzcd}$$

8.4.5. Let us now specialize to the case where $\mathrm{I}_1, \cdots, \mathrm{I}_n$ are all singletons, so $\mathrm{I} \simeq \{1, \cdots, n\}$. We write F_i for the partial Frobenius endomorphism $\mathrm{F}_{\{i\}}$ of X^{I}.

We write $(\operatorname{Ind} \mathsf{D}(\mathring{\mathrm{X}}^{\mathrm{I}}))^{\mathrm{F}_1, \cdots, \mathrm{F}_n}$ for the category of objects $\mathcal{A} \in \operatorname{Ind} \mathsf{D}(\mathring{\mathrm{X}}^{\mathrm{I}})$ equipped with n isomorphisms $(\mathrm{F}_i)^* \mathcal{A} \simeq \mathcal{A}$, which pairwise commute (in the evident sense) and such that their composite

$$\begin{aligned}
(\mathrm{Fr}_{\mathring{\mathrm{X}}^{\mathrm{I}}})^* \mathcal{A} &\simeq (\mathrm{F}_n)^* \circ \cdots \circ (\mathrm{F}_1)^* \mathcal{A} \\
&\simeq (\mathrm{F}_n)^* \circ \cdots \circ (\mathrm{F}_2)^* \mathcal{A} \simeq \cdots \simeq \mathcal{A}
\end{aligned}$$

is the canonical $\mathrm{Fr}_{\mathring{\mathrm{X}}^{\mathrm{I}}}$-equivariance on $\mathcal{A}$.

The construction of §8.4.1 (*cf.* Remark 8.4.4) thus furnishes a functor:

$$\begin{gathered}
\mathsf{Rep}(({}^{\mathrm{L}}\mathrm{H}_{\mathring{\mathrm{X}}})^{\mathrm{I}}) \to (\operatorname{Ind} \mathsf{D}(\mathring{\mathrm{X}}^{\mathrm{I}}))^{\mathrm{F}_1, \cdots, \mathrm{F}_n} \\
\mathrm{V} \mapsto \mathcal{H}_{\mathrm{I},\mathrm{V}} \text{ with } \varphi_{\{1\}}, \cdots, \varphi_{\{n\}}.
\end{gathered} \tag{8.44}$$

Furthermore, (8.44) is natural in I (*cf.* [Laf18, Proposition 4.12, Proposition 4.14]). The isomorphisms $\varphi_{\{1\}}, \cdots, \varphi_{\{n\}}$ are the *partial Frobenius-equivariance* structures on $\mathcal{H}_{\mathrm{I},\mathrm{V}}$.

Remark 8.4.6. It follows immediately from the construction that the partial Frobenius-equivariance structures on $\mathcal{H}_{\mathrm{I},\mathrm{V}}$ commute with Hecke operators (*cf.* §8.3.5).

Remark 8.4.7. Given a subset $\mathrm{J} \subset \mathrm{I}$ and a J-tuple of closed points $\{x_j\}_{j \in \mathrm{J}}$ of $\mathring{\mathrm{X}}$, we have the restriction functor

$$(\operatorname{Ind} \mathsf{D}(\mathring{\mathrm{X}}^{\mathrm{I}}))^{\mathrm{F}_1, \cdots, \mathrm{F}_n} \to (\operatorname{Ind} \mathsf{D}(\mathring{\mathrm{X}}^{\mathrm{I} \smallsetminus \mathrm{J}} \times x^{\mathrm{J}}))^{\mathrm{F}_1, \cdots, \mathrm{F}_n}, \tag{8.45}$$

where the target denotes the category of objects of $\operatorname{Ind} \mathsf{D}(\mathring{\mathrm{X}}^{\mathrm{I} \smallsetminus \mathrm{J}} \times x^{\mathrm{J}})$ equipped with partial Frobenius-equivariance defined as in §8.4.5. The composition of (8.44) and (8.45) canonically factors through a functor

$$\mathsf{Rep}(({}^{\mathrm{L}}\mathrm{H}_{\mathring{\mathrm{X}}})^{\mathrm{I} \smallsetminus \mathrm{J}} \times \prod_{j \in \mathrm{J}} {}^{\mathrm{L}}\mathrm{H}_{x_j}) \to (\operatorname{Ind} \mathsf{D}(\mathring{\mathrm{X}}^{\mathrm{I} \smallsetminus \mathrm{J}} \times x^{\mathrm{J}}))^{\mathrm{F}_1, \cdots, \mathrm{F}_n}, \tag{8.46}$$

where each ${}^{\mathrm{L}}\mathrm{H}_{x_j}$ is the unramified local L-group at x_j (*cf.* §7.3.16).

To see this, we repeat the construction of (8.44), with the paws of the Shtukas restricted to $\mathring{\mathrm{X}}^{\mathrm{I} \smallsetminus \mathrm{J}} \times x^{\mathrm{J}}$, and use the local form of the Satake equivalence (*cf.* §7.3.16).

Note that the family of functors (8.46) is natural with respect to maps of inclusions of finite sets $(\mathrm{J}_1 \subset \mathrm{I}_1) \to (\mathrm{J}_2 \subset \mathrm{I}_2)$.

8.5. Excursion operators.

8.5.1. In this subsection, we summarize Lafforgue's construction of the spectral decomposition of cusp forms, taking as input the family of functors (8.44). This corresponds to the material [Laf18, §6-11 and §12.3].

We shall start with the construction of the operators $\mathrm{S}_{\mathrm{V},x}$ (*cf.* [Laf18, §12.3.3]). Namely, for any finite set I and $\mathrm{W} \in \mathsf{Rep}(({}^{\mathrm{L}}\mathrm{H}_{\mathring{\mathrm{X}}})^{\mathrm{I}})$, we shall construct an endomorphism on the cohomology of Shtukas

$$\mathrm{S}_{\mathrm{V},x} : \mathcal{H}_{\mathrm{I},\mathrm{W}} \to \mathcal{H}_{\mathrm{I},\mathrm{W}} \tag{8.47}$$

in $\operatorname{Ind}\mathsf{D}(\mathring{\mathrm{X}}^{\mathrm{I}})$, associated to any closed point $x \in \mathring{\mathrm{X}}$ and any $\mathrm{V} \in \mathsf{Rep}({}^{\mathrm{L}}\mathrm{H}_x)$, where ${}^{\mathrm{L}}\mathrm{H}_x$ is the unramified local L-group at x (*cf.* §7.3.16).

8.5.2 *Construction of* (8.47)**.** By naturality of (8.44) with respect to the inclusion of finite sets $\mathrm{I} \to \mathrm{I} \sqcup \{1\}$, we obtain an isomorphism ("insertion of vacuum"):

$$\mathcal{H}_{\mathrm{I},\mathrm{W}}|_{\mathring{\mathrm{X}}^{\mathrm{I}}\times\mathring{\mathrm{X}}} \simeq \mathcal{H}_{\mathrm{I}\sqcup\{1\},\mathrm{W}\boxtimes\mathbf{1}} \tag{8.48}$$

where $\mathbf{1}$ denotes the trivial representation of ${}^{\mathrm{L}}\mathrm{H}_{\mathring{\mathrm{X}}}$.

Let us restrict (8.48) along the inclusion $\mathring{\mathrm{X}}^{\mathrm{I}} \times x \to \mathring{\mathrm{X}}^{\mathrm{I}} \times \mathring{\mathrm{X}}$. We may express $\mathcal{H}_{\mathrm{I}\sqcup\{1\},\mathrm{W}\boxtimes\mathbf{1}}|_{\mathring{\mathrm{X}}^{\mathrm{I}}\times x}$ as the image of $\mathrm{W} \boxtimes \mathbf{1} \in \mathsf{Rep}(({}^{\mathrm{L}}\mathrm{H}_{\mathring{\mathrm{X}}})^{\mathrm{I}} \times {}^{\mathrm{L}}\mathrm{H}_x)$ under (8.46). Applying the naturality of the latter with respect to the map of inclusions of finite sets

$$\begin{tikzcd}
\{1,2\} \arrow[r, phantom, "\subset"] \arrow[d, "f"] & \mathrm{I} \sqcup \{1,2\} \arrow[d, "\mathrm{id}_{\mathrm{I}} \sqcup f"] \\
\{1\} \arrow[r, phantom, "\subset"] & \mathrm{I} \sqcup \{1\}
\end{tikzcd}$$

with $f(1) = f(2) = 1$, we may identify $\mathcal{H}_{\mathrm{I}\sqcup\{1\},\mathrm{W}\boxtimes\mathbf{1}}|_{\mathring{\mathrm{X}}^{\mathrm{I}}\times x}$ with $\mathcal{H}_{\mathrm{I}\sqcup\{1,2\},\mathrm{W}\boxtimes\mathbf{1}}|_{\mathring{\mathrm{X}}^{\mathrm{I}}\times\Delta(x)}$, where $\mathbf{1}$ now means the trivial representation of $({}^{\mathrm{L}}\mathrm{H}_x)^{\{1,2\}}$. In other words, the restriction of (8.48) to $\mathring{\mathrm{X}}^{\mathrm{I}} \times x$ yields an isomorphism

$$\mathcal{H}_{\mathrm{I},\mathrm{W}}|_{\mathring{\mathrm{X}}^{\mathrm{I}}\times x} \simeq \mathcal{H}_{\mathrm{I}\sqcup\{1,2\},\mathrm{W}\boxtimes\mathbf{1}}|_{\mathring{\mathrm{X}}^{\mathrm{I}}\times\Delta(x)}. \tag{8.49}$$

Consider now the composition

$$\begin{aligned}\mathcal{H}_{\mathrm{I},\mathrm{W}}|_{\mathring{\mathrm{X}}^{\mathrm{I}}\times x} \simeq \mathcal{H}_{\mathrm{I}\sqcup\{1,2\},\mathrm{W}\boxtimes\mathbf{1}}|_{\mathring{\mathrm{X}}^{\mathrm{I}}\times\Delta(x)} &\xrightarrow{\text{unit}} \mathcal{H}_{\mathrm{I}\sqcup\{1,2\},\mathrm{W}\boxtimes\mathrm{V}\boxtimes\mathrm{V}^*}|_{\mathring{\mathrm{X}}^{\mathrm{I}}\times\Delta(x)} \\ &\xrightarrow{\varphi_{\{1\}}^{\deg x}} \mathcal{H}_{\mathrm{I}\sqcup\{1,2\},\mathrm{W}\boxtimes\mathrm{V}\boxtimes\mathrm{V}^*}|_{\mathring{\mathrm{X}}^{\mathrm{I}}\times\Delta(x)} \xrightarrow{\text{counit}} \mathcal{H}_{\mathrm{I}\sqcup\{1,2\},\mathrm{W}\boxtimes\mathbf{1}}|_{\mathring{\mathrm{X}}^{\mathrm{I}}\times\Delta(x)} \simeq \mathcal{H}_{\mathrm{I},\mathrm{W}}|_{\mathring{\mathrm{X}}^{\mathrm{I}}\times x},\end{aligned} \tag{8.50}$$

where unit, counit denote the morphisms induced by functoriality of (8.46) with respect to the unit $\mathbf{1} \to \mathrm{V} \boxtimes \mathrm{V}^*$, respectively counit $\mathrm{V} \boxtimes \mathrm{V}^* \to \mathbf{1}$, the first and last isomorphisms are (8.49), and the endomorphism $\varphi_{\{1\}}^{\deg x}$ is the $(\deg x)$-fold iteration of the partial Frobenius equivariance structure along the factor corresponding to $\{1\}$. (Here, $\deg x := [\mathsf{k}_x : \mathsf{k}]$, where k_x is the residue field of x.)

The composition (8.50) respects partial Frobenius equivariance along the x-factor, so it descends to an endomorphism of $\mathcal{H}_{\mathrm{I},\mathrm{W}}$, giving the desired operator $\mathrm{S}_{\mathrm{V},x}$.

Remark 8.5.3. In the construction of (8.47), we may also replace W by a representation of a product of global and local L-groups, as follows: Given a subset $\mathrm{J} \subset \mathrm{I}$, a J-tuple of closed points $\{x_j\}_{j\in\mathrm{J}}$ of $\mathring{\mathrm{X}}$, and $\mathrm{W} \in \mathsf{Rep}(({}^{\mathrm{L}}\mathrm{H}_{\mathring{\mathrm{X}}})^{\mathrm{I}\smallsetminus\mathrm{J}} \times \prod_{j\in\mathrm{J}} {}^{\mathrm{L}}\mathrm{H}_{x_j})$, we have an endomorphism

$$\mathrm{S}_{\mathrm{V},x} : \mathcal{H}_{\mathrm{I},\mathrm{W}} \to \mathcal{H}_{\mathrm{I},\mathrm{W}} \tag{8.51}$$

in $\operatorname{Ind}\mathsf{D}(\mathring{\mathrm{X}}^{\mathrm{I}\smallsetminus\mathrm{J}} \times x^{\mathrm{J}})$, associated to any $x \in \mathring{\mathrm{X}}$ and $\mathrm{V} \in \mathsf{Rep}({}^{\mathrm{L}}\mathrm{H}_x)$.

The construction of (8.51) is completely parallel to that of (8.47), except that we use the local form of the cohomology of Shtukas (8.46) also along x^{J}.

8.5.4. As in [Laf18, Proposition 6.2], the endomorphism $\mathrm{S}_{\mathrm{V},x}$ extends the Hecke operator $h_{\mathrm{V},x}$ on the restriction of $\mathcal{H}_{\mathrm{I},\mathrm{W}}$ to $(\mathring{\mathrm{X}} \smallsetminus x)^{\mathrm{I}}$ (*cf.* §7.3.15, §8.3.5).

Proposition 8.5.5 ("S = T")**.** *Given a finite set* I *and* $\mathrm{W} \in \mathsf{Rep}(({}^{\mathrm{L}}\mathrm{H}_{\mathring{\mathrm{X}}})^{\mathrm{I}})$*, as well as a closed point* $x \in \mathring{\mathrm{X}}$ *and* $\mathrm{V} \in \mathsf{Rep}({}^{\mathrm{L}}\mathrm{H}_x)$*, there is a canonical isomorphism*

$$\mathrm{S}_{\mathrm{V},x} \simeq \mathrm{T}(h_{\mathrm{V},x}) \tag{8.52}$$

of endomorphisms of $\mathcal{H}_{\mathrm{I},\mathrm{W}}|_{(\mathring{\mathrm{X}}\smallsetminus x)^{\mathrm{I}}}$.

8.5.6 *Cohomological correspondences***.** In the untwisted setting (*cf.* [Laf18, §6.3]), the isomorphism (8.52) is obtained from an identification of "cohomological correspondences" from the coefficient sheaf $\mathcal{F}_{\mathrm{I},\mathrm{W}}$ over $\mathrm{Sht}^{\mathrm{I}}_{\mathrm{G},\mathrm{D}}$ to itself (*cf.* §8.2.10). The twisted setting is similar, except that we need an appropriate notion of "cohomological correspondences."

Let $(\mathcal{X}_1, \mathcal{G}_1)$, $(\mathcal{X}_2, \mathcal{G}_2)$ be algebraic k-stacks equipped with A-gerbes. Given objects $\mathcal{A}_1 \in \mathsf{D}_{\mathcal{G}_1,\zeta}(\mathcal{X}_1)$, $\mathcal{A}_2 \in \mathsf{D}_{\mathcal{G}_2,\zeta}(\mathcal{X}_2)$, a *cohomological correspondence*

$$(\mathcal{X}_1, \mathcal{G}_1, \mathcal{A}_1) \to (\mathcal{X}_2, \mathcal{G}_2, \mathcal{A}_2)$$

consists of a span of the following data

1. a span of algebraic k-stacks $\mathcal{X}_1 \xleftarrow{\overleftarrow{\pi}} \mathcal{X} \xrightarrow{\overrightarrow{\pi}} \mathcal{X}_2$, where $\overrightarrow{\pi}$ is schematic, separated, and of finite type (so $\overrightarrow{\pi}^!$ is well-defined, *cf.* Remark B.1.4);
2. an isomorphism of A-gerbes $\gamma : \overleftarrow{\pi}^*\mathcal{G}_1 \simeq \overrightarrow{\pi}^*\mathcal{G}_2$;
3. a morphism in $\mathsf{D}_{\mathcal{G},\zeta}(\mathcal{X})$, where $\mathcal{G}$ is the A-gerbe $\overleftarrow{\pi}^*\mathcal{G}_1 \simeq \overrightarrow{\pi}^*\mathcal{G}_2$ identified via γ:
$$\overleftarrow{\pi}^*\mathcal{A}_1 \to \overrightarrow{\pi}^!\mathcal{A}_2.$$

Compositions of cohomological correspondences are defined as in [Laf18, §4.1], the only modification being that one composes the pullbacks of the isomorphisms of the A-gerbes over the fiber product of algebraic k-stacks.

Example 8.5.7. Let Y be a separated k-scheme of finite type equipped with an A-gerbe $\mathcal{G}$. For any $\mathcal{A} \in \mathsf{D}_{\mathcal{G},\zeta}(\mathrm{Y})$, we have composable cohomological correspondences

$$\begin{aligned}(\operatorname{Spec}\mathsf{k}, \mathbf{0}, \overline{\mathbf{Q}}_\ell) &\to (\mathrm{Y}\times\mathrm{Y}, \mathcal{G}\boxplus(-\mathcal{G}), \mathcal{A}\boxtimes\mathbf{D}\mathcal{A}) \\ &\to (\mathrm{Y}\times\mathrm{Y}, \mathcal{G}\boxplus(-\mathcal{G}), \mathcal{A}\boxtimes\mathbf{D}\mathcal{A}) \to (\operatorname{Spec}\mathsf{k}, \mathbf{0}, \overline{\mathbf{Q}}_\ell).\end{aligned} \tag{8.53}$$

Here, $\mathbf{0}$ denotes the trivial A-gerbe over $\operatorname{Spec}\mathsf{k}$ and $\mathbf{D}$ is the Verdier duality over Y relative to k (*cf.* §B.1.10). The spans supporting them are represented in the following diagram

$$\begin{tikzcd}
& \mathrm{Y} \arrow[dl] \arrow[dr, "\Delta"'] & & \mathrm{Y}\times\mathrm{Y} \arrow[dl, "{(\mathrm{Fr}_\mathrm{Y},\mathrm{id})}"'] \arrow[dr, equal] & & \mathrm{Y} \arrow[dl, "\Delta"'] \arrow[dr] & \\
\operatorname{Spec}\mathsf{k} & & \mathrm{Y}\times\mathrm{Y} & & \mathrm{Y}\times\mathrm{Y} & & \operatorname{Spec}\mathsf{k}
\end{tikzcd} \tag{8.54}$$

The identifications of A-gerbes are given by the canonical trivialization of $\Delta^*(\mathcal{G}\boxplus(-\mathcal{G}))$, as well as the isomorphism $(\mathrm{Fr}_\mathrm{Y}, \mathrm{id})^*(\mathcal{G}\boxplus(-\mathcal{G})) \simeq \mathcal{G}\boxplus(-\mathcal{G})$ supplied by (7.16). The morphisms on constructible complexes are given by the unit of Verdier duality, the identity on $\mathcal{A}\boxtimes\mathbf{D}\mathcal{A}$, and the co-unit of Verdier duality, respectively.

Composing the spans (8.54), we obtain the span Y^{Fr} from $\operatorname{Spec} \mathsf{k}$ to itself. The composition of isomorphisms of A-gerbes yields an automorphism of the trivial A-gerbe over Y^{Fr}: By construction, *this is the* A-*torsor* $-\operatorname{Tr}(\mathrm{Fr} \mid \mathcal{G})$ (*cf.* §7.2.4). With respect to this automorphism, the composition of the morphisms of constructible complexes yields a ζ-genuine function on $\operatorname{Tr}(\mathrm{Fr} \mid \mathcal{G})(\mathsf{k})$, which equals $\operatorname{Tr}(\mathrm{Fr} \mid \mathcal{A})$ (*cf.* §7.2.8).

8.5.8. Let us return to the context of Proposition 8.5.5. Our goal is to reformulate the isomorphism (8.52) in terms of cohomological correspondences.

We begin with the endomorphism $\mathrm{T}(h_{\mathrm{V},x})$. Recall the stack $\mathrm{Hec}_x(\mathrm{Sht}^{\mathrm{I}}_{\mathrm{G,D}})$ of Hecke modifcations of $\mathrm{Sht}^{\mathrm{I}}_{\mathrm{G,D}}$ at x (*cf.* §8.3.5). Viewed as a span, $\mathrm{Hec}_x(\mathrm{Sht}^{\mathrm{I}}_{\mathrm{G,D}})$ supports a cohomological correspondence

$$(\mathrm{Sht}^{\mathrm{I}}_{\mathrm{G,D}}, \mathbf{0}, \mathcal{F}_{\mathrm{I,W}}) \to (\mathrm{Sht}^{\mathrm{I}}_{\mathrm{G,D}}, \mathbf{0}, \mathcal{F}_{\mathrm{I,W}}),$$

where the isomorphism of the A-gerbes is given by $r^*(-\operatorname{Tr}(\mathrm{Fr} \mid \mathcal{G}_{\mathrm{Hec}}))$ and the morphism of constructible complexes is given by (8.33), which we may rewrite as

$$p^*\mathcal{F}_{\mathrm{I,W}} \otimes r^*(\mathcal{L}_{\operatorname{Tr}(\mathrm{Fr}|\mathcal{G}_{\mathrm{Hec}_{\mathrm{G}}}),\zeta})^{\otimes -1} \to q^!\mathcal{F}_{\mathrm{I,W}}. \tag{8.55}$$

The element $h_{\mathrm{V},x}$ may be viewed as a global section of $(\mathcal{L}_{\operatorname{Tr}(\mathrm{Fr}|\mathcal{G}_{\mathrm{Hec}_{\mathrm{G}}}),\zeta})^{\otimes -1}$ over $\mathrm{Hec}_{\mathrm{G},x}(\mathsf{k})$, so by evaluating (8.55) on $h_{\mathrm{V},x}$ along its second factor, we obtain a morphism

$$\widetilde{\mathrm{T}}(h_{\mathrm{V},x}) : p^*\mathcal{F}_{\mathrm{I,W}} \to q^!\mathcal{F}_{\mathrm{I,W}}. \tag{8.56}$$

The morphism (8.56) gives rise to $\mathrm{T}(h_{\mathrm{V},x})$ by adjunction and compactly supported cohomology along $\nu_{\Xi}^{\mathrm{I}_1,\cdots,\mathrm{I}_n}$.

8.5.9. As for $\mathrm{S}_{\mathrm{V},x}$, we shall consider the spans defined in [Laf18, §6.3]:

$$\begin{tikzcd}[column sep=small]
& \mathcal{Y}_\sharp \arrow[dl, "p_\sharp"'] \arrow[dr, hook, "i_\sharp"] & & \mathrm{Sht}^{\{1\},\{2\},\mathrm{I}}_{\mathrm{G,D}}|_{\mathring{\mathrm{X}}^{\mathrm{I}} \times \Delta(x)} \arrow[dl, "\mathrm{F}^{\deg x}_{\{1\}}"'] \arrow[dr, equal] & & \mathcal{Y}_\flat \arrow[dl, hook', "i_\flat"'] \arrow[dr, "p_\flat"] & \\
\mathrm{Sht}^{\mathrm{I}}_{\mathrm{G,D}} \times x & & \mathrm{Sht}^{\mathrm{I},\{2\},\{1\}}_{\mathrm{G,D}}|_{\mathring{\mathrm{X}}^{\mathrm{I}} \times \Delta(x)} & & \mathrm{Sht}^{\{1\},\{2\},\mathrm{I}}_{\mathrm{G,D}}|_{\mathring{\mathrm{X}}^{\mathrm{I}} \times \Delta(x)} & & \mathrm{Sht}^{\mathrm{I}}_{\mathrm{G,D}} \times x
\end{tikzcd} \tag{8.57}$$

Here, $\mathcal{Y}_\sharp$ is the closed substack of $\mathrm{Sht}^{\mathrm{I},\{2\},\{1\}}_{\mathrm{G,D}}|_{\mathring{\mathrm{X}}^{\mathrm{I}} \times \Delta(x)}$ consisting of R-points

$$(\mathrm{P}^0, \phi^0) \overset{x^{\mathrm{I}}}{\sim} (\mathrm{P}^1, \phi^1) \overset{x}{\sim} (\mathrm{P}^2, \phi^2) \overset{x}{\sim} ({}^\tau\mathrm{P}^0, {}^\tau\phi^0)$$

where the composition of the two last modifications extends to an isomorphism $(\mathrm{P}^1, \phi^1) \simeq ({}^\tau\mathrm{P}^0, {}^\tau\phi^0)$ of R-points of $\mathrm{Bun}_{\mathrm{G,D}}$. The closed substack $\mathcal{Y}_\flat$ of $\mathrm{Sht}^{\{1\},\{2\},\mathrm{I}}_{\mathrm{G,D}}|_{\mathring{\mathrm{X}}^{\mathrm{I}} \times \Delta(x)}$ is similarly defined. The morphism $\mathrm{F}^{\deg x}_{\{1\}}$ is the $(\deg x)$-fold iteration of the lift of the partial Frobenius endomorphism along $\{1\}$ (*cf.* §8.4.2).[18]

According to [Laf18, Lemme 6.10], the composition of the spans (8.57) is canonically isomorphic to $\mathrm{Hec}_x(\mathrm{Sht}^{\mathrm{I}}_{\mathrm{G,D}}) \times x$, *i.e.* we have a canonical isomorphism

$$\mathcal{Y}_\sharp \underset{\mathrm{Sht}^{\mathrm{I},\{2\},\{1\}}_{\mathrm{G,D}}|_{\mathring{\mathrm{X}}^{\mathrm{I}} \times \Delta x}}{\times} \mathrm{Sht}^{\{1\},\{2\},\mathrm{I}}_{\mathrm{G,D}}|_{\mathring{\mathrm{X}}^{\mathrm{I}} \times \Delta(x)} \underset{\mathrm{Sht}^{\{1\},\{2\},\mathrm{I}}_{\mathrm{G,D}}|_{\mathring{\mathrm{X}}^{\mathrm{I}} \times \Delta(x)}}{\times} \mathcal{Y}_\flat \simeq \mathrm{Hec}_x(\mathrm{Sht}^{\mathrm{I}}_{\mathrm{G,D}}) \times x, \tag{8.58}$$

where the structural morphisms p, q correspond to the projection onto $\mathcal{Y}_\sharp$ followed by $p_\sharp$, respectively the projection onto $\mathcal{Y}_\flat$, followed by $p_\flat$.

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[18] The formation of $\mathrm{F}^{\deg x}_{\{1\}}$ requires a re-ordering of $\{1\}, \{2\}, \mathrm{I}$; see [Laf18, §6.3(c)] for details.

8.5.10. Let us now invoke the A-gerbes $\mathcal{G}_{\mathrm{Sht}_{\mathrm{G,D}}}$ over the various moduli stacks of Shtukas, temporarily ignoring their canonical trivializations (*cf.* §8.2.4). Note that we have a canonical identification of A-gerbes

$$(p_\sharp)^* \mathcal{G}_{\mathrm{Sht}^{\mathrm{I}}_{\mathrm{G,D}}} \simeq (i_\sharp)^* \mathcal{G}_{\mathrm{Sht}^{\mathrm{I},\{2\},\{1\}}_{\mathrm{G,D}}} \tag{8.59}$$

arising from the multiplicative structure on $\mathcal{G}_{\mathrm{Hec}_{\mathrm{G}}}$ (*cf.* Lemma 1.1.14).

We shall extend (8.59) to a cohomological correspondence supported on $\mathcal{Y}_\sharp$:

$$(\mathrm{Sht}^{\mathrm{I}}_{\mathrm{G,D}} \times x, \mathcal{G}_{\mathrm{Sht}_{\mathrm{G,D}}} \boxplus \mathbf{0}, \mathcal{F}_{\mathrm{I,W}}) \to (\mathrm{Sht}^{\mathrm{I},\{2\},\{1\}}_{\mathrm{G,D}}|_{\mathring{\mathrm{X}}^{\mathrm{I}} \times \Delta(x)}, \mathcal{G}_{\mathrm{Sht}_{\mathrm{G,D}}}, \mathcal{F}_{\mathrm{I},\{2\},\{1\},\mathrm{W} \boxtimes \mathrm{V}^* \boxtimes \mathrm{V}}). \tag{8.60}$$

Indeed, what remains to be constructed is a morphism of constructible complexes over $\mathcal{Y}_\sharp$ twisted by the A-gerbes (8.59):

$$(p_\sharp)^* \mathcal{F}_{\mathrm{I,W}} \to (i_\sharp)^! \mathcal{F}_{\mathrm{I},\{2\},\{1\},\mathrm{W} \boxtimes \mathrm{V}^* \boxtimes \mathrm{V}}. \tag{8.61}$$

The morphism (8.61) is the pullback of the unit morphism in the Satake category (*cf.* Proposition 1.6.4). To be more precise, we consider the Cartesian square

$$\begin{tikzcd}
\mathcal{Y}_\sharp \arrow[r, hook, "i_\sharp"] \arrow[d, "(r^{\mathrm{I}}, r^{\{2\}})"] & \mathrm{Sht}^{\mathrm{I},\{2\},\{1\}}_{\mathrm{G,D}}|_{\mathring{\mathrm{X}}^{\mathrm{I}} \times \Delta(x)} \arrow[d, "(r^{\mathrm{I}}, r^{\{2\},\{1\}})"] \\
\mathrm{Hec}_{\mathrm{G,I}} \times \mathrm{Hec}_{\mathrm{G},x} \arrow[r, hook, "(\mathrm{id}, \delta)"] & \mathrm{Hec}_{\mathrm{G,I}} \times \mathrm{Hec}^{\{2\},\{1\}}_{\mathrm{G},x}
\end{tikzcd} \tag{8.62}$$

where the vertical arrows are the products of restriction maps (*cf.* §8.2.6) and δ sends $\mathrm{P}^0 \overset{x}{\sim} \mathrm{P}^1$ to the concatenation with its own inverse $\mathrm{P}^0 \overset{x}{\sim} \mathrm{P}^1 \overset{x}{\sim} \mathrm{P}^0$. The unit morphism of the Satake category (1.58) (applied to the objects $\mathcal{S}_{\mathrm{V}}, \mathcal{S}_{\mathrm{V}^*} \in \mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},x})$ corresponding to V and V^*) yields a morphism in $\mathsf{D}(\mathrm{Hec}_{\mathrm{G},x})$:

$$\overline{\mathbf{Q}}_\ell \to \delta^! (p^*_{\{2\}} \mathcal{S}_{\mathrm{V}^*} \otimes p^*_{\{1\}} \mathcal{S}_{\mathrm{V}}), \tag{8.63}$$

where $p_{\{2\}}$, $p_{\{1\}}$ are the projections onto the first, respectively the second modification. The desired morphism (8.61) is the pullback along $(r^{\mathrm{I}}, r^{\{2\}})$ of the external tensor product of (8.63) with the identity on $\mathcal{S}_{\mathrm{I,W}} \in \mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G,I}})$. Here, we use the base change property for $*$- and !-pullbacks along (8.62), which holds by smoothness after passing to an ind-presentation of the local Hecke stacks (*cf.* Remark 8.2.7).

Let us now invoke the canonical trivialization of $\mathcal{G}_{\mathrm{Sht}_{\mathrm{G,D}}}$ (*cf.* §8.2.4). It is compatible with (8.59) in the following sense: After trivializing both $\mathcal{G}_{\mathrm{Sht}^{\mathrm{I}}_{\mathrm{G,D}}}$ and $\mathcal{G}_{\mathrm{Sht}^{\mathrm{I},\{2\},\{1\}}_{\mathrm{G,D}}}$, the isomorphism (8.59) becomes the *identity* automorphism of the trivial A-gerbe. In particular, we may view (8.61) as a morphism of *untwisted* constructible complexes over $\mathcal{Y}_\sharp$, and thus (8.60) as a cohomological correspondence in the usual sense:

$$(\mathrm{Sht}^{\mathrm{I}}_{\mathrm{G,D}} \times x, \mathbf{0}, \mathcal{F}_{\mathrm{I,W}}) \to (\mathrm{Sht}^{\mathrm{I},\{2\},\{1\}}_{\mathrm{G,D}}|_{\mathring{\mathrm{X}}^{\mathrm{I}} \times \Delta(x)}, \mathbf{0}, \mathcal{F}_{\mathrm{I},\{2\},\{1\},\mathrm{W} \boxtimes \mathrm{V}^* \boxtimes \mathrm{V}}), \tag{8.64}$$

which gives rise to the morphism labeled "unit" in (8.50).

Dually, we have the cohomological correspondence supported on $\mathcal{Y}_\flat$:

$$(\mathrm{Sht}^{\{1\},\{2\},\mathrm{I}}_{\mathrm{G,D}}|_{\mathring{\mathrm{X}}^{\mathrm{I}} \times \Delta(x)}, \mathbf{0}, \mathcal{F}_{\{1\},\{2\},\mathrm{I},\mathrm{V} \boxtimes \mathrm{V}^* \boxtimes \mathrm{W}}) \to (\mathrm{Sht}^{\mathrm{I}}_{\mathrm{G,D}} \times x, \mathbf{0}, \mathcal{F}_{\mathrm{I,W}}), \tag{8.65}$$

which gives rise to teh morphism "counit" in (8.50).

8.5.11. Finally, we have the cohomological correspondence from the target of (8.64) to the source of (8.65), supported on $\mathrm{Sht}_{\mathrm{G},\mathrm{D}}^{\{1\},\{2\},\mathrm{I}}|_{\mathring{\mathrm{X}}^{\mathrm{I}}\times\Delta(x)}$:

$$(\mathrm{Sht}_{\mathrm{G},\mathrm{D}}^{\mathrm{I},\{2\},\{1\}}|_{\mathring{\mathrm{X}}^{\mathrm{I}}\times\Delta(x)}, \mathbf{0}, \mathcal{F}_{\mathrm{I},\{2\},\{1\},\mathrm{W}\boxtimes\mathrm{V}^*\boxtimes\mathrm{V}}) \to (\mathrm{Sht}_{\mathrm{G},\mathrm{D}}^{\{1\},\{2\},\mathrm{I}}|_{\mathring{\mathrm{X}}^{\mathrm{I}}\times\Delta(x)}, \mathbf{0}, \mathcal{F}_{\{1\},\{2\},\mathrm{I},\mathrm{V}\boxtimes\mathrm{V}^*\boxtimes\mathrm{W}}) \tag{8.66}$$

defined by the $\mathrm{F}_{\{1\}}^*$-equivariance of $\mathcal{F}_{\{1\},\{2\},\mathrm{I},\mathrm{V}\boxtimes\mathrm{V}^*\boxtimes\mathrm{W}}$ (*cf.* §8.4.2).

The composition of the three cohomological correspondences (8.64), (8.66), (8.65) then corresponds to a morphism of (untwisted) constructible complexes over $\mathrm{Hec}_x(\mathrm{Sht}_{\mathrm{G},\mathrm{D}}^{\mathrm{I}})$:

$$\widetilde{\mathrm{S}}_{\mathrm{V},x} : p^*\mathcal{F}_{\mathrm{I},\mathrm{W}} \to q^!\mathcal{F}_{\mathrm{I},\mathrm{W}}, \tag{8.67}$$

which gives rise to $\mathrm{S}_{\mathrm{V},x}$ by adjunction and compactly supported cohomology.

Remark 8.5.12. Let us point out an aspect of the construction of (8.67) which may cause confusion. Indeed, (8.67) invokes the composition of three identifications of A-gerbes, after pulling back to $\mathrm{Hec}_x(\mathrm{Sht}_{\mathrm{G},\mathrm{D}}^{\mathrm{I}}) \times x$ along (8.58):

$$\begin{aligned}(p_\sharp)^*\mathcal{G}_{\mathrm{Sht}_{\mathrm{G},\mathrm{D}}^{\mathrm{I}}} &\simeq (i_\sharp)^*\mathcal{G}_{\mathrm{Sht}_{\mathrm{G},\mathrm{D}}^{\mathrm{I},\{2\},\{1\}}},\\ (\mathrm{F}_{\{1\}}^{\deg x})^*\mathcal{G}_{\mathrm{Sht}_{\mathrm{G},\mathrm{D}}^{\mathrm{I},\{2\},\{1\}}} &\simeq \mathcal{G}_{\mathrm{Sht}_{\mathrm{G},\mathrm{D}}^{\{1\},\{2\},\mathrm{I}}},\\ (i_\flat)^*\mathcal{G}_{\mathrm{Sht}_{\mathrm{G},\mathrm{D}}^{\{1\},\{2\},\mathrm{I}}} &\simeq (p_\flat)^*\mathcal{G}_{\mathrm{Sht}_{\mathrm{G},\mathrm{D}}^{\mathrm{I}}},\end{aligned}$$

where the first isomorphism is (8.59), the third isomorphism is similarly defined, and the second isomorphism comes from the $(\mathrm{F}_{\{1\}}^{\deg x})^*$-equivariance (*cf.* §8.4.3).

Their composition yields an isomorphism of A-gerbes

$$p^*\mathcal{G}_{\mathrm{Sht}_{\mathrm{G},\mathrm{D}}^{\mathrm{I}}} \simeq q^*\mathcal{G}_{\mathrm{Sht}_{\mathrm{G},\mathrm{D}}^{\mathrm{I}}}. \tag{8.68}$$

This isomorphism is *different* from the canonical isomorphism (8.26) defined over $(\mathring{\mathrm{X}} \smallsetminus x)^{\mathrm{I}} \times x$. Indeed, under the canonical trivialization of $\mathcal{G}_{\mathrm{Sht}_{\mathrm{G},\mathrm{D}}^{\mathrm{I}}}$ (*cf.* §8.2.4), (8.68) corresponds to the identity automorphism of the trivial A-gerbe while (8.26) corresponds to the multiplication by the pullback of $-\mathrm{Tr}(\mathrm{Fr} \mid \mathcal{G}_{\mathrm{Hec}_{\mathrm{G},x}})$ (*cf.* Lemma 8.3.4).

8.5.13. With the above constructions in place, we may prove Proposition 8.5.5 by repeating the proof of [Laf18, Proposition 6.2].

Proof of Proposition 8.5.5. It suffices to identify the morphisms of (untwisted) constructible complexes (8.56), (8.67) over $\mathrm{Hec}_x(\mathrm{Sht}_{\mathrm{G},\mathrm{D}}^{\mathrm{I}})$:

$$\widetilde{\mathrm{S}}_{\mathrm{V},x} \simeq \widetilde{\mathrm{T}}(h_{\mathrm{V},x}), \tag{8.69}$$

as it will give rise to (8.52) by taking compactly supported cohomology.

The isomorphism (8.69) is our version of [Laf18, Lemme 6.11], and is proved in the same way.[19] Namely, we may reduce this assertion to the "vacuum case" $\mathrm{I} = \varnothing$ and $\mathrm{W} = \mathbf{1}$ by rewriting the construction of (8.67) in terms local to x (*cf.* [Laf18, §6.4]). For the vacuum case, we argue as in the proof of [Laf18, Lemme 6.13], reducing to the fact that the morphism (8.61), which here is a morphism

$$\overline{\mathbf{Q}}_\ell \to (i_\sharp)^!\mathcal{F}_{\{2\},\{1\},\mathrm{V}^*\boxtimes\mathrm{V}}$$

---

[19] However, for the reason mentioned in Remark 8.5.12, it is not formulated as an identification of cohomological correspondences.

of (untwisted) constructible complexes over $\mathcal{Y}^\sharp$, is uniquely determined by its restriction to the smooth locus of $\mathcal{Y}^\sharp$ (*cf.* [Laf18, Lemme 6.15]). ☐

8.5.14. Next, we turn to the Eichler–Shimura relation for the partial Frobenius-equivariance structure on the cohomology of Shtukas.

More precisely, let I, W be as in §8.5.1, x be a closed point of $\mathring{\mathrm{X}}$, and $\mathrm{V} \in \mathsf{Rep}({}^{\mathrm{L}}\mathrm{H}_x)$. Consider the object $\mathcal{H}_{\mathrm{I}\sqcup\{0\},\mathrm{W}\boxtimes\mathrm{V}}$ of $\operatorname{Ind}\mathsf{D}(\mathring{\mathrm{X}}^{\mathrm{I}}\times x)$ equipped with its partial Frobenius-equivariance structure $\varphi_{\{0\}}$ along the x-factors (*cf.* Remark §8.4.7). Since $\mathrm{F}_{\{0\}}^{\deg x}$ is the identity on $\mathring{\mathrm{X}}^{\mathrm{I}}\times x$, we may view $\varphi_{\{0\}}^{\deg x}$ as an endomorphism

$$\varphi_{\{0\}}^{\deg x} : \mathcal{H}_{\mathrm{I}\sqcup\{0\},\mathrm{W}\boxtimes\mathrm{V}} \to \mathcal{H}_{\mathrm{I}\sqcup\{0\},\mathrm{W}\boxtimes\mathrm{V}}. \tag{8.70}$$

The Eicher–Shimura relation is a polynomial equation satisfied by (8.70) with coefficients in the operators $\mathrm{S}_{\wedge^{\dim\mathrm{V}-k}\mathrm{V},x}$ (in its local form (8.51) for the inclusion $\{0\}\subset\mathrm{I}\sqcup\{0\}$).

Proposition 8.5.15 (The Eichler–Shimura relation)**.** *There is an isomorphism of endomorphisms of* $\mathcal{H}_{\mathrm{I}\sqcup\{0\},\mathrm{W}\boxtimes\mathrm{V}}$*:*

$$\sum_{k=0}^{\dim\mathrm{V}}(-1)^k\mathrm{S}_{\wedge^{\dim\mathrm{V}-k}\mathrm{V},x}\circ(\varphi_{\{0\}}^{\deg x})^k \simeq 0. \tag{8.71}$$

Proof. This follows formally from the proof of [Laf18, Proposition 7.1]. ☐

8.5.16 *Hecke-finiteness***.** Next, we shall focus on the degree-0 part of the cohomology of Shtukas, *i.e.* the ind-constructible sheaf

$$\mathcal{H}^0_{\mathrm{I},\mathrm{V}} \coloneqq \mathrm{H}^0\mathcal{H}_{\mathrm{I},\mathrm{V}} \in \operatorname{Ind}\mathsf{D}(\mathring{\mathrm{X}}^{\mathrm{I}})^{\heartsuit}$$

naturally associated to any finite set I and $\mathrm{V}\in\mathsf{Rep}({}^{\mathrm{L}}\mathrm{H}_{\mathring{\mathrm{X}}})$. Recall that $\mathcal{H}^0_{\varnothing,\mathbf{1}}$ is canonically identified with the vector space of ζ-genuine automorphic forms (*cf.* §8.2.12).

Let $\overline{\eta}$ be a geometric point of $\mathring{\mathrm{X}}^{\mathrm{I}}$ lying over the generic point of each factor. We shall call an element of $\mathcal{H}^0_{\mathrm{I},\mathrm{V}}|_{\overline{\eta}}$ *Hecke-finite* if it belongs to a finite-dimensional $\overline{\mathbf{Q}}_\ell$-vector space which is stable under actions of the spherical Hecke algebra at any $x\in\mathring{\mathrm{X}}$ (*cf.* §8.3.5). Write

$$(\mathcal{H}^0_{\mathrm{I},\mathrm{V}}|_{\overline{\eta}})^{\mathrm{Hf}} \subset \mathcal{H}^0_{\mathrm{I},\mathrm{V}}|_{\overline{\eta}}$$

for the space of Hecke-finite elements.

As in [Laf18, Proposition 8.23], we identify Hecke-finite elements of $\mathcal{H}^0_{\varnothing,\mathbf{1}}$ with cuspidal ζ-genuine automorphic forms.

Proposition 8.5.17. *An element of* $\mathcal{H}^0_{\varnothing,\mathbf{1}}$ *is Hecke-finite if and only if the corresponding* ζ*-genuine automorphic form is cuspidal.*

Proof. The fact that cusp forms are Hecke-finite follows from Lemma 7.3.13. For the opposite containment, one applies the proof of [Laf18, Lemma 8.25] to each summand in (7.34). ☐

8.5.18. For any finite set I, we write η_{I} for the generic point of X^{I}. Choose a geometric point $\overline{\eta}_{\mathrm{I}}$ lying over η_{I}, together with a lift of $\overline{\eta}_{\mathrm{I}}\to\mathrm{X}^{\mathrm{I}}$ to the Henselian local ring of X^{I} along $\Delta(\overline{\eta})$ (called a *specialization map* $\mathrm{sp}:\overline{\eta}_{\mathrm{I}}\rightsquigarrow\Delta(\overline{\eta})$).

As in [Laf18, Proposition 8.27] (whose proof uses the "S = T" identity and the Eichler–Shimura relation, for which we have the substitutes Proposition 8.5.5, Proposition 8.5.15), we have a presentation

$$(\mathcal{H}^0_{\mathrm{I},\mathrm{V}}|_{\overline{\eta}_{\mathrm{I}}})^{\mathrm{Hf}} \simeq \bigcup_k \mathrm{M}_k$$

where each M_k is a finite-dimensional $\overline{\mathbf{Q}}_\ell$-vector space on which the $\pi_1^{\text{ét}}(\eta_{\mathrm{I}}, \bar{\eta}_{\mathrm{I}})$-action factors through $\pi_1^{\text{ét}}(\eta, \bar{\eta})^{\mathrm{I}}$ via the partial Frobenius-equivariance structure. Likewise, by [Laf18, Corollaire 8.34], the specialization map induces an isomorphism

$$\mathrm{sp}^* : (\mathcal{H}^0_{\mathrm{I,V}}|_{\Delta(\bar{\eta})})^{\mathrm{Hf}} \simeq (\mathcal{H}^0_{\mathrm{I,V}}|_{\bar{\eta}_{\mathrm{I}}})^{\mathrm{Hf}}. \tag{8.72}$$

Remark 8.5.19. Xue's work [Xue20] yields a stronger result: The ind-constructible sheaf $\mathcal{H}^k_{\mathrm{I,V}} := \mathrm{H}^k\mathcal{H}_{\mathrm{I,V}}$ is ind-lisse for any $k \in \mathbf{Z}$. Indeed, this is established in the untwisted setting in [Xue20, Theorem 4.2.3], but the argument is general: It uses the family of functors (8.46), the "S = T" identity, and the Eichler–Shimura relation.

In particular, we may realize (8.72) as the Hecke-finite part of the isomorphism

$$\mathrm{sp}^* : \mathcal{H}^0_{\mathrm{I,V}}|_{\Delta(\bar{\eta})} \simeq \mathcal{H}^0_{\mathrm{I,V}}|_{\bar{\eta}_{\mathrm{I}}}, \tag{8.73}$$

although (8.73) (for arbitrary degree $k \in \mathbf{Z}$) is established as a step in the proof that $\mathcal{H}^k_{\mathrm{I,V}}$ is ind-lisse (*cf.* [Xue20, Proposition 1.4.3]).

8.5.20. We are now ready to construct the spectral decomposition of ζ-genuine cusp forms (8.4) by repeating [Laf18, §12.3.4].

Proof of Theorem 8.1.6. For each finite set I and $\mathrm{V} \in \mathsf{Rep}((^{\mathrm{L}}\mathrm{H}_{\mathring{\mathrm{X}}})^{\mathrm{I}})$, we denote by $\mathrm{H}_{\mathrm{I,V}}$ the left-hand-side of (8.72), equipped with the continuous $\mathrm{Gal}(\bar{\mathrm{F}}/\mathrm{F})^{\mathrm{I}}$-action induced from the right-hand-side. This defines a family of functors

$$\begin{gathered}\mathsf{Rep}((^{\mathrm{L}}\mathrm{H}_{\mathring{\mathrm{X}}})^{\mathrm{I}}) \to \mathsf{Rep}(\mathrm{Gal}(\bar{\mathrm{F}}/\mathrm{F})^{\mathrm{I}}), \\ \mathrm{V} \mapsto \mathrm{H}_{\mathrm{I,V}},\end{gathered} \tag{8.74}$$

natural in I, together with a canonical identification (*cf.* Proposition 8.5.17)

$$\mathrm{H}_{\varnothing,\mathbf{1}} \simeq \mathrm{Fun}_{\mathrm{cusp},\zeta}(\widetilde{\mathrm{Bun}}_{\mathrm{G,D}}/\Xi, \overline{\mathbf{Q}}_\ell). \tag{8.75}$$

Note that the naturality of (8.74) in I implies that $\mathrm{H}_{\mathrm{I},\mathbf{1}}$ is the vector space (8.75) equipped with the trivial $\mathrm{Gal}(\bar{\mathrm{F}}/\mathrm{F})^{\mathrm{I}}$-action.

Given $x \in \mathrm{V}$ and $\xi \in \mathrm{V}^*$ which are $\mathrm{H}_{\bar{\eta}}$-invariant under the diagonal action, and an I-tuple $\gamma^{\mathrm{I}} \in \mathrm{Gal}(\bar{\mathrm{F}}/\mathrm{F})^{\mathrm{I}}$, we obtain an endomorphism $\mathrm{S}_{\mathrm{I,V},x,\xi,\gamma^{\mathrm{I}}}$ of $\mathrm{H}_{\{0\},\mathbf{1}}$ as the composition

$$\begin{aligned}&\mathrm{H}_{\{0\},\mathbf{1}} \xrightarrow{x} \mathrm{H}_{\{0\},\mathrm{V}^{\mathrm{H}_{\bar{\eta}}}} \to \mathrm{H}_{\{0\},\mathrm{V}} \simeq \mathrm{H}_{\mathrm{I,V}} \\ &\qquad\qquad \xrightarrow{\gamma^{\mathrm{I}}} \mathrm{H}_{\mathrm{I,V}} \simeq \mathrm{H}_{\{0\},\mathrm{V}} \to \mathrm{H}_{\{0\},\mathrm{V}_{\mathrm{H}_{\bar{\eta}}}} \xrightarrow{\xi} \mathrm{H}_{\{0\},\mathbf{1}}.\end{aligned}$$

where the isomorphisms are induced from the naturality of (8.74) with respect to the constant map $\mathrm{I} \to \{0\}$, and $\mathrm{V}^{\mathrm{H}_{\bar{\eta}}}$ (respectively $\mathrm{V}_{\mathrm{H}_{\bar{\eta}}}$) denotes the space of $\mathrm{H}_{\bar{\eta}}$-invariants (respectively, $\mathrm{H}_{\bar{\eta}}$-coinvariants) of V with respect to the diagonal action.

Under the identification of $\mathrm{H}_{\{0\},\mathbf{1}}$ with (8.75), we may view each $\mathrm{S}_{\mathrm{I,V},x,\xi,\gamma^{\mathrm{I}}}$ as an endomorphisms of $\mathrm{Fun}_{\mathrm{cusp},\zeta}(\widetilde{\mathrm{Bun}}_{\mathrm{G,D}}/\Xi, \overline{\mathbf{Q}}_\ell)$. The proof of [Laf18, Lemme 10.6] shows that $\mathrm{S}_{\mathrm{I,V},x,\xi,\gamma^{\mathrm{I}}}$ depends only on I, γ^{I}, and the function

$$\begin{gathered}\mathrm{H}_{\bar{\eta}}\backslash(^{\mathrm{L}}\mathrm{H}_{\mathring{\mathrm{X}}})^{\mathrm{I}}/\mathrm{H}_{\bar{\eta}} \to \overline{\mathbf{Q}}_\ell \\ h^{\mathrm{I}} \mapsto \langle \xi, h^{\mathrm{I}} \cdot x \rangle,\end{gathered} \tag{8.76}$$

so one may define $\mathrm{S}_{\mathrm{I},f,\gamma^{\mathrm{I}}}$, for any locally constant $\mathrm{H}_{\bar{\eta}}$-bi-invariant function f on $^{\mathrm{L}}\mathrm{H}_{\mathring{\mathrm{X}}}$, to be $\mathrm{S}_{\mathrm{I,V},x,\xi,\gamma^{\mathrm{I}}}$ for any V, x, ξ whose induced function (8.76) equals f. We refer to $\mathrm{S}_{\mathrm{I},f,\gamma^{\mathrm{I}}}$ as the *excursion operator* associated to I, f, and γ^{I}. They form a mutually commuting family of endomorphisms of $\mathrm{Fun}_{\mathrm{cusp},\zeta}(\widetilde{\mathrm{Bun}}_{\mathrm{G,D}}/\Xi, \overline{\mathbf{Q}}_\ell)$, commuting with the Hecke operators at each $x \in \mathring{\mathrm{X}}$ (*cf.* [Laf18, Lemme 10.1, Lemme 10.2]). Furthermore, $\mathrm{S}_{\mathrm{I},f,\gamma^{\mathrm{I}}}$ depends only on the image of γ^{I} in $\pi_1^{\text{ét}}(\mathring{\mathrm{X}}, \overline{\eta})^{\mathrm{I}}$ (*cf.* [Laf18, Proposition 10.10]).

Denote by $\mathcal{B}$ the subalgebra of $\mathrm{End}\,\mathrm{Fun}_{\mathrm{cusp},\zeta}(\widetilde{\mathrm{Bun}}_{\mathrm{G,D}}/\Xi, \overline{\mathbf{Q}}_\ell)$ generated by the excursion operators. It follows from Lemma 7.3.13 that $\mathcal{B}$ is Artinian. Consider the decomposition of (8.75) into generalized $\mathcal{B}$-eigenspaces:

$$\mathrm{Fun}_{\mathrm{cusp},\zeta}(\widetilde{\mathrm{Bun}}_{\mathrm{G,D}}/\Xi, \overline{\mathbf{Q}}_\ell) \simeq \bigoplus_{\chi:\mathcal{B}\to\overline{\mathbf{Q}}_\ell} \mathbf{H}_{\mathrm{D},\chi} \tag{8.77}$$

The desired decomposition (8.4) is defined by coarsening (8.77), using a map $\chi \mapsto [\sigma]$ from the set of characters of $\mathcal{B}$ to the set of L-parameters obtained from invariant theory. Namely, assigning the function $\gamma^{\mathrm{I}} \mapsto \mathrm{S}_{\mathrm{I},f,\gamma^{\mathrm{I}}}$ to f defines a map

$$\mathrm{C}^0(\mathrm{H}_{\overline{\eta}}\backslash({}^{\mathrm{L}}\mathrm{H}_{\mathring{\mathrm{X}}})^{\mathrm{I}}/\mathrm{H}_{\overline{\eta}}, \overline{\mathbf{Q}}_\ell) \to \mathrm{C}^0(\pi_1^{\text{ét}}(\mathring{\mathrm{X}}, \overline{\eta})^{\mathrm{I}}, \mathcal{B}) \tag{8.78}$$

where $\mathcal{B}$ is endowed with the ℓ-adic topology. Thus, any character $\chi : \mathcal{B} \to \overline{\mathbf{Q}}_\ell$ determines a continuous pseudo-representation of $\pi_1^{\text{ét}}(\mathring{\mathrm{X}}, \overline{\eta})$ with values in ${}^{\mathrm{L}}\mathrm{H}_{\mathring{\mathrm{X}}}$ by composition with (8.78) (*cf.* [Laf18, Définition-Proposition 11.3]), which corresponds to a semisimple L-parameter $[\sigma]$ by [Laf18, Proposition 11.7].

The L-parameters arising this way are defined over finite extensions of $\mathbf{Q}_\ell$. The Hecke eigen-property of $\mathbf{H}_{\mathrm{D},[\sigma]}$ follows from the generalized eigen-property of $\mathbf{H}_{\mathrm{D},\chi}$ with respect to $\mathcal{B}$ and the fact that Hecke operators are diagonalizable (*cf.* [Laf18, Remarque 11.6]). □

Remark 8.5.21. At least for split G, one can use [Xue20] to streamline the proof of Theorem 8.1.6 as follows. We define the *cuspidal subsheaf* $\mathcal{H}^0_{\mathrm{I,V,cusp}} \subset \mathcal{H}^0_{\mathrm{I,V}}$ as in [Xue20, Definition 7.0.3]. Then by [Xue20, Proposition 7.0.5] (*cf.* Remark 8.5.19), $\mathcal{H}^0_{\mathrm{I,V,cusp}}$ is a lisse $\overline{\mathbf{Q}}_\ell$-sheaf over $\mathring{\mathrm{X}}^{\mathrm{I}}$. By [Xue20, Proposition 1.3.4], its $\pi_1^{\text{ét}}(\mathring{\mathrm{X}}^{\mathrm{I}}, \overline{\eta}_{\mathrm{I}})$-action factors through $\pi_1^{\text{ét}}(\mathring{\mathrm{X}}, \overline{\eta})^{\mathrm{I}}$, so by taking fibers at $\overline{\eta}^{\mathrm{I}}$, we obtain a family of functors

$$\begin{gathered}\mathsf{Rep}(({}^{\mathrm{L}}\mathrm{H}_{\mathring{\mathrm{X}}})^{\mathrm{I}}) \to \mathsf{Rep}(\pi_1^{\text{ét}}(\mathring{\mathrm{X}}, \overline{\eta})^{\mathrm{I}}), \\ \mathrm{V} \mapsto \mathrm{H}_{\mathrm{I,V}} := \mathcal{H}^0_{\mathrm{I,V,cusp}}|_{\overline{\eta}^{\mathrm{I}}},\end{gathered} \tag{8.79}$$

natural in I. The analogue of (8.75) is now tautological. The rest of the proof of Theorem 8.1.6 proceeds as in §8.5.20.

Part 3. Appendices

Appendix A. Twisting ∞-categories

The goal of this section is to define twisted sections of a sheaf of ∞-categories, which is used both for the definition of twisted ℓ-adic constructible complexes (*cf.* §B.1.3) and for twisting the category of H-representations (*cf.* §2.2).

The material of §A.1 is similar to [GL18, §1.7], but we supply more details than *loc.cit.* on matters concerning homotopy coherence.

A.1. Construction.

A.1.1. Let X be a site. Let e be a ring and $\mathsf{Perf}_{\mathsf{e}}$ be the ∞-category of perfect complexes of e-modules. Let $\mathcal{C}$ be a sheaf of (small) Karoubi $\mathsf{Perf}_{\mathsf{e}}$-module categories over X.

Write $\mathcal{X}$ for the ∞-category of Spc-valued sheaves over X. Write $\mathsf{e}^\times$ for the group of units of e and $\mathsf{B}^2\mathsf{e}^\times$ its double deloop as an object of $\mathcal{X}$.

In this subsection, we shall explain that given a morphism $\mathcal{G} : x \to \mathsf{B}^2\mathsf{e}^\times$ in $\mathcal{X}$, we obtain a sheaf $\mathcal{C}_{\mathcal{G}}$ of $\mathsf{Perf}_{\mathsf{e}}$-module categories over the slice site $\mathrm{X}_{/x}$ functorial in $(x, \mathcal{G})$. Objects of its global section $\Gamma(x, \mathcal{C}_{\mathcal{G}})$ are called $\mathcal{G}$-*twisted sections of* $\mathcal{C}$ *over* x.

Remark A.1.2. Informally, an object of $\Gamma(x, \mathcal{C}_{\mathcal{G}})$ is a compatible family of objects $c_{a,\tau_a} \in \Gamma(a, \mathcal{C})$ for every $a \in \mathrm{X}_{/x}$ equipped with a neutralization τ_a of the restriction $\mathcal{G}_a \in \Gamma(a, \mathsf{B}^2\mathsf{e}^\times)$ of $\mathcal{G}$, such that for any two neutralizations τ_a, τ'_a related by $\tau'_a \simeq \tau_a \cdot \ell_a$ for some $\ell_a \in */\mathsf{e}^\times$ (which always exists locally on a), there is an isomorphism

$$c_{a,\tau'_a} \simeq c_{a,\tau_a} \otimes \ell_a,$$

where ℓ_a is viewed as a free e-module of rank 1.

A.1.3. In order to construct $\mathcal{C}_{\mathcal{G}}$, we need to explain how to glue sheaves in a homotopy-coherent manner. This involves a bit of formalism.

Let $\mathcal{O}$ be an ∞-category admitting limits. Denote by $\mathsf{Shv}(\mathrm{X}, \mathcal{O})$ the ∞-category of $\mathcal{O}$-valued sheaves over X (*cf.* [Lur18, Definition 1.3.1.1]). Tautologically, we have

$$\mathcal{X} \simeq \mathsf{Shv}(\mathrm{X}, \mathsf{Spc}).$$

Given an ∞-topos $\mathcal{Y}$, we write $\mathsf{Shv}(\mathcal{Y}, \mathcal{O})$ for the ∞-category of $\mathcal{O}$-valued sheaves over $\mathcal{Y}$, *i.e.* limit-preserving functors $\mathcal{Y}^{\mathrm{op}} \to \mathcal{O}$. By [Lur18, Proposition 1.3.1.7], restriction along the sheafification of the Yoneda embedding determines an equivalence of ∞-categories:

$$\mathsf{Shv}(\mathcal{X}, \mathcal{O}) \simeq \mathsf{Shv}(\mathrm{X}, \mathcal{O}).$$

Note that if $\mathcal{O}$ is presentable, then $\mathsf{Shv}(\mathcal{X}, \mathcal{O})$ is canonically identified with the Lurie tensor product $\mathcal{X} \otimes \mathcal{O}$ (*cf.* [Lur17, Proposition 4.8.1.17]).

A.1.4. Denote by Pr^{L} (respectively, Pr^{R}) the ∞-category of presentable ∞-categories with colimit-preserving functors (respectively, limit-preserving functors).

Given a presentable ∞-category $\mathcal{O}$, we shall construct a functor

$$\mathsf{Shv}(\mathcal{X}_{/(\cdot)}, \mathcal{O}) : \mathcal{X}^{\mathrm{op}} \to \mathsf{Pr}^{\mathrm{R}}, \tag{A.1}$$

which assigns to $x \in \mathcal{X}^{\mathrm{op}}$ the ∞-category $\mathsf{Shv}(\mathcal{X}_{/x}, \mathcal{O})$.

Indeed, the co-Cartesian fibration $\mathrm{ev}_1 : \mathrm{Fun}(\Delta^1, \mathcal{X}) \to \mathcal{X}$ defines a functor

$$\mathcal{X} \to \mathsf{Pr}^{\mathrm{L}}, \quad x \mapsto \mathcal{X}_{x/} \tag{A.2}$$

Its composition with the endofunctor $(\cdot)\otimes\mathcal{O}$ of Pr^{L} yields a functor

$$\mathcal{X}\to\mathsf{Pr}^{\mathrm{L}},\quad x\mapsto\mathcal{X}_{/x}\otimes\mathcal{O}\simeq\mathsf{Shv}(\mathcal{X}_{/x},\mathcal{O}) \tag{A.3}$$

The functor (A.1) is defined to be the opposite of (A.3), using the canonical equivalence $(\mathsf{Pr}^{\mathrm{L}})^{\mathrm{op}}\simeq\mathsf{Pr}^{\mathrm{R}}$ (*cf.* [Lur09, Corollary 5.5.3.4]).

The following assertion can be viewed as a precise version of the statement that the ∞-category of sheaves itself satisfies gluing. Its proof is contained in [Lur18, Remark 2.1.0.5], but we reproduce the argument for the reader's conveninence.

Lemma A.1.5. *The functor* (A.1) *preserves limits.*

Proof. It suffices to show that (A.3) preserves colimits. Since the endofunctor $(\cdot)\otimes\mathcal{O}$ of Pr^{L} preserves colimits, it is enough to prove that (A.2) preserves colimits. Passing again to the opposite ∞-categories, we obtain the functor

$$\mathcal{X}^{\mathrm{op}}\to\mathsf{Pr}^{\mathrm{R}},\quad x\mapsto\mathcal{X}_{x/} \tag{A.4}$$

classifying $\mathrm{ev}_1:\mathrm{Fun}(\Delta^1,\mathcal{X})\to\mathcal{X}$ as a *Cartesian* fibration.

Since limits in Pr^{L} and Pr^{R} are both computed by limits in Cat, the fact that (A.4) preserves limits follows from [Lur09, Theorem 6.1.3.9]. ☐

A.1.6. We now specialize to the case $\mathcal{O}:=\mathsf{Perf}_{\mathsf{e}}\text{-}\mathsf{Mod}$, the ∞-category of (small) Karoubi $\mathsf{Perf}_{\mathsf{e}}$-module ∞-categories.

Given a $\mathsf{Perf}_{\mathsf{e}}\text{-}\mathsf{Mod}$-valued sheaf $\mathcal{C}$ on X, we obtain a $\mathsf{Perf}_{\mathsf{e}}\text{-}\mathsf{Mod}$-valued sheaf over the ∞-topos $\mathcal{X}$ (*cf.* §A.1.3), and consequently a morphism in $\mathsf{Shv}(\mathcal{X},\mathsf{Pr}^{\mathrm{R}})$:

$$\mathcal{C}:*\to\mathsf{Shv}(\mathcal{X}_{/(\cdot)},\mathsf{Perf}_{\mathsf{e}}\text{-}\mathsf{Mod}). \tag{A.5}$$

Here, the target is an object of $\mathsf{Shv}(\mathcal{X},\mathsf{Pr}^{\mathrm{R}})$ thanks to Lemma A.1.5. Let us view (A.5) as a morphism in $\mathsf{Shv}(\mathcal{X},\mathsf{Cat})$ via the forgetful functor $\mathsf{Pr}^{\mathrm{R}}\to\mathsf{Cat}$.

Proposition A.1.7. *The morphism* (A.5) *canonically extends to*

$$\widetilde{\mathcal{C}}:\mathsf{B}^2\mathsf{e}^{\times}\to\mathsf{Shv}(\mathcal{X}_{/(\cdot)},\mathsf{Perf}_{\mathsf{e}}\text{-}\mathsf{Mod}) \tag{A.6}$$

as a morphism in $\mathsf{Shv}(\mathcal{X},\mathsf{Cat})$.

A.1.8. In order to construct the extension (A.6), we note that $\mathsf{Perf}_{\mathsf{e}}$ admits a commutative algebra structure. Consequently, the ∞-category $\mathsf{Perf}_{\mathsf{e}}\text{-}\mathsf{Mod}$ admits a symmetric monoidal structure, given by Lurie tensor product of the ind-completed ∞-categories.[20]

The tensor product functor on $\mathsf{Perf}_{\mathsf{e}}\text{-}\mathsf{Mod}$ gives rise to a functor

$$\mathsf{Perf}_{\mathsf{e}}\text{-}\mathsf{Mod}\to\mathrm{End}(\mathsf{Perf}_{\mathsf{e}}\text{-}\mathsf{Mod}),\quad \mathrm{M}\mapsto(\cdot)\otimes_{\mathsf{Perf}_{\mathsf{e}}}\mathrm{M}$$

by adjunction. It sends $\mathsf{Perf}_{\mathsf{e}}$ to $\mathrm{id}_{\mathsf{Perf}_{\mathsf{e}}\text{-}\mathsf{Mod}}$. By taking endomorphisms, we obtain a monoidal morphism

$$\mathsf{Perf}_{\mathsf{e}}\to\mathrm{End}(\mathrm{id}_{\mathsf{Perf}_{\mathsf{e}}}\text{-}\mathsf{Mod}). \tag{A.7}$$

Remark A.1.9. The $\mathbb{E}_2$-monoidal structure on $\mathsf{Perf}_{\mathsf{e}}$ suffices for the construction of (A.7). However, the associative algebra structure does *not* suffice: Informally, (A.7) sends $\mathrm{A}\in\mathsf{Perf}_{\mathsf{e}}$ to the compatible family of endomorphisms of M, for each $\mathrm{M}\in\mathsf{Perf}_{\mathsf{e}}\text{-}\mathsf{Mod}$, given by multiplication by A, which is only $\mathsf{Perf}_{\mathsf{e}}$-linear by its $\mathbb{E}_2$-monoidal structure.

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[20] The functor of ind-completion realizes $\mathsf{Perf}_{\mathsf{e}}\text{-}\mathsf{Mod}$ as the ∞-category of e-linear presentable ∞-categories with functors preserving colimits and compact objects.

A.1.10. We are now ready to construct the morphism (A.6).

Proof of Proposition A.1.7. Since $\mathsf{B}^2\mathsf{e}^\times$ is the constant sheaf associated to the double deloop $\mathrm{K}(\mathsf{e}^\times, 2)$ in the ∞-category Spc, it suffices to construct a morphism in Cat

$$\mathrm{K}(\mathsf{e}^\times, 2) \to \mathsf{Shv}(\mathcal{X}, \mathsf{Perf}_\mathsf{e}\text{-}\mathsf{Mod}) \tag{A.8}$$

sending the neutral point to $\mathcal{C}$.

The pointed morphism (A.8) is equivalent to a monoidal morphism $*/\mathsf{e}^\times \to \mathrm{End}(\mathcal{C})$, where the endomorphism monoid is computed in the ∞-category $\mathrm{Fun}(\mathcal{X}^{\mathrm{op}}, \mathsf{Perf}_\mathsf{e}\text{-}\mathsf{Mod})$. The desired monoidal morphism is given by the composition

$$*/\mathsf{e}^\times \subset \mathsf{Perf}_\mathsf{e} \to \mathrm{End}(\mathrm{id}_{\mathsf{Perf}_\mathsf{e}\text{-}\mathsf{Mod}}) \to \mathrm{End}(\mathcal{C}),$$

where $*/\mathsf{e}^\times$ is identified as the full subcategory of Perf_e spanned by e, the middle morphism is (A.7), while the last morphism is defined by the functor

$$(\cdot) \circ \mathcal{C} : \mathrm{End}(\mathsf{Perf}_\mathsf{e}\text{-}\mathsf{Mod}) \to \mathrm{Fun}(\mathcal{X}^{\mathrm{op}}, \mathsf{Perf}_\mathsf{e}\text{-}\mathsf{Mod})$$

by evaluating on the endomorphisms of the object $\mathrm{id}_{\mathsf{Perf}_\mathsf{e}\text{-}\mathsf{Mod}}$. □

A.1.11 *Construction of* $\mathcal{C}_\mathcal{G}$**.** Let us now fix $x \in \mathcal{X}$ and a morphism $\mathcal{G} : x \to \mathsf{B}^2\mathsf{e}^\times$. Having (A.6) at our disposal, we define $\mathcal{C}_\mathcal{G}$ to be the image of $\mathcal{G}$ under (A.6):

$$\mathcal{C}_\mathcal{G} := \Gamma(x, \widetilde{\mathcal{C}})(\mathcal{G})$$

By construction, $\mathcal{C}_\mathcal{G}$ is an object of $\mathsf{Shv}(\mathcal{X}_{/x}, \mathsf{Perf}_\mathsf{e}\text{-}\mathsf{Mod})$. We obtain a sheaf on the slice site $\mathrm{X}_{/x}$ by restriction along the sheafification of the Yoneda embedding (*cf.* §A.1.3).

A.2. Monoidality.

A.2.1. We shall now introduce a variant of the twisting construction for sheaves of *symmetric monoidal* categories, needed for the definition of "twisted H-representations" (*cf.* §2.2).

Let $\mathcal{X}$ be an ∞-topos and $x \in \mathcal{X}$. Then $\mathsf{Shv}(\mathcal{X}_{/x}, \mathsf{Perf}_\mathsf{e}\text{-}\mathsf{Mod})$ inherits a symmetric monoidal structure from that of $\mathsf{Perf}_\mathsf{e}\text{-}\mathsf{Mod}$, functorially in x.

The construction $\mathcal{G}, \mathcal{C} \mapsto \mathcal{C}_\mathcal{G}$ (*cf.* §A.1.11) may be regarded as a functor

$$\Gamma(x, \mathsf{B}^2\mathsf{e}^\times) \times \mathsf{Shv}(\mathcal{X}, \mathsf{Perf}_\mathsf{e}\text{-}\mathsf{Mod}) \to \mathsf{Shv}(\mathcal{X}_{/x}, \mathsf{Perf}_\mathsf{e}\text{-}\mathsf{Mod}). \tag{A.9}$$

Lemma A.2.2. *The functor* (A.9) *is naturally symmetric monoidal.*

Proof. By construction, (A.9) is induced via descent from the functor

$$\mathrm{K}(\mathsf{e}^\times, 2) \times \mathsf{Perf}_\mathsf{e}\text{-}\mathsf{Mod} \to \mathsf{Perf}_\mathsf{e}\text{-}\mathsf{Mod} \tag{A.10}$$

adjoint to the deloop of $*/\mathsf{e}^\times \subset \mathsf{Perf}_\mathsf{e} \to \mathrm{End}(\mathrm{id}_{\mathsf{Perf}_\mathsf{e}\text{-}\mathsf{Mod}})$ (*cf.* §A.1.10). It suffices to endow (A.10) with the structure of a symmetric monoidal functor.

However, (A.10) is identified with the composition

$$\begin{aligned}&\mathrm{K}(\mathsf{e}^\times, 2) \times \mathsf{Perf}_\mathsf{e}\text{-}\mathsf{Mod} \\ &\qquad \to \mathsf{Perf}_\mathsf{e}\text{-}\mathsf{Mod} \times \mathsf{Perf}_\mathsf{e}\text{-}\mathsf{Mod} \xrightarrow{\otimes} \mathsf{Perf}_\mathsf{e}\text{-}\mathsf{Mod},\end{aligned}$$

where both functors are symmetric monoidal. For the second functor, we have invoked the fact that the multiplication map of any commutative algebra in a symmetric monoidal ∞-category naturally lifts to a map of commutative algebras, which is a consequence of [Lur17, Proposition 3.2.4.7]. □

Remark A.2.3. Informally, the symmetric monoidal structure on (A.9) yields a canonical isomorphism of $\mathsf{Perf}_{\mathsf{e}}\text{-}\mathsf{Mod}$-valued sheaves over $\mathcal{X}_{/x}$:

$$(\mathcal{C}_1)_{\mathcal{G}_1} \otimes (\mathcal{C}_2)_{\mathcal{G}_2} \simeq (\mathcal{C}_1 \otimes \mathcal{C}_2)_{\mathcal{G}_1 + \mathcal{G}_2} \tag{A.11}$$

for any $\mathsf{Perf}_{\mathsf{e}}\text{-}\mathsf{Mod}$-valued sheaves $\mathcal{C}_1, \mathcal{C}_2$ over $\mathcal{X}$ and A-gerbes $\mathcal{G}_1$, $\mathcal{G}_2$ over x (*i.e.* morphisms $x \to \mathsf{B}^2\mathsf{e}^\times$), together with homotopy coherence.

A.2.4. Let Ξ be a commutative monoid, whose binary operation is written additively.

By a Ξ-*graded sheaf of* e-*linear symmetric monoidal* ∞-*categories over* $\mathcal{X}$, we shall mean a (right) lax symmetric monoidal functor (*cf.* [Lur17, §2.1.3]):

$$\mathcal{C}^{\Xi} : \Xi \to \mathsf{Shv}(\mathcal{X}, \mathsf{Perf}_{\mathsf{e}}\text{-}\mathsf{Mod}), \quad \xi \mapsto \mathcal{C}^{\xi}.$$

Such functors form an ∞-category which we denote by $\mathsf{Shv}^{\Xi,\otimes}(\mathcal{X}, \mathsf{Perf}_{\mathsf{e}}\text{-}\mathsf{Mod})$.

Remark A.2.5. Informally, the lax symmetric monoidal structure supplies morphisms

$$\mathbf{1} : \mathsf{Perf}_{\mathsf{e}} \to \mathcal{C}^0$$
$$\otimes : \mathcal{C}^{\xi_1} \otimes \mathcal{C}^{\xi_2} \to \mathcal{C}^{\xi_1+\xi_2}$$

together with homotopy coherence.

Note that for $\Xi \simeq 0$ the trivial commutative monoid, an object of $\mathsf{Shv}^{0,\otimes}(\mathcal{X}, \mathsf{Perf}_{\mathsf{e}}\text{-}\mathsf{Mod})$ is precisely a sheaf of e-linear symmetric monoidal categories over $\mathcal{X}$, with $\mathbf{1}$ given by its unit and $\otimes$ given by its monoidal product.

A.2.6. Restriction along $\Xi \to 0$ yields a functor

$$\mathsf{Shv}^{0,\otimes}(\mathcal{X}, \mathsf{Perf}_{\mathsf{e}}\text{-}\mathsf{Mod}) \to \mathsf{Shv}^{\Xi,\otimes}(\mathcal{X}, \mathsf{Perf}_{\mathsf{e}}\text{-}\mathsf{Mod}),$$

which admits a left adjoint $\mathrm{LKE}^{\otimes}$ (*cf.* [AGK+20, Theorem B.1.3], which is a special case of [Lur17, Corollary 3.1.3.5]).

Furthermore, the underlying $\mathsf{Perf}_{\mathsf{e}}$-valued sheaf over $\mathcal{X}$ of $\mathrm{LKE}^{\otimes}(\mathcal{C}^{\Xi})$ is computed by

$$\mathrm{LKE}^{\otimes}(\mathcal{C}^{\Xi}) \simeq \bigoplus_{\xi \in \Xi} \mathcal{C}^{\xi},$$

where $\bigoplus$ is the colimit in $\mathsf{Shv}(\mathcal{X}, \mathsf{Perf}_{\mathsf{e}}\text{-}\mathsf{Mod})$. Thus, any Ξ-graded sheaf of e-linear symmetric monoidal categories has an associated sheaf of e-linear symmetric monoidal categories, by taking sum over its graded components.

For our purposes, it is convenient to view the datum of $\mathcal{C}^{\Xi}$ as additional structure on the sheaf of e-linear symmetric monoidal categories $\mathcal{C} := \mathrm{LKE}^{\otimes}(\mathcal{C}^{\Xi})$, and simply refer to $\mathcal{C}$ as a Ξ-graded sheaf of e-linear symmetric monoidal ∞-categories.

A.2.7 *Symmetric monoidal twists***.** Let $\mathcal{X}$ be an ∞-topos with $x \in \mathcal{X}$. We shall consider a variant of §A.1.11 which takes symmetric monoidal structures into account.

Our input is a triple $(\Xi, \mathcal{C}^{\Xi}, \nu)$, where

1. Ξ is a commutative monoid;
2. $\mathcal{C}^{\Xi}$ is a Ξ-graded sheaf of e-linear symmetric monoidal ∞-categories over $\mathcal{X}$;
3. $\nu : \Xi \to \mathsf{B}^2\mathsf{e}^\times$ is a symmetric monoidal morphism over $\mathcal{X}_{/x}$.

We define the Ξ-graded sheaf of e-linear symmetric monoidal ∞-categories $\mathcal{C}^{\Xi}_{\nu}$ over $\mathcal{X}_{/x}$ as the composition of

$$(\nu, \mathcal{C}^{\Xi}) : \Xi \to \Gamma(x, \mathsf{B}^2\mathsf{e}^\times) \times \mathsf{Shv}(\mathcal{X}, \mathsf{Perf}_{\mathsf{e}}\text{-}\mathsf{Mod})$$

with (A.9), which is symmetric monoidal by Lemma A.2.2.

Remark A.2.8. By construction, for each $\xi \in \Xi$, the ξ-graded component $\mathcal{C}^{\xi}_{\nu}$ of $\mathcal{C}^{\Xi}_{\nu}$ is the twist of $\mathcal{C}^{\xi}$ by $\nu(\xi)$ in the sense of §A.1.11.

On the other hand, the lax symmetric monoidal structure on $\mathcal{C}^{\Xi}_{\nu}$ depends on the symmetric monoidal structure on ν. For example, the monoidal product $\otimes$ may be expressed as the composition

$$\begin{aligned}(\mathcal{C}^{\xi_1})_{\nu(\xi_1)} \otimes (\mathcal{C}^{\xi_2})_{\nu(\xi_2)} &\simeq (\mathcal{C}^{\xi_1} \otimes \mathcal{C}^{\xi_2})_{\nu(\xi_1)+\nu(\xi_2)} \\ &\xrightarrow{\otimes} (\mathcal{C}^{\xi_1+\xi_2})_{\nu(\xi_1)+\nu(\xi_2)} \simeq (\mathcal{C}^{\xi_1+\xi_2})_{\nu(\xi_1+\xi_2)},\end{aligned}$$

where the first isomorphism is (A.11) and the last isomorphism is supplied by the monoidal structure on ν.

Appendix B. Twisted ℓ-adic sheaves

The goal of this section is to define and study ℓ-adic constructible complexes twisted by a gerbe, *cf.* §B.1.3. We will extend a number of standard results to the twisted setting. These mainly concern universal local acyclicity (*cf.* Definition B.1.7) and hyperbolic localization (*cf.* Theorem B.3.4).

Although the main body of this article takes place over a field, we work over more general base schemes in this section.

B.1. Definitions.

B.1.1. We work over a base scheme S, which is assumed quasi-compact quasi-separated with finitely many irreducible components.[21] Denote by Sch the category of such schemes.

Let ℓ be a prime invertible in S and e be a finite extension of $\mathbf{Q}_\ell$. Let A be a finite subgroup of $\mathsf{e}^\times$ of order invertible in S and write $\zeta : \mathrm{A} \to \mathsf{e}^\times$ for the inclusion.

Consider the sheaf theory of constructible étale sheaves of e-vector spaces, which we understand as a functor of ∞-categories (*cf.* [HRS23, §3])

$$\mathsf{Sch}^{\mathrm{op}} \to \mathsf{Perf}_{\mathsf{e}}\text{-}\mathsf{Mod}, \quad \mathrm{X} \mapsto \mathsf{D}(\mathrm{X}) := \mathsf{D}_{\mathrm{cons}}(\mathrm{X}, \mathsf{e}). \tag{B.1}$$

To a morphism $f : \mathrm{X}_1 \to \mathrm{X}_2$ in Sch, (B.1) assigns the functor $f^* : \mathsf{D}(\mathrm{X}_2) \to \mathsf{D}(\mathrm{X}_1)$.

Remark B.1.2. By [HRS23, Theorem 7.7], the ∞-category $\mathsf{D}(\mathrm{X})$ coincides with the classically defined ∞-category of ℓ-adic constructible complexes over X.

In particular, we have the functor $f_! : \mathsf{D}(\mathrm{X}_1) \to \mathsf{D}(\mathrm{X}_2)$, whenever f is separated and of finite presentation: We refer the reader to [LZ24, §3.4] for its construction as a functor between ∞-categories and to [AGV73, Exposé XVII, Théorème 5.3.6] for the preservation of constructibility. The functor $f_!$ admits a right adjoint $f^!$.

B.1.3 *Twisted ℓ-adic sheaves***.** Denote by $\mathsf{B}^2\mathrm{A}$ the double deloop of A, viewed as a sheaf on the big étale site of S.

The assignment (B.1) satisfies étale descent (*cf.* [HRS23, Corollary 4.7]), so the twisting construction of §A.1 yields a functor of ∞-categories

$$(\mathsf{Sch}_{/\mathsf{B}^2\mathrm{A}})^{\mathrm{op}} \to \mathsf{Perf}_{\mathsf{e}}\text{-}\mathsf{Mod}, \quad (\mathrm{X}, \mathcal{G}) \mapsto \mathsf{D}_{\mathcal{G},\zeta}(\mathrm{X}) := \Gamma(\mathrm{X}, \mathsf{D}(\cdot)_{\mathcal{G}_\zeta}), \tag{B.2}$$

where $\mathcal{G}_\zeta$ denotes the composition of $\mathcal{G} : \mathrm{X} \to \mathsf{B}^2\mathrm{A}$ with the morphism $\zeta : \mathsf{B}^2\mathrm{A} \to \mathsf{B}^2\mathsf{e}^\times$.

We refer to objects of $\mathsf{D}_{\mathcal{G},\zeta}(\mathrm{X})$ as $(\mathcal{G},\zeta)$*-twisted constructible complexes over* X.

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[21] We use these conditions to ensure the existence of the relative perverse t-structure (*cf.* §B.1.12). For our applications, it is (almost) sufficient to restrict to schemes of finite type over a field.

Remark B.1.4. Since $\mathsf{D}_{\mathcal{G},\zeta}(\mathrm{X})$ is identified with $\mathsf{D}(\mathrm{X})$ once a neutralization of $\mathcal{G}$ is fixed, and neutralizations of $\mathcal{G}$ exist étale locally, all étale-local notions about constructible complexes extend to $(\mathcal{G},\zeta)$-twisted constructible complexes.

In particular, given a morphism $(\mathrm{X}_1,\mathcal{G}_1)\to(\mathrm{X}_2,\mathcal{G}_2)$ in $\mathsf{Sch}_{/\mathsf{B}^2\mathrm{A}}$ whose first component $f:\mathrm{X}_1\to\mathrm{X}_2$ is separated and of finite presentation, we have the exceptional adjunction

$$\begin{tikzcd} \mathsf{D}_{\mathcal{G}_1,\zeta}(\mathrm{X}_1) \arrow[r, shift left, "f_!"] & \mathsf{D}_{\mathcal{G}_2,\zeta}(\mathrm{X}_2) \arrow[l, shift left, "f^!"] \end{tikzcd}$$

constructed by descent along an étale cover of X_2 trivializing $\mathcal{G}_2$.

B.1.5 *The* 2*-category* K_{S}**.** Next, we shall define universal local acyclicity for twisted complexes. Since this is a étale-local notion, we can define it using Remark B.1.4. However, we will offer a definition intrinsic to twisted complexes, by adapting [HS23, §3].

Write K_{S} for the 2-category[22] whose objects are triples $(\mathrm{X},\mathcal{G},\mathcal{A})$, where X is a separated S-scheme of finite presentation, $\mathcal{G}$ is an A-gerbe over X, and $\mathcal{A}\in\operatorname{Ho}\mathsf{D}_{\mathcal{G},\zeta}(\mathrm{X})$.

A morphism

$$(\mathrm{X}_1,\mathcal{G}_1,\mathcal{A}_1)\to(\mathrm{X}_2,\mathcal{G}_2,\mathcal{A}_2) \tag{B.3}$$

in K_{S} consists of an object $\mathcal{B}\in\operatorname{Ho}\mathsf{D}_{\overleftarrow{\pi}^*\mathcal{G}_1-\overrightarrow{\pi}^*\mathcal{G}_2,\zeta}(\mathrm{X}_1\times_{\mathrm{S}}\mathrm{X}_2)$ together with a morphism

$$\overrightarrow{\pi}_!(\overleftarrow{\pi}^*\mathcal{A}_1\otimes\mathcal{B})\to\mathcal{A}_2,$$

where $\overleftarrow{\pi}$, $\overrightarrow{\pi}$ denote the projections of $\mathrm{X}_1\times_{\mathrm{S}}\mathrm{X}_2$ onto X_1, respectively X_2. Note that morphisms (B.3) form a category in the evident way, and their compositions are defined by convolution product in $\mathcal{B}$.

B.1.6. The 2-category K_{S} admits a symmetric monoidal structure, with monoidal product

$$(\mathrm{X}_1,\mathcal{G}_1,\mathcal{A}_1)\otimes(\mathrm{X}_2,\mathcal{G}_2,\mathcal{A}_2):=(\mathrm{X}_1\times_{\mathrm{S}}\mathrm{X}_2,\overleftarrow{\pi}^*\mathcal{G}_1+\overrightarrow{\pi}^*\mathcal{G}_2,\mathcal{A}_1\boxtimes\mathcal{A}_2)$$

and monoidal unit $(\mathrm{S},\mathbf{0},\mathsf{e})$, where $\mathbf{0}$ stands for the trivial A-gerbe.

Internal Homs exist in K_{S} and are given by

$$\mathcal{H}om_{\mathsf{K}_{\mathrm{S}}}((\mathrm{X}_1,\mathcal{G}_1,\mathcal{A}_1),(\mathrm{X}_2,\mathcal{G}_2,\mathcal{A}_2))\simeq(\mathrm{X}_1\times_{\mathrm{S}}\mathrm{X}_2,-\overleftarrow{\pi}^*\mathcal{G}_1+\overrightarrow{\pi}^*\mathcal{G}_2,\mathcal{H}om(\overleftarrow{\pi}^*\mathcal{A}_1,\overrightarrow{\pi}^!\mathcal{A}_2)). \tag{B.4}$$

Definition B.1.7. Given a separated S-scheme X of finite presentation and an A-gerbe $\mathcal{G}$ over X, an object $\mathcal{A}\in\mathsf{D}_{\mathcal{G},\zeta}(\mathrm{X})$ is *universally locally acyclic* (ULA) relative to S if $(\mathrm{X},\mathcal{G},\mathcal{A})$ is dualizable as an object of K_{S}.

Remark B.1.8. The dualizability of $(\mathrm{X},\mathcal{G},\mathcal{A})$ can be checked étale locally over X, where we may assume that $\mathcal{G}$ is neutral. In that case, Definition B.1.7 coincides with [HS23, Definition 3.2]. Thus, our notion of ULA is equivalent to the classical notion over an étale cover of X neutralizing $\mathcal{G}$.

Remark B.1.9. The analogue of [HS23, Theorem 4.4] holds in our context, by repeating the proof of *loc.cit.* or by applying Remark B.1.8.

In particular, the criterion of [HS23, Theorem 4.4(iv)] shows that when S is the spectrum of an algebraically closed field, any $\mathcal{A}\in\mathsf{D}_{\mathcal{G},\zeta}(\mathrm{X})$ is ULA relative to S.

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[22] It is sufficient for our purposes to construct K_{S} as an *ordinary* 2-category.

B.1.10 *Verdier duality.* Given a separated S-scheme X of finite presentation equipped with an A-gerbe $\mathcal{G}$, we define Verdier duality to be the functor

$$\mathbf{D}_{\mathrm{X/S}} : \mathsf{D}_{\mathcal{G},\zeta}(\mathrm{X}) \to \mathsf{D}_{-\mathcal{G},\zeta}(\mathrm{X})$$
$$\mathcal{A} \mapsto \mathcal{H}om(\mathcal{A}, \pi^! \mathsf{e}),$$

where $\pi : \mathrm{X} \to \mathrm{S}$ denotes the structural morphism.

If $\mathcal{A}$ is ULA relative to S, then the dual of $(\mathrm{X}, \mathcal{G}, \mathcal{A}) \in \mathsf{K}_{\mathrm{S}}$ is computed by (B.4) to be $(\mathrm{X}, -\mathcal{G}, \mathbf{D}_{\mathrm{X/S}}(\mathcal{A}))$. It follows that $\mathbf{D}_{\mathrm{X/S}}(\mathcal{A})$ is also ULA relative to S and we have an canonical isomorphism $\mathcal{A} \simeq \mathbf{D}_{\mathrm{X/S}}(\mathbf{D}_{\mathrm{X/S}}\mathcal{A})$.

Since K_{S} is functorial with respect to ($*$-)pullbacks along S, we see that the formation of $\mathbf{D}_{\mathrm{X/S}}(\mathcal{A})$ for an ULA object $\mathcal{A}$ commutes with any base change along S.

Remark B.1.11. Given two ULA objects $\mathcal{A}, \mathcal{B} \in \mathsf{D}_{\mathcal{G},\zeta}(\mathrm{X})$, we may express their internal Hom in terms of duality and tensor product:

$$\mathcal{H}om(\mathcal{A}, \mathcal{B}) \simeq \mathbf{D}_{\mathrm{X/S}}(\mathcal{A} \otimes \mathbf{D}_{\mathrm{X/S}}\mathcal{B}). \tag{B.5}$$

Indeed, the canonical isomorphism $\mathcal{H}om_{\mathsf{K}_{\mathrm{S}}}(a, b) \simeq \mathcal{H}om_{\mathsf{K}_{\mathrm{S}}}(a, \mathbf{1}) \otimes b$, for $a := (\mathrm{X}, \mathcal{G}, \mathcal{A}), b := (\mathrm{X}, \mathcal{G}, \mathcal{B}) \in \mathsf{K}_{\mathrm{S}}$, yields an isomorphism in $\operatorname{Ho} \mathsf{D}(\mathrm{X})$:

$$\mathcal{H}om(\mathcal{A}, \mathcal{B}) \simeq \Delta^!(\mathbf{D}_{\mathrm{X/S}}(\mathcal{A}) \boxtimes \mathcal{B}), \tag{B.6}$$

where $\Delta : \mathrm{X} \to \mathrm{X} \times_{\mathrm{S}} \mathrm{X}$ is the diagonal embedding. Since $\mathbf{D}_{\mathrm{X/S}}(\mathcal{A})$ and $\mathcal{B}$ are both ULA, so is $\mathbf{D}_{\mathrm{X/S}}(\mathcal{A}) \boxtimes \mathcal{B}$ (as it corresponds to a monoidal product in K_{S}). Then (B.5) follows by rewriting the right-hand-side of (B.6) using the isomorphism $\Delta^! \mathbf{D}_{\mathrm{X} \times_{\mathrm{S}} \mathrm{X/S}} \simeq \mathbf{D}_{\mathrm{X/S}} \Delta^*$.

B.1.12 *Relative perversity.* Finally, we recall the relative perverse t-structure (*cf.* [HS23, Theorem 6.1]), which adapts to twisted complexes without change.

Namely, given a finitely presented S-scheme X together with an A-gerbe $\mathcal{G}$ over X, there is a t-structure on $\mathsf{D}_{\mathcal{G},\zeta}(\mathrm{X})$ such that for any $\mathcal{A} \in \mathsf{D}_{\mathcal{G},\zeta}(\mathrm{X})$,

$$\mathcal{A} \in {}^p\mathsf{D}^{\le 0}_{\mathcal{G},\zeta}(\mathrm{X}) \Leftrightarrow (i_{\bar{s}})^*\mathcal{A} \in {}^p\mathsf{D}^{\le 0}_{\mathcal{G},\zeta}(\mathrm{X}_{\bar{s}}) \text{ for any geometric point } \bar{s} \to \mathrm{S}$$
$$\mathcal{A} \in {}^p\mathsf{D}^{\ge 0}_{\mathcal{G},\zeta}(\mathrm{X}) \Leftrightarrow (i_{\bar{s}})^*\mathcal{A} \in {}^p\mathsf{D}^{\ge 0}_{\mathcal{G},\zeta}(\mathrm{X}_{\bar{s}}) \text{ for any geometric point } \bar{s} \to \mathrm{S}$$

where $i_{\bar{s}} : \mathrm{X}_{\bar{x}} := \mathrm{X} \times_{\mathrm{S}} \bar{x} \to \mathrm{X}$ denotes the base change of $\bar{s} \to \mathrm{S}$.

Here, the fiberwise perverse t-structure is defined either by repeating the classical definition, or equivalently by appealing to Remark B.1.8.

Remark B.1.13. The analogue of [HS23, Theorem 6.7] holds in our context. Note that its proof shows that the perverse truncation functors preserve universal local acyclicity.

B.2. Vanishing by equivariance.

B.2.1. We remain in the context of §B.1.1. Let G be a group S-scheme of finite type with connected geometric fibers.

Let $\mathcal{G}$ be an A-gerbe over BG. Denote by $\mathsf{D}_{\mathcal{G},\zeta}(\mathsf{BG})$ the ∞-category of G-equivariant objects of $\mathsf{D}_{\mathcal{G},\zeta}(\mathrm{S})$. For any geometric point $\bar{s}$ of S, we write $\mathcal{G}_{\bar{s}}$ for the fiber of $\mathcal{G}$ at $\bar{s}$, which is an A-gerbe over $\mathsf{BG}_{\bar{s}}$.

Lemma B.2.2. *Suppose that $\mathcal{G}_{\bar{s}}$ is nontrivial for all geometric points $\bar{s}$ of* S. *Then*

$$\mathsf{D}_{\mathcal{G},\zeta}(\mathsf{BG}) \simeq 0.$$

Proof. It suffices to prove that $\mathcal{A} \in \mathsf{D}_{\mathcal{G},\zeta}(\mathsf{BG})$ has vanishing fiber at any geometric point $\bar{s}$ of S. Thus, we may replace S by $\bar{s}$ and assume that S is itself a geometric point. Fixing a neutralization of $\mathcal{G}$ along the base point of BG, we may assume that $\mathcal{G}$ is the deloop of a nontrivial monoidal morphism $\chi : \mathrm{G} \to \mathsf{B}\mathrm{A}$.

Note that χ is nontrivial as a plain morphism: Any morphism $\mathrm{G} \times \mathrm{G} \to \mathrm{A}$ which is trivial along $e \times \mathrm{G}$ and $\mathrm{G} \times e$ is itself trivial, by connectedness of G. Denote by $\mathcal{L}_{\chi,\zeta}$ the 1-dimensional e-local system over G induced from χ along ζ. We then have

$$\mathrm{H}^0(\mathrm{G}, \mathcal{L}_{\chi,\zeta}) \simeq 0, \tag{B.7}$$

as it is the space of invariants of a nontrivial character of $\pi_1^{\text{ét}}(\mathrm{G}, e)$.

Let us describe an object $\mathcal{A} \in \mathsf{D}_{\mathcal{G},\zeta}(\mathsf{BG})$ by its fiber $\mathcal{A}_e$ along e, which is a constructible complex of e-vector spaces, together with (the first piece of) its descent data:

$$\underline{\mathsf{e}} \otimes \mathcal{A}_e \simeq \mathcal{L}_{\chi,\zeta} \otimes \mathcal{A}_e \text{ in } \mathsf{D}(\mathrm{G}). \tag{B.8}$$

If $\mathrm{H}^i(\mathcal{A}_e) \neq 0$ for some $i \in \mathbf{Z}$, then by taking ith cohomology of (B.8) and choosing a surjective character $\mathrm{H}^i(\mathcal{A}_e) \twoheadrightarrow \mathsf{e}$, we obtain nonzero global sections of $\mathcal{L}_{\chi,\zeta}$, contradicting (B.7). Thus, $\mathrm{H}^i(\mathcal{A}_e) \simeq 0$ for all $i \in \mathbf{Z}$ and the complex $\mathcal{A}$ vanishes. ☐

B.2.3. We shall strengthen Lemma B.2.2 to a vanishing statement about compactly supported cohomology of twisted constructible complexes.

Let X be a separated S-scheme of finite presentation equipped with a G-action. Denote by $\mathsf{D}_{\mathcal{G},\zeta}(\mathrm{X}/\mathrm{G})$ the ∞-category of G-equivariant objects of $\mathsf{D}_{\mathcal{G},\zeta}(\mathrm{X})$, formed with respect to the pullback of $\mathcal{G}$ along the structural morphism $\mathrm{X}/\mathrm{G} \to \mathsf{BG}$.

The structural morphism $f : \mathrm{X} \to \mathrm{S}$ defines a functor

$$f_! : \mathsf{D}_{\mathcal{G},\zeta}(\mathrm{X}) \to \mathsf{D}_{\mathcal{G},\zeta}(\mathrm{S}), \tag{B.9}$$

where the target is formed with respect to the pullback of $\mathcal{G}$ along $e : \mathrm{S} \to \mathsf{BG}$.

The following statement includes Lemma B.2.2 as the special case for X = S.

Proposition B.2.4. *Suppose that $\mathcal{G}_{\bar{s}}$ is nontrivial for all geometric points $\bar{s}$ of* S. *Then* (B.9) *vanishes on the essential image of* $\mathsf{D}_{\mathcal{G},\zeta}(\mathrm{X}/\mathrm{G})$.

Proof. Consider the Cartesian square of stacks

$$\begin{tikzcd} \mathrm{X} \arrow[r, "p"] \arrow[d, "f"] & \mathrm{X}/\mathrm{G} \arrow[d, "g"] \\ \mathrm{S} \arrow[r, "e"] & \mathsf{BG} \end{tikzcd}$$

By descent and base change, there is a functor

$$g_! : \mathsf{D}_{\mathcal{G},\zeta}(\mathrm{X}/\mathrm{G}) \to \mathsf{D}_{\mathcal{G},\zeta}(\mathsf{BG})$$

related to (B.9) by the isomorphism $e^* g_! \simeq f_! p^*$. Since $\mathsf{D}_{\mathcal{G},\zeta}(\mathsf{BG}) \simeq 0$ (*cf.* Lemma B.2.2), the functor $f_! p^*$ vanishes, as desired. ☐

B.3. Hyperbolic localization.

B.3.1. We remain in the context of §B.1.1. Let X be a quasi-separated S-scheme locally of finite presentation equipped with a $\mathbb{G}_m$-action. Let $\mathcal{G}$ be an A-gerbe over $\mathrm{X}/\mathbb{G}_m$, which we shall often view as a $\mathbb{G}_m$-equivariant A-gerbe over X.

The $\mathbb{G}_m$-action on X defines the *attractor* X^+, the *repeller* X^-, and the *fix point locus* X^0, as S-presheaves parametrizing $\mathbb{G}_m$-equivariant morphisms

$$\begin{aligned}
\mathrm{X}^+ &:= \mathcal{M}aps_{\mathbb{G}_m}(\mathbb{A}^{1,+}, \mathrm{X}) \\
\mathrm{X}^- &:= \mathcal{M}aps_{\mathbb{G}_m}(\mathbb{A}^{1,-}, \mathrm{X}) \\
\mathrm{X}^0 &:= \mathcal{M}aps_{\mathbb{G}_m}(\operatorname{Spec} \mathsf{k}, \mathrm{X})
\end{aligned}$$

were $\mathbb{A}^{1,+}$ (respectively $\mathbb{A}^{1,-}$) denotes the affine line over S endowed with the $\mathbb{G}_m$-action $z \cdot a := za$ (respectively $z \cdot a := z^{-1}a$).

Our assumption on X implies that the $\mathbb{G}_m$-action on X is étale locally linearizable, *i.e.* it admits a $\mathbb{G}_m$-equivariant étale cover by affine S-schemes (*cf.* [AHR25, Corollary 10.2]). It follows that X^+, X^-, X^0 are represented by S-schemes locally of finite presentation (*cf.* [Ric19, Theorem 1.8]), with X^0 being a closed subscheme of X.

B.3.2. We shall consider the following morphisms of S-schemes

$$\begin{tikzcd}
& \mathrm{X}^+ \arrow[ld, "q^+"'] \arrow[rd, shift left, "p^+"] & \\
\mathrm{X} & & \mathrm{X}^0 \arrow[lu, shift left, "i^+"] \arrow[ld, shift left, "i^-"] \\
& \mathrm{X}^- \arrow[lu, "q^-"] \arrow[ru, shift left, "p^-"] &
\end{tikzcd} \tag{B.10}$$

where q^+, q^- (respectively p^+, p^-) are defined by evaluation at 1 (respectively 0), and i^+, i^- are defined by pullback along $\mathbb{A}^{1,+} \to \mathrm{S}$, $\mathbb{A}^{1,-} \to \mathrm{S}$.

The morphisms i^+, i^- are closed immersions (*cf.* [Ric19, Proposition 1.17]), and $q^+i^+ = q^-i^-$ is the natural closed immersion of X^0 into X.

B.3.3. Writing $\mathcal{G}^0$ for the restriction of $\mathcal{G}$ to X^0, we have canonical isomorphisms of A-gerbes over X^+ and X^-:

$$(q^+)^*\mathcal{G} \simeq (p^+)\mathcal{G}^0, \tag{B.11}$$

$$(q^-)^*\mathcal{G} \simeq (p^-)\mathcal{G}^0. \tag{B.12}$$

Indeed, given an R-point of X^+ represented by the $\mathbb{G}_m$-equivariant morphism $\mathbb{A}^{1,+} \times \operatorname{Spec} \mathrm{R} \to \mathrm{X}$, the pullback of $\mathcal{G}$ to $\mathbb{A}^{1,+} \times \operatorname{Spec} \mathrm{R}$ canonically descends to $\operatorname{Spec} \mathrm{R}$, so we obtain an identification of its pullbacks along $\{1\} \times \operatorname{Spec} \mathrm{R}$ and $\{0\} \times \operatorname{Spec} \mathrm{R}$. This yields (B.11). The isomorphism (B.12) arises from the same analysis for X^-.

Define the full subcategory of $\mathbb{G}_m$*-monodromic* objects

$$\mathsf{D}_{\mathcal{G},\zeta}(\mathrm{X})^{\mathbb{G}_m\text{-mon}} \subset \mathsf{D}_{\mathcal{G},\zeta}(\mathrm{X})$$

to be the essential image of $\mathsf{D}_{\mathcal{G},\zeta}(\mathrm{X}/\mathbb{G}_m)$ under the pullback functor.

The main result of this subsection is the following twisted version of Braden's hyperbolic localization theorem (*cf.* [Bra03]). Its proof will be supplied in §B.3.10.

Theorem B.3.4. *Suppose that* $\mathrm{X} \to \mathrm{S}$ *is separated and of finite presentation. There is a canonical isomorphism of functors from* $\mathsf{D}_{\mathcal{G},\zeta}(\mathrm{X})^{\mathbb{G}_m\text{-mon}}$ *to* $\mathsf{D}_{\mathcal{G}^0,\zeta}(\mathrm{X}^0)$*:*

$$(p^+)_!(q^+)^* \simeq (p^-)_*(q^-)^!. \tag{B.13}$$

Remark B.3.5. The formation of $(p^+)_!$ and $(q^-)^!$ require p^+ and q^- to be separated and of finite presentation. The morphism p^+ is affine (*cf.* [Ric19, Corollary 1.12]), hence separated. Our assumption on X implies that q^- is a monomorphism (*cf.* [Ric19, Remark 1.19]), hence separated as well. Finite presentation follows from [Ric19, Theorem 1.8(iii)].

B.3.6. As in the untwisted setting (*cf.* [DG14, §3.4]), Theorem B.3.4 can be strengthened to a statement concerning quotient stacks.

Namely, we take the quotient of (B.10) by $\mathbb{G}_m$:

$$\begin{tikzcd}
& \mathrm{X}^+/\mathbb{G}_m \arrow[dl, "q^+"'] \arrow[dr, shift left, "p^+"] & \\
\mathrm{X}/\mathbb{G}_m & & \mathrm{X}^0/\mathbb{G}_m \arrow[ul, shift left, "i^+"] \arrow[dl, shift left, "i^-"] \\
& \mathrm{X}^-/\mathbb{G}_m \arrow[ul, "q^-"] \arrow[ur, shift left, "p^-"] &
\end{tikzcd} \tag{B.14}$$

without changing notation for the morphisms. There is a canonical morphism of functors from $\mathsf{D}_{\mathcal{G},\zeta}(\mathrm{X}/\mathbb{G}_m)$ to $\mathsf{D}_{\mathcal{G}^0,\zeta}(\mathrm{X}^0/\mathbb{G}_m)$:

$$(p^+)_!(q^+)^* \to (p^-)_*(q^-)^!, \tag{B.15}$$

where $\mathsf{D}_{\mathcal{G},\zeta}(\mathrm{X}/\mathbb{G}_m)$ denotes the ∞-category of $\mathbb{G}_m$-equivariant objects of $\mathsf{D}_{\mathcal{G},\zeta}(\mathrm{X})$ (and likewise for $\mathsf{D}_{\mathcal{G}^0,\zeta}(\mathrm{X}^0/\mathbb{G}_m)$). The construction of (B.15) is a repetition of its untwisted analogue (*cf.* [Ric19, Construction 2.2]).

To prove Theorem B.3.4, it suffices to show that (B.15) is an isomorphism. (More precisely, one constructs the functor (B.13) in one direction as in *loc.cit.* and uses its compatibility with (B.15) to achieve this reduction.)

B.3.7. Since $\mathbb{G}_m$ acts trivially on X^0, we have $\mathrm{X}^0/\mathbb{G}_m \simeq \mathrm{X}^0 \times \mathsf{B}\mathbb{G}_m$, so the quotient map $\mathrm{X}^0 \to \mathrm{X}^0/\mathbb{G}_m$ admits a section $s : \mathrm{X}^0/\mathbb{G}_m \to \mathrm{X}^0$.

The Kummer $\hat{\mathbf{Z}}(1)$-gerbe over $\mathsf{B}\mathbb{G}_m$ (*cf.* Remark 1.1.10) induces an isomorphism

$$\mathrm{Maps}(\mathrm{X}^0, \mathrm{A}(-1)) \simeq \mathrm{Maps}_{\mathrm{X}^0/}(\mathrm{X}^0/\mathbb{G}_m, \mathsf{B}^2\mathrm{A}). \tag{B.16}$$

The A-gerbe $\mathcal{G}^0 \otimes s^*(\mathcal{G}^0|_{\mathrm{X}^0})^{-1}$ over $\mathrm{X}^0/\mathbb{G}_m$ defines a morphism $\mathrm{X}^0 \to \mathrm{A}(-1)$ under (B.16), whose vanishing locus $\mathrm{Y}^0 \subset \mathrm{X}^0$ is an open and closed subscheme.

By definition, the restriction of $\mathcal{G}^0$ to $\mathrm{Y}^0/\mathbb{G}_m$ is pulled back along $s : \mathrm{Y}^0/\mathbb{G}_m \to \mathrm{Y}^0$. The following observation is an application of [AHR25, Theorem 1.1].

Lemma B.3.8. *Given a point $y \in \mathrm{Y}^0$, there exists an affine* S*-scheme* U *with a $\mathbb{G}_m$-action and a $\mathbb{G}_m$-equivariant étale morphism $f : \mathrm{U} \to \mathrm{X}$ such that*

(1) the image of f contains y;
(2) the pullback $f^*\mathcal{G}$ is $\mathbb{G}_m$-equivariantly trivial.

Proof. The (total space of the) A-gerbe $\mathcal{G}$ over $\mathrm{X}/\mathbb{G}_m$ is an algebraic S-stack satisfying the hypotheses of [AHR25, Theorem 1.1]. Replacing y by a finite étale extension, we may assume that it lifts to a point $\tilde{y}$ of $\mathcal{G}$. The residue gerbe $\mathcal{G}_{\tilde{y}}$ is non-canonically isomorphic to $\mathsf{B}\mathbb{G}_m \times \mathsf{B}\mathrm{A}$. Applying *loc.cit.* to the étale morphism

$$\mathrm{id} \times e : \mathsf{B}\mathbb{G}_m \to \mathsf{B}\mathbb{G}_m \times \mathsf{B}\mathrm{A}$$

we may extend it to a Cartesian square of algebraic stacks

$$\begin{tikzcd}
\mathsf{B}\mathbb{G}_m \arrow[r, phantom, "\subset"] \arrow[d, "\mathrm{id}\times e"] & \mathcal{U} \arrow[d, "h"] \\
\mathsf{B}\mathbb{G}_m \times \mathsf{BA} \arrow[r, phantom, "\subset"] & \mathcal{G}
\end{tikzcd}$$

where h is étale and $\mathcal{U}$ contains $\mathsf{B}\mathbb{G}_m$ as a closed substack.

The composition $\mathcal{U} \xrightarrow{h} \mathcal{G} \to \mathsf{B}\mathbb{G}_m$ restricts to the identity along $\mathsf{B}\mathbb{G}_m \subset \mathcal{U}$, so by [AHR25, Proposition 5.3], it becomes affine after shrinking $\mathcal{U}$. The base change of h along the neutral point $\mathrm{S} \to \mathsf{B}\mathbb{G}_m$ then yields a $\mathbb{G}_m$-equivariant morphism $\mathrm{U} \to \mathcal{G}|_{\mathrm{X}}$, where U is S-affine. The composition $f : \mathrm{U} \to \mathcal{G}|_{\mathrm{X}} \to \mathrm{X}$ satisfies the desired properties. □

Remark B.3.9. The condition $y \in \mathrm{Y}^0$ (as opposed to an arbitrary point of X^0) in Lemma B.3.8 is crucial: Even for X a geometric point equipped with the trivial $\mathbb{G}_m$-action, a $\mathbb{G}_m$-equivariant A-gerbe over X may not be $\mathbb{G}_m$-equivariantly trivial.

B.3.10. We are now ready to prove Theorem B.3.4, or rather, reduce it to hyperbolic localization for *untwisted* constructible complexes (*cf.* [Ric19, Theorem 2.6]).

Proof of Theorem B.3.4. We shall prove that (B.15) is an isomorphism.

Recall the open and closed subscheme $\mathrm{Y}^0 \subset \mathrm{X}^0$ (*cf.* §B.3.7). By Lemma B.2.2, restriction along $\mathrm{Y}^0 \subset \mathrm{X}^0$ yields an equivalence of ∞-categories

$$\mathsf{D}_{\mathcal{G}^0,\zeta}(\mathrm{X}^0/\mathbb{G}_m) \simeq \mathsf{D}_{\mathcal{G}^0,\zeta}(\mathrm{Y}^0/\mathbb{G}_m). \tag{B.17}$$

By Lemma B.3.8, we find a family of $\mathbb{G}_m$-equivariant étale morphisms $f_i : \mathrm{U}_i \to \mathrm{X}$ $(i \in \mathrm{I})$ satisfying the following properties:

(1) each U_i is S-affine;
(2) the pullback $f_i^*\mathcal{G}$ is $\mathbb{G}_m$-equivariantly trivial;
(3) the union $\mathrm{Y} := \bigcup_{i\in\mathrm{I}} f_i(\mathrm{U}_i)$ is an open subscheme of X containing Y^0.

To prove that (B.15) is an isomorphism, we may replace X by any $\mathbb{G}_m$-stable open subscheme containing Y^0 (for instance, Y): In view of (B.17), it suffices to show that both sides of (B.15) are unchanged after this replacement. This can be seen from the "contraction lemma" (*cf.* [DG15, Proposition 5.3.2]), *i.e.* isomorphisms of functors

$$(p^+)_! \simeq (i^+)^!, \quad (p^-)_* \simeq (i^-)^*,$$

which remain valid for twisted complexes because they are of étale local nature over Y^0, and $\mathcal{G}^0$ is $\mathbb{G}_m$-equivariantly trivial over an étale cover of Y^0.

After replacing X by Y, we may show that (B.15) is an isomorphism over the étale cover $f_i : \mathrm{U}_i \to \mathrm{Y}$ $(i \in \mathrm{I})$, where the $\mathbb{G}_m$-equivariant triviality of each $f_i^*\mathcal{G}$ allows us to reduce the statement to the untwisted (affine) case (*cf.* [Ric19, §2.8]). □

B.4. Properties of $\mathrm{L}_{\mathrm{X}/\mathrm{S}}$.

B.4.1. We continue in the context of §B.1.1 and let X be a separated S-scheme of finite presentation equipped with a $\mathbb{G}_m$-action. Let $\mathcal{G}$ be an A-gerbe over $\mathrm{X}/\mathbb{G}_m$.

Having proved Theorem B.3.4, we denote either side of (B.13) by

$$\mathrm{L}_{\mathrm{X}/\mathrm{S}} : \mathsf{D}_{\mathcal{G},\zeta}(\mathrm{X})^{\mathbb{G}_m\text{-mon}} \to \mathsf{D}_{\mathcal{G}^0,\zeta}(\mathrm{X}^0)$$

and call it the *hyperbolic localization* functor. In this subsection, we establish some properties of $\mathrm{L}_{\mathrm{X}/\mathrm{S}}$, which are all standard consequences of Theorem B.3.4.

Lemma B.4.2. *The formation of* $\mathrm{L}_{\mathrm{X/S}}$ *commutes with* $*$*-and* $!$*-base changes in* S.

Proof. Given a morphism $\mathrm{S}' \to \mathrm{S}$ of schemes satisfying the conditions of §B.1.1, with induced morphisms $f : \mathrm{X}' := \mathrm{X} \times_{\mathrm{S}} \mathrm{S}' \to \mathrm{X}$ and $f^0 : (\mathrm{X}^0)' := \mathrm{X}^0 \times_{\mathrm{S}} \mathrm{S}' \to \mathrm{X}^0$, we need to construct canonical isomorphisms

$$(f^0)^* \mathrm{L}_{\mathrm{X/S}} \simeq \mathrm{L}_{\mathrm{X'/S'}} f^*,$$
$$(f^0)^! \mathrm{L}_{\mathrm{X/S}} \simeq \mathrm{L}_{\mathrm{X'/S'}} f^!.$$

These are immediate, by presenting $\mathrm{L}_{\mathrm{X/S}}$ as $(p^+)_!(q^+)^*$, respectively $(p^-)_*(q^-)^!$. □

Proposition B.4.3. *The functor* $\mathrm{L}_{\mathrm{X/S}}$ *preserves universal local acyclicity relative to* S.

Proof. Let $\mathcal{A}$ be an object of $\mathsf{D}_{\mathcal{G},\zeta}(\mathrm{X})^{\mathbb{G}_m\text{-mon}}$ which is ULA relative to S. We want to prove that $\mathrm{L}_{\mathrm{X/S}}(\mathcal{A}) \in \mathsf{D}_{\mathcal{G}^0,\zeta}(\mathrm{X}^0)$ is also ULA relative to S.

By Lemma B.4.2 and [HS23, Theorem 4.4] (which remains valid in our setting, *cf.* Remark B.1.9), we may assume that S is the spectrum of of a rank-1 valuation ring with algebraically closed fraction field and prove that the natural map ("cospecialization")

$$i^* \mathrm{L}_{\mathrm{X/S}}(\mathcal{A}) \to i^* j_* j^* \mathrm{L}_{\mathrm{X/S}}(\mathcal{A}) \tag{B.18}$$

is an isomorphism. Here, i, j are the base changes of the inclusion of the special, respectively generic point of S.

Recall that the formation of $\mathrm{L}_{\mathrm{X/S}}$ commutes with i^* and j^* (*cf.* Lemma B.4.2). It also commutes with j_* by presenting $\mathrm{L}_{\mathrm{X/S}}$ as $(p^-)_*(q^-)^!$. Therefore, (B.18) is identified with the image of $i^*\mathcal{A} \to i^* j_* j^* \mathcal{A}$ under $\mathrm{L}_{\mathrm{X/S}}$, which is an isomorphism because $\mathcal{A}$ is ULA. □

B.4.4. Next, we study the interaction between $\mathrm{L}_{\mathrm{X/S}}$ and Verdier duality (*cf.* §B.1.10).

Denote by $\mathrm{L}^-_{\mathrm{X/S}}$ the hyperbolic localization functor for X endowed with the *inverted* $\mathbb{G}_m$-action. Note that $-\mathcal{G}$ is naturally $\mathbb{G}_m$-equivariant with respect to the inverted action, and we may present $\mathrm{L}^-_{\mathrm{X/S}}$ as either of the isomorphic functors

$$\mathrm{L}^-_{\mathrm{X/S}} \simeq (p^-)_!(q^-)^* \simeq (p^+)_*(q^+)^!.$$

The Verdier duality functor $\mathbf{D}_{\mathrm{X/S}} : \mathsf{D}_{\mathcal{G},\zeta}(\mathrm{X}) \to \mathsf{D}_{-\mathcal{G},\zeta}(\mathrm{X})$ restricts to a functor on the full subcategories of $\mathbb{G}_m$-monodromic objects.

Proposition B.4.5. *The following diagram is canonically commutative:*

$$\begin{tikzcd}
\mathsf{D}_{\mathcal{G},\zeta}(\mathrm{X})^{\mathbb{G}_m\text{-mon}} \arrow[r, "\mathrm{L}_{\mathrm{X/S}}"] \arrow[d, "\mathbf{D}_{\mathrm{X/S}}"] & \mathsf{D}_{\mathcal{G}^0,\zeta}(\mathrm{X}^0) \arrow[d, "\mathbf{D}_{\mathrm{X}^0/\mathrm{S}}"] \\
\mathsf{D}_{-\mathcal{G},\zeta}(\mathrm{X})^{\mathbb{G}_m\text{-mon}} \arrow[r, "\mathrm{L}^-_{\mathrm{X/S}}"] & \mathsf{D}_{-\mathcal{G}^0,\zeta}(\mathrm{X}^0)
\end{tikzcd}$$

Proof. For any object $\mathcal{A}$ of $\mathsf{D}_{\mathcal{G},\zeta}(\mathrm{X})^{\mathbb{G}_m\text{-mon}}$, we have natural isomorphisms

$$\begin{aligned}
\mathbf{D}_{\mathrm{X}^0/\mathrm{S}} \mathrm{L}_{\mathrm{X/S}}(\mathcal{A}) &\simeq \mathcal{H}om((p^+)_!(q^+)^*\mathcal{A}, \omega_{\mathrm{X}^0/\mathrm{S}}) \\
&\simeq (p^+)_* \mathcal{H}om((q^+)^*\mathcal{A}, (p^+)^! \omega_{\mathrm{X}^0/\mathrm{S}}) \\
&\simeq (p^+)_* \mathcal{H}om((q^+)^*\mathcal{A}, (q^+)^! \omega_{\mathrm{X/S}}) \\
&\simeq (p^+)_*(q^+)^! \mathcal{H}om(\mathcal{A}, \omega_{\mathrm{X/S}}) \simeq \mathrm{L}^-_{\mathrm{X/S}} \mathbf{D}_{\mathrm{X/S}}(\mathcal{A})
\end{aligned}$$

as desired. (Here, $\omega_{\mathrm{Z/S}}$ denotes the relative dualizing complex $\pi^! \mathsf{e}$ for any separated morphism $\pi : \mathrm{Z} \to \mathrm{S}$ of finite presentation.) □

Appendix C. Related works

In this section, we indicate the relation between our version of the twisted geometric Satake equivalence (*cf.* Theorem 2.3.2) with its predecessors: [FL10, Theorem 2.9], [Rei12, Theorem IV.8.3], [Lys14, Theorem 2.1], and [GL18, §9.2].

Since the works [FL10, Lys14] use the K-theoretic parametrization of covers of [BD01], we shall first recall this formalism in §C.1 in order to set the stage. Much of this material already appears in [GL18, §3.4]. In §C.2-C.3, we discuss the aforementioned works on the twisted geometric Satake equivalence in relation to our version. We also point out some shortcomings in them.[23] Finally, in §C.4, we sketch another proof of the semisimplicity of the Satake category (*cf.* Corollary 3.3.7), communicated to the author by Gaitsgory.[24]

C.1. Brylinski–Deligne covers.

C.1.1. Let S be a smooth scheme over a field. Denote by K_2 the Zariski sheafification of the S-presheaf sending $\operatorname{Spec} \mathrm{R} \to \mathrm{S}$ to the second algebraic K-group $\mathrm{K}_2(\mathrm{R})$.

Given a smooth affine group S-scheme G, a *Brylinski–Deligne cover* (or an *integral level*) of G is a central extension

$$1 \to \mathsf{K}_2 \to \mathrm{E} \to \mathrm{G} \to 1 \tag{C.1}$$

of sheaves of groups on the (big) Zariski site of S.

Equivalently, a Brylinski–Deligne cover of G is a morphism of pointed Zariski stacks $\mathsf{B}_{\mathrm{Zar}}\mathrm{G} \to \mathsf{B}^2_{\mathrm{Zar}}\mathsf{K}_2$, where $\mathsf{B}_{\mathrm{Zar}}$ is the deloop functor for Zariski stacks (*cf.* §0.4.3).

C.1.2. Let G be a reductive group S-scheme and N be an integer invertible over S.

Gaitsgory constructed an *étale realization* functor (*cf.* [Gai20, §6], [Zha22, §2.3])

$$\mathrm{R}_{\text{ét}} : \operatorname{Maps}_*(\mathsf{B}_{\mathrm{Zar}}\mathrm{G}, \mathsf{B}^2_{\mathrm{Zar}}\mathsf{K}_2) \to \operatorname{Maps}_*(\mathsf{B}\mathrm{G}, \mathsf{B}^4\mu_{\mathrm{N}}^{\otimes 2}), \tag{C.2}$$

where B denotes the deloop functor for étale stacks, as in the main body of the text. In particular, when μ_{N} is constant over S, every Brylinski–Deligne cover defines an étale level valued in the abelian group $\mathrm{A} := \mu_{\mathrm{N}}(\mathrm{S})$.

Notably, when S is the spectrum of a local or global field F containing N distinct Nth roots of unity, μ_{N} is constant over S. This brings us to the setting of the Langlands–Weissman program (*cf.* [GGW18]). In this setting, the construction of topological covers factors through the étale realization functor (C.2) (*cf.* [Zha22, Proposition 2.3.13]).

Remark C.1.3. In the setting of the Langlands–Weissman program, it is natural to compare Weissman's construction of the L-group (*cf.* [Wei18], assuming that G is quasi-split) with our construction for its étale realization (formulated for function fields in §2.2.8).

As in [Wei18], our L-group is obtained from a Baer sum of extensions (*cf.* Remark 2.2.9), where the "first twist" is identified with Weissman's meta-Galois twist (*cf.* Theorem 7.4.6) and the "second twist" is obtained from the **Z**-linear morphism (2.30), whose construction is valid over an arbitrary base scheme S over which μ_{N} is constant.

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[23] Some of these shortcomings are known to experts. However, in putting them in writing, the author assumes sole responsibility for any potential misrepresentation of these works.

[24] Any deficiency in the presentation of this proof is due to the author.

In particular, our construction carries over to number fields, by replacing the "first twist" by Weissman's meta-Galois twist. What remains to be done is thus a comparison of (2.30) with the construction of [Wei18, §3] over a general base. We leave this for future work.

C.1.4. Let us now work over a smooth curve X over a field k (*cf.* §1.1.1). Let N be an integer invertible in k. Denote by Ψ the reduction mod N of the Kummer map (*cf.* Remark 1.1.10), assigning to a line bundle the μ_{N}-gerbe parametrizing its Nth roots:

$$\Psi : \mathsf{B}\mathbb{G}_m \to \mathsf{B}^2\mu_{\mathrm{N}}.$$

We fix a Brylinski–Deligne cover (C.1) of a reductive group X-scheme G. According to [Zha23, Corollary 3.4.7], there is an associated factorization central extension

$$1 \to \mathbb{G}_{m,\mathrm{Ran}} \to \widetilde{\mathrm{LG}} \to \mathrm{LG} \to 1 \tag{C.3}$$

equipped with a splitting over $\mathrm{L}^+\mathrm{G}$. Here, $\mathrm{L}^+\mathrm{G}$, LG are the arc, respectively loop groups over the Ran space (*cf.* §1.1.3).

Taking quotients of $\widetilde{\mathrm{LG}}$ by the left and right $\mathrm{L}^+\mathrm{G}$-actions defined by the above section, we obtain a factorization line bundle

$$\mathcal{L}_{\mathrm{Hec}_{\mathrm{G}}} := \mathrm{L}^+\mathrm{G}\backslash\widetilde{\mathrm{LG}}/\mathrm{L}^+\mathrm{G} \tag{C.4}$$

over the local Hecke stack $\mathrm{Hec}_{\mathrm{G}}$, equipped with a multiplicative structure (in the sense of §1.1.13, after replacing $\mathsf{B}^2\mathrm{A}$ by $\mathsf{B}\mathbb{G}_m$) induced from $\widetilde{\mathrm{LG}}$.[25]

C.1.5. Let us now assume further that μ_{N} is constant over k and fix an injective character $\zeta : \mu_{\mathrm{N}}(\mathsf{k}) \hookrightarrow \overline{\mathbf{Q}}_\ell^\times$, where ℓ is a prime invertible in k.

Applying the construction of the Satake category (*cf.* §1.3.7) with $\mathcal{G}_{\mathrm{Hec}_{\mathrm{G}}}$ replaced by $\Psi(\mathcal{L}_{\mathrm{Hec}_{\mathrm{G}}})$, we obtain a monoidal category

$$\mathsf{Sat}_{\Psi(\mathcal{L}),\zeta}(\mathrm{Hec}_{\mathrm{G},\underline{x}}) \tag{C.5}$$

associated to any S-point $\underline{x}$ of Ran.

Concretely, objects of (C.5) are perverse ULA sheaves over the total space of $\mathcal{L}_{\mathrm{Hec}_{\mathrm{G},\underline{x}}}$ (relative to S) which are $\mathbb{G}_m$-equivariant against the character local system induced from ζ. The monoidal structure on (C.5) is given by convolution with respect to the multiplicative structure on $\mathcal{L}_{\mathrm{Hec}_{\mathrm{G}}}$.

Using the factorization structure on $\mathcal{L}_{\mathrm{Hec}_{\mathrm{G}}}$, we may lift the monoidal category (C.5), for $\underline{x}$ the canonical map $\mathrm{X}^{\mathrm{I}} \to \mathrm{Ran}$ corresponding to a finite set I, to a symmetric monoidal category following §1.3.12–§1.3.16. This results in a symmetric monoidal category

$$^+\mathsf{Sat}_{\Psi(\mathcal{L}),\zeta}(\mathrm{Hec}_{\mathrm{G},\mathrm{I}}) \tag{C.6}$$

naturally associated to any finite set I.

C.1.6. In order to compare the Satake category (C.6) with the one introduced in §1.3 (for the étale realization of the given Brylinski–Deligne cover), we need to compare $\Psi(\mathcal{L}_{\mathrm{Hec}_{\mathrm{G}}})$ with the μ_{N}-gerbe $\mathcal{G}_{\mathrm{Hec}_{\mathrm{G}}}$ (*cf.* §1.1.7).

The following version of [GL18, Conjecture 3.4.5] is expected to be valid without the assumption that μ_{N} be constant, as long as N is invertible in k.[26]

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[25] We remark that the construction of $\widetilde{\mathrm{LG}}$, and consequently that of $\mathcal{L}_{\mathrm{Hec}_{\mathrm{G}}}$, is significantly less direct than that of $\mathcal{G}_{\mathrm{Hec}_{\mathrm{G}}}$ (*cf.* §1.1.7). This makes it much harder to prove properties about $\mathcal{L}_{\mathrm{Hec}_{\mathrm{G}}}$.

[26] Although we constructed $\mathcal{G}_{\mathrm{Hec}_{\mathrm{G}}}$ for coefficients in the constant sheaf A, the construction generalizes *verbatim* for the locally constant sheaf of coefficients $\mathrm{A} := \mu_{\mathrm{N}}$.

Conjecture C.1.7. *There is a canonical isomorphism of* μ_{N}*-gerbes over* $\mathrm{Hec}_{\mathrm{G}}$

$$\Psi(\mathcal{L}_{\mathrm{Hec}_{\mathrm{G}}}) \simeq \mathcal{G}_{\mathrm{Hec}_{\mathrm{G}}}$$

*compatibly with the factorization and multiplicative structures (*cf. *Proposition 1.2.11).*

C.1.8. More functorially, we expect the following diagram to commute:

$$\begin{tikzcd}
\mathrm{Maps}_*(\mathsf{B}_{\mathrm{Zar}}\mathrm{G}, \mathsf{B}^2_{\mathrm{Zar}}\mathsf{K}_2) \arrow[r, "\mathrm{R}_{\acute{e}t}"] \arrow[d] & \mathrm{Maps}_*(\mathsf{B}\mathrm{G}, \mathsf{B}^4\mu_{\mathrm{N}}^{\otimes 2}) \arrow[d] \\
\mathrm{Maps}_{\mathrm{fact}}(\mathrm{Hec}_{\mathrm{G}}^{[\cdot]}, \mathsf{B}_{\mathrm{Ran}}\mathbb{G}_m^{[\cdot]}) \arrow[r, "\Psi"] & \mathrm{Maps}_{\mathrm{fact}}(\mathrm{Hec}_{\mathrm{G}}^{[\cdot]}, \mathsf{B}^2_{\mathrm{Ran}}\mu_{\mathrm{N}}^{[\cdot]})
\end{tikzcd} \tag{C.7}$$

where the lower row consists of the spaces of factorization multiplicative line bundles, respectively μ_{N}-gerbes over $\mathrm{Hec}_{\mathrm{G}}$, and the vertical arrows are given by [Zha23, Corollary 3.4.7], respectively Proposition 1.2.11.

Let us indicate how one may obtain the commutativity of (C.7) by somewhat indirect means. Note that the vertical arrows in (C.7) are equivalences: For the left vertical arrow, this follows from [Zha23, Corollary 3.4.7], while for the right vertical arrow, this follows from [GL18, Proposition 3.1.9] and [GL18, Proposition 7.2.5].

Therefore, in order to prove that (C.7) commutes, we may replace the vertical arrows by their inverses. Then we may replace the top row by their classification data using [BD01, Theorem 7.2], respectively [Zha22, Theorem 5.1.13]. It then remains to show that the resulting square commutes, which one reduces to the case of tori.

Remark C.1.9. The reason for the indirectness of this argument is that the left vertical arrow in (C.7) lacks a "natural" definition (*cf.* the discussion in [Zha23, §4.5]). Namely, if it can be constructed in the same spirit as our construction of $\mathcal{G}_{\mathrm{Hec}_{\mathrm{G}}}$, then the commutativity of (C.7) would be tautological.

C.1.10. Let us return to the context of §C.1.5 and assume the validity of Conjecture C.1.7. We then have an equivalence of symmetric monoidal categories:

$${}^{+}\mathsf{Sat}_{\Psi(\mathcal{L}),\zeta}(\mathrm{Hec}_{\mathrm{G},\mathrm{I}}) \simeq {}^{+}\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G},\mathrm{I}}) \tag{C.8}$$

naturally in the finite set I.

In particular, one can reformulate our geometric Satake equivalence (*cf.* Theorem 2.3.2) in terms of the factorization line bundle $\mathcal{L}_{\mathrm{Hec}_{\mathrm{G}}}$:

$${}^{+}\mathsf{Sat}_{\Psi(\mathcal{L}),\zeta}(\mathrm{Hec}_{\mathrm{G},\mathrm{I}}) \simeq \mathsf{Rep}_{\mathrm{H}^{\boxtimes \mathrm{I}},(\nu+\vartheta)^{\boxplus \mathrm{I}}} \tag{C.9}$$

by composing (2.36) with (C.8).

C.2. [FL10] and [Lys14].

C.2.1. The general context for [FL10, Lys14] is that of §C.1.4-§C.1.5. However, these texts present more assumptions. Let us first briefly indicate their roles.

In [FL10], two further assumptions are imposed:

(1) G is almost simple and simply connected;
(2) the characteristic p of k does not divide $2\check{h}/d$, where $\check{h}$ is the dual Coxeter number of G and d is defined as in [FL10, §2.1].

The reason for (1) is that [FL10] constructs the central extension (C.3) only in this context (*cf.* [FL10, Proposition 2.2]). Note that *op.cit.* does not construct the factorization structure on (C.4), but rather uses the factorization structure of its $(2\check{h}/d)$-power to construct the commutativity constraint on the Satake category. This is responsible for assumption (2) which, however, can be removed following [FL10, Remark 2.10].

In [Lys14], one encounters a series of input data concerning central extensions of LG by $\mathbb{G}_{m,\mathrm{Ran}}$ (*cf.* [Lys14, §2.3]). The reason for this additional complication again has to do with (C.3), which was not constructed in general at the time [Lys14] was written.

C.2.2 *Fiber functor***.** Next, we mention an issue in the proof of [FL10, Theorem 2.9], which is pointed out by Reich (*cf.* [Rei12, §V.1]). Namely, the twisted Satake equivalence for tori stated in [FL10, §4.2] is *not* a symmetric monoidal equivalence.

Let us explain the issue more precisely. Assume that G is split simple and simply connected, endowed with a pinning $\mathrm{T} \subset \mathrm{B} \subset \mathrm{G}$, $\mathfrak{g}_{\check{\alpha}} \simeq \mathbb{G}_a$ ($\check{\alpha} \in \check{\Delta}$) and a Brylinski–Deligne cover (C.1). Let us also fix a square root $\Omega^{1/2}$ of the canonical line bundle of X.

In this context, it is effectively stated in [FL10, §4.2] that we have an equivalence

$$\mathsf{Sat}_{\Psi(\mathcal{L}),\zeta}(\mathrm{Hec}_{\mathrm{T},\{1\}}) \simeq \mathsf{Rep}_{\mathrm{T_H}}, \tag{C.10}$$

where $\mathrm{T_H}$ is the canonical maximal torus of H (*i.e.* the Langlands dual of $\mathrm{T}^\sharp$), given by restrictions along $\varpi^\lambda : \mathrm{X} \to \mathrm{Hec}_{\mathrm{T},\{1\}}$ for $\lambda \in \Lambda^\sharp$ (*cf.* §3.1.2).[27]

The same statement appears in [Lys14, §3.2] with weaker assumptions on G.

C.2.3. Let us specialize to the example $\mathrm{G} = \mathrm{SL}_2$ equipped with the unique Brylinski–Deligne cover whose quadratic form Q takes value 1 on the simple coroot $\alpha \in \Lambda$.

The factorization line bundle $\mathcal{L}_{\mathrm{Hec_G}}$ is then identified with the *inverse* of the determinant line bundle. Namely, the pullback of $\mathcal{L}_{\mathrm{Hec}}$ along an R-point $\mathrm{P}^0 \overset{x}{\simeq} \mathrm{P}^1$ of $\mathrm{Hec_G}$ is the line bundle $\det(\mathrm{P}^1 \mid \mathrm{P}^0)^{\otimes -1}$, where P^0, P^1 are viewed as rank-2 vector bundles over $\mathrm{D}_{\underline{x}}$ identified over $\mathring{\mathrm{D}}_{\underline{x}}$ (*cf.* [Fal03, §2]). Here, the notation $\det(\mathrm{P}^1 \mid \mathrm{P}^0)$ means the relative determinant

$$\det(\mathrm{P}^1 \mid \mathrm{P}^0) \coloneqq \det(\mathrm{L}^1/\mathrm{L}) \otimes \det(\mathrm{L}^0/\mathrm{L})^{-1},$$

where L^0, L^1 are the lattices in the Tate R-module $\Gamma(\mathring{\mathrm{D}}_{\underline{x}}, \mathrm{P}^0) \simeq \Gamma(\mathring{\mathrm{D}}_{\underline{x}}, \mathrm{P}^1)$ defined by P^0, respectively P^1, and L is a lattice contained in $\mathrm{L}^0 \cap \mathrm{L}^1$.

The restriction $\mathcal{L}_{\mathrm{Hec_T}} \coloneqq \mathcal{L}_{\mathrm{Hec_G}}|_{\mathrm{Hec_T}}$ (where $\mathrm{T} \coloneqq \mathbb{G}_m$ is the diagonal maximal torus) admits a canonical square-root as a factorization *super* line bundle:

$$\mathcal{L}_{\mathrm{Hec_T}} \simeq (\mathcal{L}_{\mathrm{Tate}})^{\otimes 2}, \tag{C.11}$$

where $\mathcal{L}_{\mathrm{Tate}}$ is the super line bundle over $\mathrm{Hec}_{\mathbb{G}_m}$ assigning $\det(\mathcal{L}^1 \otimes \Omega^{1/2} \mid \mathcal{L}^0 \otimes \Omega^{1/2})^{\otimes -1}$ with its natural $\mathbf{Z}/2$-grading to an R-point $\mathcal{L}^0 \overset{x}{\simeq} \mathcal{L}^1$ of $\mathrm{Hec}_{\mathbb{G}_m}$ (*cf.* [GL18, §5.2.4]). The latter has grading λ mod 2 over the component of $\mathrm{Hec}_{\mathbb{G}_m,\{1\}}$ corresponding to $\lambda \in \Lambda \simeq \mathbf{Z}$.

Let us take $\mathrm{N} = 2$, so $\Lambda^\sharp = \Lambda$, which we will identify with $\mathbf{Z}$. It follows from (C.11) that $\Psi(\mathcal{L}_{\mathrm{Hec_T}})$ is canonically isomorphic to the pullback of the sign gerbe along the reduction

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[27] We use the notation $\mathrm{T}^\sharp$, $\mathrm{T_H}$, and H as in §2.1, but here these objects are obtained directly from the combinatorial classification of Brylinski–Deligne covers. Namely, to (C.1), one may attach an integral Weyl-invariant quadratic form Q on Λ (*cf.* [BD01, Theorem 6.2]). Applying the constructions of §2.1 to Q mod N then yields $\mathrm{T}^\sharp$, $\mathrm{T_H}$, and H.

Note that Q mod N coincides with the strict Weyl-invariant quadratic form associated to the étale realization of (C.1) (*cf.* §2.1.3), in view of [BD01, §6.1] and [Zha22, §5.2].

mod 2 map $\epsilon : \Lambda \to \mathbf{Z}/2$ (*cf.* [GL18, §4.9.1]). Consequently, we obtain an equivalence of symmetric monoidal categories

$$\mathsf{Sat}_{\Psi(\mathcal{L}),\zeta}(\mathrm{Hec}_{\mathrm{T},\{1\}}) \simeq {}^{\epsilon}\mathsf{Rep}_{\mathrm{T}_{\mathrm{H}}}, \tag{C.12}$$

where ${}^{\epsilon}\mathsf{Rep}_{\mathrm{T}_{\mathrm{H}}}$ is equivalent to $\mathsf{Rep}_{\mathrm{T}_{\mathrm{H}}}$ as a Λ-graded monoidal category, but its commutativity constraint is given by the isomorphism

$$\mathrm{V}^{\lambda_1} \otimes \mathrm{V}^{\lambda_2} \simeq \mathrm{V}^{\lambda_2} \otimes \mathrm{V}^{\lambda_1}$$
$$a \otimes b \mapsto (-1)^{\epsilon(\lambda_1)\epsilon(\lambda_2)} b \otimes a$$

for any two objects $\mathrm{V}^{\lambda_1}, \mathrm{V}^{\lambda_2}$ with Λ-gradings λ_1, respectively λ_2.

There are no symmetric monoidal equivalences between ${}^{\epsilon}\mathsf{Rep}_{\mathrm{T}_{\mathrm{H}}}$ and $\mathsf{Rep}_{\mathrm{T}_{\mathrm{H}}}$ compatible with the Λ-gradings, so the same applies to the two sides of (C.10).

Remark C.2.4. Assuming the validity of Conjecture C.1.7, we may also obtain an equivalence (C.12) from (C.9) (for G = T and I = {1}).

Indeed, let μ be the étale realization of the Brylinski–Deligne cover of §C.2.3. The restriction of μ to $\mathrm{T} \simeq \mathbb{G}_m$ is canonically identified with the étale level of §7.4.5 (*cf.* [Zha22, Lemma 5.1.19]). It follows that the $\mathbb{E}_\infty$-monoidal morphism $\nu : \Lambda \to \mathsf{B}^2\{\pm 1\}$ admits an $\mathbb{E}_1$-monoidal trivialization, in terms of which the commutativity constraint of its fiber is given by the pairing $\lambda_1 \otimes \lambda_2 \mapsto (-1)^{\epsilon(\lambda_1)\epsilon(\lambda_2)}$ on $\Lambda \otimes \Lambda$ (*cf.* [SZ25, §2.2.9]).

Thus, we obtain a symmetric monoidal equivalence

$$\mathsf{Rep}_{\mathrm{T}_{\mathrm{H}},\nu} \simeq {}^{\epsilon}\mathsf{Rep}_{\mathrm{T}_{\mathrm{H}}},$$

and consequently (C.12), by composing with (C.9) and identifying $\nu + \vartheta$ with ν using the choice of $\Omega^{1/2}$ (*cf.* Remark 2.1.22). We have not checked that this construction of (C.12) coincides with the one of §C.2.3.

C.2.5. While the issue of §C.2.2 may seem innocuous, it causes a gap in the proof of [FL10, Theorem 2.9]. Namely, because (C.10) is not symmetric monoidal, it is not possible to apply the Tannakian formalism to ${}^{+}\mathsf{Sat}_{\Psi(\mathcal{L}),\zeta}(\mathrm{Hec}_{\mathrm{G},\{1\}})$.

We do not see a way to fill this gap without invoking the results of this text.

C.2.6 *Semisimplicity***.** The work [FL10] refers to the proof of [Lys06, Proposition 11] for the semisimplicity of the twisted Satake category.

However, the proof of [Lys06, Proposition 11] essentially states the the A-gerbe $\Psi(\mathcal{L}_{\mathrm{Hec}_{\mathrm{G}}})$ (at least in the special case $\mathrm{G} = \mathrm{Sp}_{2n}$ and N = 2) is trivial over the Demazure resolutions of closures of Bruhat cells in the affine flag variety. This is *not* true.

For a concrete example, we may again consider the one of §C.2.3. Denote by $\mathrm{Fl}^{w}_{\mathrm{G},x}$ the Bruhat cell in the affine flag variety $\mathrm{Fl}_{\mathrm{G},x}$ at a k-point x of X, associated to an element w of the (extended) affine Weyl group $\mathrm{W}^{\mathrm{aff}}$ (*cf.* §3.5). Write $\overline{\mathrm{Fl}}^{w}_{\mathrm{G},x}$ for its closure.

If w is a simple reflection, then $\overline{\mathrm{Fl}}^{w}_{\mathrm{G},x}$ coincides with its own Demazure resolution, and is isomorphic to $\mathbb{P}^1$ (*cf.* [Fal03, §3]). If we take w to be the simple reflection s associated to the *affine* simple root, then the pullback of the determinant line bundle $\mathcal{L}_{\mathrm{Hec}_{\mathrm{G}}}$ to $\overline{\mathrm{Fl}}^{s}_{\mathrm{G},x}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(1)$ (*cf.* [Fal03, Theorem 7]). This line bundle does not have a square root, so $\Psi(\mathcal{L}_{\mathrm{Hec}_{\mathrm{G}}})$ is nontrivial over $\overline{\mathrm{Fl}}^{s}_{\mathrm{G},x}$.

Remark C.2.7. The above computation is consistent with Lemma 3.5.3, as the $\{\pm 1\}$-gerbe $\mathcal{G}^{\mathrm{aff}}$ there coincides with the pullback of $\Psi(\mathcal{L}_{\mathrm{Hec}_{\mathrm{G}}})$. (This is a special case of Conjecture C.1.7, but for the example at hand, it can be established directly.)

C.3. [Rei12] and [GL18].

C.3.1. We now comment on Reich's treatment of the twisted geometric Satake equivalence [Rei12], along with further developments due to Gaitsgory and Lysenko [GL18]. In what follows, we shall take G to be a split reductive group.

We first remark that Reich's text [Rei12] is written in the complex analytic context, so "gerbes" in *op.cit.* refers to $\mathbf{C}^{\times}$-gerbes in the analytic topology.

The text [GL18] is written in the same context as ours (*i.e.* étale cohomology with torsion coefficients), but its treatment of the Satake equivalence requires the characteristic p of the ground field k to be coprime to the order of $\pi_1(\mathrm{G}_{\mathrm{der}})$ (*cf.* Remark C.3.4 below).

C.3.2. The text [Rei12] contains three main results:

1. the classification of factorization gerbes over Gr_{G} (*cf.* [Rei12, Theorem II.7.3]);
2. the proof that any factorization gerbe over Gr_{G} admits a canonical $\mathrm{L}^{+}\mathrm{G}$-equivariance structure, satisfying certain properties (*cf.* [Rei12, Theorem III.2.10]);
3. the construction of the geometric Satake equivalence [Rei12, Theorem IV.8.3] associated to each factorization gerbe over Gr_{G}.

Note that in both [Rei12] and [GL18], results (1) & (2) are needed for the formulation of the geometric Satake equivalence, because their authors use factorization gerbes over Gr_{G} to parametrize covering groups (as opposed to étale levels, *cf.* §1.1.1).[28]

In what follows, we will discuss the status of these three results in order.

C.3.3 *Classification of factorization gerbes***.** The result [Rei12, Theorem II.7.3] is incorrect as stated, because the fiber sequence of *loc.cit.* does not split. Since its proof relies on an incorrectly defined splitting, we do not see a way to fix it.

The correct classification of factorization gerbes over Gr_{G} appears as [GL18, Proposition 3.1.9]: This is the assertion that factorization gerbes over Gr_{G} are *equivalent* to étale levels. In particular, it implies that the input data of our version of the Satake equivalence (*cf.* Theorem 2.3.2) are equivalent to those of [GL18, §9], except that we do *not* need the condition that p is coprime to $\pi_1(\mathrm{G}_{\mathrm{der}})$.

Remark C.3.4. In [GL18], the condition on p is needed for the construction of the metaplectic dual data (*cf.* [GL18, §6.2, Appendix A]).

It is unnecessary for us since our construction of the metaplectic dual data is different from that of [GL18] and does not make use of the affine Grassmannian (*cf.* §2.1).

C.3.5 $\mathrm{L}^{+}\mathrm{G}$*-equivariance structure***.** The result [Rei12, Theorem III.2.10] does not appear to be adequately justified.

More precisely, in the proofs of [Rei12, Proposition III.2.8] and [Rei12, Theorem III.2.10], the author applies the notion of "orders" of gerbes (*cf.* [Rei12, §I.5]) along certain Cartier divisors in indschemes to justify their extension properties. This relies on cohomological purity of the Cartier divisors in question, which is not justified.

However, [GL18, §7.3] offers two constructions of the $\mathrm{L}^{+}\mathrm{G}$-equivariance structure: One of them is essentially restated in §1.1 of this text, but the other one uses different ideas.

C.3.6 *Geometric Satake equivalence***.** Taking into account the corrections §C.3.3-C.3.5 offered in [GL18], one may state Reich's version of the geometric Satake equivalence as in

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[28] As mentioned in the introduction, the idea of using étale levels to parametrize covering groups dates much earlier: It first appeared in Deligne's work [Del96].

[GL18, §9.2]. However, *op.cit.* refers to [Rei12, Theorem IV.8.3] for its proof and there are at least two issues with Reich's proof:

(1) The proof of the semisimplicity of the pointwise Satake category (*cf.* [Rei12, Proposition IV.6.13]) contains an error.
(2) It does not offer a correct construction of the fiber functor.

The issue with semisimplicity occurs in [GL18, Lemma IV.6.9], where the criterion for equality there does *not* hold in general.

The issue with the fiber functor occurs in [GL18, Lemma IV.7.8], whose proof indicates that the $\hat{\mathrm{Z}}_{\mathrm{H}}$-grading (denoted by $\mathrm{X}^*(\mathrm{Z}(\check{\mathrm{G}}_{\mathrm{Q}}))$ in *loc.cit.*) on the twisted Satake category is induced from its $\pi_1\mathrm{G}$-grading. We cannot make sense of this statement since the natural map $\hat{\mathrm{Z}}_{\mathrm{H}} \to \pi_1\mathrm{G}$ is *not* injective in general.

On the other hand, assuming the semisimplicity of the Satake category, one can obtain a $\hat{\mathrm{Z}}_{\mathrm{H}}$-grading by coarsening its $\Lambda^\sharp$-grading. However, the $\hat{\mathrm{Z}}_{\mathrm{H}}$-grading obtained this way is not obviously compatible with the monoidal structure. Thus, we do not know how to fix Reich's construction of the fiber functor without invoking the results of this text.

Remark C.3.7. As mentioned in §C.2.2 and §C.2.6, these two issues of the proof are already present in [FL10], and at least one of them is explicitly pointed out by Reich.

Let us also acknowledge that Reich first observed the need to construct a $\hat{\mathrm{Z}}_{\mathrm{H}}$-grading on the Satake category compatible with the monoidal structure, even though this may not have been achieved in [Rei12].

C.4. Gaitsgory's proof of semisimplicity.

C.4.1. Finally, let us sketch an unpublished proof due to Gaitsgory of the semisimplicity of the twisted Satake category at a point (*cf.* Corollary 3.3.7). It avoids parity vanishing (*cf.* Proposition 3.4.8) and instead uses the $\hat{\mathrm{Z}}_{\mathrm{H}}$-grading of §5.3, combined with the method of the proof of [FGV01, Theorem 3].

Let us return to the context of §3. Furthermore, we shall fix a k-point x of X and consider the local Hecke stack $\mathrm{Hec}_{\mathrm{G}}$ at x, suppressing x from the notation. Our goal is to prove that $\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G}})$ is semisimple.

C.4.2. First, we establish the following vanishing result: Given $\lambda \in \Lambda^{\sharp,+}$ and $(\mathcal{G}_{\varpi^\lambda}, \zeta)$-twisted lisse sheaves $\mathcal{E}_1$, $\mathcal{E}_2$ over x, we have

$$\mathrm{Ext}^1(\mathrm{IC}^\lambda(\mathcal{E}_1), \mathrm{IC}^\lambda(\mathcal{E}_2)) \simeq 0, \tag{C.13}$$

where IC^λ is the functor of §3.3.4.

To prove (C.13), we may choose a trivialization of $\mathcal{G}_{\varpi^\lambda}$ at x and reduce to the case where $\mathcal{E}_1$, $\mathcal{E}_2$ are both the constant sheaf e. Then the argument of [Gai01, Proposition 1] applies. (Details may also be found in [Rei12, Lemma IV.6.6]).

Next, we need the following consequence of the Decomposition Theorem.

Proposition C.4.3 (Reich)**.** *Given $\lambda_1, \lambda_2 \in \Lambda^{\sharp,+}$ and $\mathcal{G}_{\varpi^{\lambda_1}}$, $\mathcal{G}_{\varpi^{\lambda_2}}$-twisted lisse sheaves $\mathcal{E}_1$, $\mathcal{E}_2$ over x, the convolution product $\mathrm{IC}^{\lambda_1}(\mathcal{E}_1) \star \mathrm{IC}^{\lambda_2}(\mathcal{E}_2)$ is a direct sum of objects of the form $\mathrm{IC}^\lambda(\mathcal{E})$, for $\lambda \in \Lambda^{\sharp,+}$ and $\mathcal{G}_{\varpi^\lambda}$-twisted lisse sheaves $\mathcal{E}$ over x.*

C.4.4. Proposition C.4.3 is a reformulation of [Rei12, Lemma IV.6.11]. Note, however, that the proof of *loc.cit.* uses a corollary of the problematic [Rei12, Lemma IV.6.9] (*cf.* §C.3.6), but the problematic part (the criterion for equality) is not used.

Let us write the conclusion of Proposition C.4.3 as an isomorphism

$$\mathrm{IC}^{\lambda_1}(\mathcal{E}_1)\star\mathrm{IC}^{\lambda_2}(\mathcal{E}_2)\simeq\bigoplus_{\lambda}\mathrm{IC}^{\lambda}(\mathcal{E}) \tag{C.14}$$

in $\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G}})$. We now need a consequence of the $\hat{\mathrm{Z}}_{\mathrm{H}}$-grading on the Satake category: By Corollary 5.3.4 (and the proof of Proposition 5.3.3), all elements $\lambda\in\Lambda^{\sharp,+}$ appearing in the sum (C.14) share the same equivalence class in $\hat{\mathrm{Z}}_{\mathrm{H}}$ as $\lambda_1+\lambda_2$.

C.4.5. As in §6.1, we construct the untwisted Satake category ${}^{+}\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G}})_{-(\nu+\vartheta)}$ together with the fiber functor ω valued in finite-dimensional e-vector spaces.

Applying Tannakian formalism, we obtain a symmetric monoidal equivalence

$${}^{+}\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G}})_{-(\nu+\vartheta)}\simeq\mathsf{Rep}_{\check{\mathrm{G}}}, \tag{C.15}$$

for some affine monoid e-scheme $\check{\mathrm{G}}$. The rigidity of the Satake category (*cf.* Proposition 1.6.4) implies that $\check{\mathrm{G}}$ is an affine group e-scheme.

The construction of §6.4.4 yields a group subscheme $\mathrm{T}_{\mathrm{H}}\subset\check{\mathrm{G}}$, where T_{H} is the canonical maximal torus of H. Write $\check{\mathrm{B}}$ for the group subscheme of $\check{\mathrm{G}}$ consisting of automorphisms of ω preserving the filtration defined by $2\check{\rho}$, *i.e.* $\mathrm{F}^{\geq n}\omega\subset\omega$ $(n\in\mathbf{Z})$ is the sum of components of T_{H}-weights λ satisfying $\langle 2\check{\rho},\lambda\rangle\geq n$.

C.4.6. Write $\mathcal{C}$ for the full subcategory of ${}^{+}\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G}})_{-(\nu+\vartheta)}$ generated by the images of (the untwisted forms of) IC^{λ} $(\lambda\in\Lambda^{\sharp,+})$ under *direct sums*.

By (C.13), $\mathcal{C}$ is semisimple. Since ${}^{+}\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G}})_{-(\nu+\vartheta)}$ is generated by the images of IC^{λ} $(\lambda\in\Lambda^{\sharp,+})$ under extensions, $\mathcal{C}$ coincides with its *maximal* semisimple subcategory. By Proposition C.4.3, $\mathcal{C}$ inherits a symmetric monoidal structure.

By Tannakian formalism, we have a symmetric monoidal equivalence $\mathcal{C}\simeq\mathsf{Rep}_{\check{\mathrm{G}}_{\mathrm{red}}}$ for some affine group scheme $\check{\mathrm{G}}_{\mathrm{red}}$, equipped with a natural homomorphism

$$\check{\mathrm{G}}\to\check{\mathrm{G}}_{\mathrm{red}}. \tag{C.16}$$

The morphism (C.16) is faithfully flat by [DM82, Proposition 2.21(a)]. Furthermore, the proof of Lemma 6.4.3 shows that $\check{\mathrm{G}}_{\mathrm{red}}$ is reductive and the proof of Lemma 6.4.5 shows that the composition $\mathrm{T}_{\mathrm{H}}\hookrightarrow\check{\mathrm{G}}\to\check{\mathrm{G}}_{\mathrm{red}}$ realizes T_{H} as a maximal torus of $\check{\mathrm{G}}_{\mathrm{red}}$. Denote by $\check{\mathrm{B}}_{\mathrm{red}}$ the image of $\check{\mathrm{B}}$ in $\check{\mathrm{G}}_{\mathrm{red}}$, which is a Borel subgroup.

The following implication is the main insight of the proof.

Lemma C.4.7. *Assuming that* $\mathrm{T}_{\mathrm{H}}\subset\check{\mathrm{B}}_{\mathrm{red}}\subset\check{\mathrm{G}}_{\mathrm{red}}$ *and* $\mathrm{T}_{\mathrm{H}}\subset\mathrm{B}_{\mathrm{H}}\subset\mathrm{H}$ *define the same root data, then* (C.16) *is an isomorphism.*

Proof. It suffices to proving the vanishing statement

$$\mathrm{Ext}^1(\mathrm{IC}^{\lambda_1}(\mathsf{e}),\mathrm{IC}^{\lambda_2}(\mathsf{e}))\simeq 0, \tag{C.17}$$

for any $\lambda_1,\lambda_2\in\Lambda^{\sharp,+}$, where the Ext group is computed in ${}^{+}\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G}})_{-(\nu+\vartheta)}$. Note that the special case for $\lambda_1=\lambda_2$ is known by §C.4.2. Moreover, by Proposition 5.3.3, (C.17) also holds when λ_1, λ_2 have distinct images in $\hat{\mathrm{Z}}_{\mathrm{H}}$, so let us assume that λ_1, λ_2 have the same image in $\hat{\mathrm{Z}}_{\mathrm{H}}$ in what follows.

Note that we have an isomorphism

$$\mathrm{Ext}^1(\mathrm{IC}^{\lambda_1}(\mathsf{e}),\mathrm{IC}^{\lambda_2}(\mathsf{e}))\simeq\mathrm{Ext}^1(\mathrm{IC}^{0}(\mathsf{e}),\mathrm{IC}^{\lambda_1}(\mathsf{e})^{\vee}\otimes\mathrm{IC}^{\lambda_2}(\mathsf{e})), \tag{C.18}$$

where the monoidal dual $\mathrm{IC}^{\lambda_1}(\mathsf{e})^\vee$ is a simple object of ${}^{+}\mathsf{Sat}_{\mathcal{G},\zeta}(\mathrm{Hec}_{\mathrm{G}})_{-(\nu+\vartheta)}$ with $\hat{\mathrm{Z}}_{\mathrm{H}}$-grading the image of $-\lambda_1$ (*cf.* Corollary 5.3.4). This implies that $\mathrm{IC}^{\lambda_1}(\mathsf{e})^\vee \otimes \mathrm{IC}^{\lambda_2}(\mathsf{e})$ is a direct sum of objects of the form $\mathrm{IC}^{\lambda}(\mathsf{e})$, where $\lambda \in \Lambda^{\sharp,+}$ has vanishing image in $\hat{\mathrm{Z}}_{\mathrm{H}}$ (*cf.* §C.4.4).

It thus remains to prove (C.17) when $\lambda_1 = 0$ and λ_2 belonging to the root lattice of H. Since $\mathrm{T}_{\mathrm{H}} \subset \check{\mathrm{B}}_{\mathrm{red}} \subset \check{\mathrm{G}}_{\mathrm{red}}$ and $\mathrm{T}_{\mathrm{H}} \subset \mathrm{B}_{\mathrm{H}} \subset \mathrm{H}$ are assumed to have the same root data, λ_2 belongs to the root lattice of $\hat{\mathrm{G}}_{\mathrm{red}}$. Applying [FGV01, Lemma 6.1.3] to $\check{\mathrm{G}}_{\mathrm{red}}$, we may realize $\mathrm{IC}^{\lambda_2}(\mathsf{e})$ as a summand of $\mathrm{IC}^{\lambda}(\mathsf{e})^\vee \star \mathrm{IC}^{\lambda}(\mathsf{e})$, for some $\lambda \in \Lambda^{\sharp,+}$. Applying (C.18) again, we reduce to the assertion (C.17) for $\lambda_1 = \lambda_2$, which is established in §C.4.2. ☐

C.4.8. We shall now show that the assumption of Lemma C.4.7 holds unconditionally.

This requires a minor modification of the proof of Proposition 6.4.7, because the construction of $\check{\mathrm{G}}_{\mathrm{red}}$ is *a priori* incompatible with constant term functors.

Lemma C.4.9. *The triples* $\mathrm{T}_{\mathrm{H}} \subset \check{\mathrm{B}}_{\mathrm{red}} \subset \check{\mathrm{G}}_{\mathrm{red}}$, $\mathrm{T}_{\mathrm{H}} \subset \mathrm{B}_{\mathrm{H}} \subset \mathrm{H}$ *define the same root data.*

Proof. The case where G has a unique simple root is handled as in the proof of Proposition 6.4.7. For the general case, we fix a simple root $\check{\alpha}$ of G and write P for the corresponding subminimal parabolic with Levi quotient M. The constant term functor induces a morphism of affine group e-schemes

$$\check{\mathrm{M}} \to \check{\mathrm{G}}. \tag{C.19}$$

Applying Lemma C.4.7 to M, we find that $\check{\mathrm{M}}$ is a reductive group e-scheme with unique simple root $\alpha^\sharp$ and simple coroot $\check{\alpha}^\sharp$. Then, the argument of Proposition 6.4.7 shows that the composition $\check{\mathrm{M}} \to \check{\mathrm{G}}_{\mathrm{red}}$ of (C.19) and (C.16) is a closed immersion, and consequently yields the computation of the root data of $\mathrm{T}_{\mathrm{H}} \subset \check{\mathrm{B}}_{\mathrm{red}} \subset \check{\mathrm{G}}_{\mathrm{red}}$. ☐

Proposition C.4.10. *The morphism* (C.16) *is an isomorphism.*

Proof. This follows by combining Lemma C.4.7 and Lemma C.4.9. ☐

C.4.11. Proposition C.4.10 implies that $\check{\mathrm{G}}$ is reductive, thereby giving a new proof of the semisimplicity of the Satake category (*cf.* Corollary 3.3.7).

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